Establishing a Number System Between $\mathbb{C}$ and $\mathbb{H}$ with Defined Division by Zero and a 5D Extension of the Partial Algebra

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September 7, 2026

Abstract

In this paper we formally construct $\mathbb{M}$, a number system that contains three dimensions and forms a bridge between the complex numbers and the quaternions. In this system we define the operation of division by zero $1/0 = \infty$. The third dimension of the system is the scalar infinity s, which does not mix with real or complex numbers.

Historically, a system with exactly three dimensions was impossible because of the multiplication of two new axes. In the quaternions these are i and j, whose product must give k, creating a new dimension because ij cannot be displayed in those three dimensions. Historically, taking the square root of -1 was not allowed. This was solved by the new number i, which introduced the complex numbers and achieved the transition from $\mathbb{R}$ to $\mathbb{C}$ by adding the new number i such that $i^2 = -1$. $\mathbb{R}$ is one-dimensional, and $\mathbb{C}$ is a two-dimensional number system, so the question arises whether there are number systems above $\mathbb{C}$.

The answer was first found by William Rowan Hamilton when he tried to construct a three-dimensional number system. By adding one new axis j such that $j^2 = i^2 = -1$, he encountered the problem of what the product of i and j would give. He realized that it would give k, a new spatial axis, and so he constructed a four-dimensional number system with axes $1, i, j, k$ such that $k^2 = j^2 = i^2 = -1$. Thus the three-dimensional algebra was skipped because no matter which axis we add, the product of two axes must always give something new, namely a fourth dimension.

Later Frobenius and Hurwitz formalized this and proved that number systems are possible only in dimensions 2^n, where n is a natural number: namely 1 ($\mathbb{R}$), 2 ($\mathbb{C}$), 4 ($\mathbb{H}$), 8 ($\mathbb{O}$), and so on, up to 256 dimensions and beyond. This is the Cayley-Dickson construction, and the number systems obtained after the sedenions are no longer particularly interesting or useful for study because they lose critical properties such as associativity, alternativity, and commutativity.

All these systems do not solve the problems for which new systems are introduced, namely the definition of undefined numbers. Just as the complex numbers define the square root of -1, which was undefined in the real numbers, so the definition of $n/0$ brings with it the definitions of other undefined numbers, such as the zeroth root and zero to the zeroth power.

1 Introduction to $\mathbb{M}$

In classical mathematics $n/0$ is undefined because the null multiplication axiom $(0 \cdot x = 0)$ is an unchangeable rule. Also, infinity in classical mathematics is only a concept, not a number, and there are many paradoxes in which our understanding of infinity breaks down. In this paper we formalize the number system $\mathbb{M}$ that lies between $\mathbb{C}$ and $\mathbb{H}$.

The system is carried by the number s as a scalar object that is the holder of the third axis. We modify the null multiplication axiom $(0 \cdot x = 0)$ and allow $0 \times \infty = 1$, thereby constructing a three-dimensional number system that does not collapse into the fourth dimension, and later we also construct the fifth dimension. We also distinguish two types of infinity, ∞ and s, with different roles.

The system $\mathbb{M}$ successfully coexists with the theorems established by Frobenius and Hurwitz because it operates outside their set axioms, especially outside $0 \cdot x = 0$. By abolishing the null multiplication axiom and establishing a partial algebra with representation-defined multiplication, $\mathbb{M}$ opens a new path that is not within the domain of those restrictions. $\mathbb{M}$ modifies zero, not dimensions, as Hamilton did. $\mathbb{M}$ preserves commutativity, unlike the quaternions, during the transition from the second to the fourth dimension.

2 Related Work and Contextualization

Before continuing, it is necessary to place $\mathbb{M}$ within existing nonstandard arithmetics:

- **Wheel Theory:** Formalized by Carlström (2004). Wheel Theory represents a mathematical structure in which division by zero is universally defined by adding two elements: $\infty = 1/0$ and $\perp = 0/0$. Wheels successfully remove problems of division by zero, but they sacrifice distributivity and introduce an additional element $\perp$ ("bottom") as the result of $0/0$. In contrast, $\mathbb{M}$ retains ∞ and retains distributivity except for ∞i (which we will define later) and has no need for the element $\perp$ — the expression $0/0$ is computed as $0 \times \infty = 1$, consistent with the limit $\lim_{x \to 0} x/x = 1$. Although Wheel Theory manages to define division by zero, it loses many other properties and is therefore not a particularly practical system for use, and apart from division by zero it has no other advantages.

- **IEEE 754 Standard:** In computer science it is sometimes allowed to use `1.0 / 0.0 = Inf`, but as soon as that infinity is triggered, all three-dimensional graphics crash and fall into that undefined place. $\mathbb{M}$ provides a structured algebraic alternative that supports such coordinates through the infinite axis. In computer 3D graphics, programs crash when they reach the position $(0, 0, 0)$. In $\mathbb{M}$ that point is a normal element of the three-dimensional algebra and there would be no crash.

- **Projective Geometry and Riemann Spheres:** In complex analysis the Riemann sphere places infinity at an undefined point at the North Pole. Although very useful for visualizing complex functions that have infinite coordinates, it does not solve the main problem of division by zero, even though infinity is defined as a number. In complex analysis $\infty - \infty$ and $\frac{\infty}{\infty}$ remain undefined. Furthermore, numbers such as 2∞ do not exist because infinity is not a scalar object. The Riemann sphere, along with Wheel Theory, comes closest to the system we will define here because it allows infinity to be a number. In $\mathbb{M}$ we will obtain the same effect of function overview, avoiding infinite coordinates and remaining in three dimensions without bending the entire space.

Wheel Theory and Riemann spheres have a curved space and do not have infinity as a scalar object. In $\mathbb{M}$ space remains flat and allows infinity to be a scalar.

3 Axioms and Mathematical Framework

We define $\mathbb{M}$ over the real numbers and two special non-real axes, i and s.

Definition 3.1. Let $\mathbb{M}$ be a three-dimensional vector space over the field of real numbers, with basis $\{1, s, i\}$. A hypercomplex element $m \in \mathbb{M}$ is written as:

$$m = x + ys + zi, \tag{1}$$

where $x, y, z \in \mathbb{R}$. The element x represents the real part, z the standard imaginary part, and y the coefficient of the scalar infinity. The set $\mathbb{M}$ is the set of all such linear combinations.

We now list the basic axioms of the set $\mathbb{M}$.

Axiom 1 (Scalar infinity as holder of the axis)**.** s is a new basis vector, linearly independent of 1 and i. s is not a real number and does not behave like classical infinity: $s \neq s+1$ and $2s \neq s$ hold, because $(0,1,0)$ and $(0,2,0)$ are different vectors in the basis.

Note on possible interpretations of the number s. The number s itself is defined exclusively as a basis element with the properties stated above. The following are possible concrete interpretations:

- $s = 1/\varepsilon$, where ε is an infinitesimal element that is not part of the basic algebra, but can be introduced as an extension;
- s as a measure of the set of natural numbers $\mathbb{N}$, where s is not directly $\aleph_0$, but an algebraic element that only formally corresponds to that quantity.

These interpretations are not part of the basic axioms and serve only as intuition.

Axiom 2 (Absolute infinity as a new element)**.** In $\mathbb{M}$ we introduce a new element ∞, which is the solution of the equation:

$$0 \times \infty = 1. \tag{2}$$

It follows that $\frac{1}{0} = \infty$.

Axiom 3 (Difference between ∞ and s)**.** ∞ and s are different elements of the algebra. s is subject to the null multiplication axiom:

$$0 \times s = 0. \tag{3}$$

This is necessary for resolving paradoxes (see Subsection 3.1).

Axiom 4 ($s^2 = -1$)**.** Every new axis must square to -1 if we want to preserve rotations and if we want $\mathbb{M}$ to lie between $\mathbb{C}$ and $\mathbb{H}$.

$$s^2 = -1, \quad i^2 = -1. \tag{4}$$

Just as in the quaternions $i^2 = j^2 = k^2 = -1$ holds, so s must behave. If $s^2 \neq -1$, the algebra would contain two non-zero numbers whose product is zero: $(1+s)(1-s) = 1 - s^2 = 0$.

A brief look at s under exponents Before continuing, let us analyze the powers of s because this will be important later. We defined $s^2 = -1$ because of the new axis and consistency. $s^2 = -1$ also has an interesting rotation property: s^2 rotates by half a circle and reverses the orientation of any point on the axis. This property will be important later. Now let us examine how s behaves under exponentiation. The powers of s behave analogously to the powers of i: $s^2 = -1$, $s^3 = -s$, $s^4 = 1$, and generally the cycle repeats with period 4. s has the same properties as i, which is used for rotation in robotics, so s is identical to i also under exponentiation. We will discuss this in more detail later; for now this is enough for understanding the work.

3.1 Paradox Check

Since we are working in a system in which the equation $0 \cdot x = 1$ has a solution, we need to check paradoxes. Let us try the classical way of forcing a paradox:

$$\frac{1}{0} = \infty, \tag{5}$$

$$0 \times \left(\frac{1}{0}\right) = 0 \times \infty = 1. \tag{6}$$

There is no paradox like $1 = 0$. Now let us check the same with scalars:

$$\frac{2}{0} = 2\left(\frac{1}{0}\right) = 2\infty, \tag{7}$$

$$0 \times (2\infty) = 2 \times (0 \times \infty) = 2 \times 1 = 2. \tag{8}$$

Here we multiplied by zero to try to force $2 = 1$. Since 2∞ and 1∞ are completely different linear elements, linearity is preserved and every multiplication by zero is remembered. The system remains free of the classical division-by-zero paradoxes.

3.2 Comparison of two numbers of the form $x + ys$

Two numbers from $\mathbb{M}$ can be compared if the complex part is zero. We know that complex numbers cannot be compared, that is, it cannot be said which one is larger. If we have numbers of the form $a+bs$ or $a+b\infty$, they can be compared. Since infinity is a concept that we have turned into a very large number, we can compare two numbers because infinity can be considered a real number.

Theorem 3.1. Let $a_1 + b_1 s$ and $a_2 + b_2 s$ be two triplets without imaginary coordinates, where neither a nor b is zero. These two triplets can be compared. We establish that only the number next to s is considered. If b_1 is larger, then $a_1 + b_1 s$ is also larger; if b_1 is smaller, then $a_1 + b_1 s$ is also smaller. The comparison of two such numbers is considered only with respect to s. There is an exception when b is epsilon, but since epsilon is not an element of this algebra nor a real number, we need not burden ourselves with it here, although we will analyze it at the end of the paper.

4 Associativity

In $\mathbb{M}$ associativity holds everywhere except in the specific interaction between ∞ and 0:

$$(0 \times 0) \times \frac{1}{0} = 0 \times \infty = 1.$$

We see that the result is 1. If we change the order of operations, we obtain:

$$0 \times \left(0 \times \frac{1}{0}\right) = 0 \times 1 = 0.$$

We see that in that specific interaction between ∞ and 0, associativity does not hold. In the pure $\mathbb{M}$ system associativity holds universally because the element ∞ never appears alone. Since division by zero is defined as $1/0 = \infty$, every calculation involving division by zero can be non-associative. Associativity is violated only in that specific interaction between ∞ and 0. For this reason the algebra does not collapse and is therefore called a partial algebra.

5 Algebraic Operations and the Basic Multiplication Table

Addition of two triplets in $\mathbb{M}$ is componentwise:

$$(x_1 + y_1 s + z_1 i) + (x_2 + y_2 s + z_2 i) = (x_1 + x_2) + (y_1 + y_2)s + (z_1 + z_2)i. \tag{9}$$

Multiplication of two triplets is given by the formula:

$$m \cdot n = (ax - by - cz) + (ay + bx)s + (az + cx)i + (bz + cy)i\infty, \tag{10}$$

where $m = a + bs + ci$ and $n = x + ys + zi$. Here a new element ∞i appears. Before explaining its role, we give the multiplication table of the basic basis elements:

$\times$	1	s	i
1	1	s	i
s	s	-1	∞i
i	i	∞i	-1

Table 1: Basic multiplication table for $\mathbb{M}$

We define the product of the infinite and imaginary axes as

$$s \cdot i = \infty i.$$

We abbreviate this as ∞i. The notation ∞i instead of si emphasizes that we are dealing with absolute infinity ∞, not scalar infinity s. Thus the product of two non-real axes does not create a new dimension, but is displayed as an infinite point on the already existing imaginary axis.

If we multiply step by step using this definition, we obtain the element $(bz + cy)\infty i$. This element cannot be displayed in the coordinates $(1, s, i)$. It can therefore be concluded that the result of multiplying two standard triplets does not belong to the basic three-dimensional basis; this is exactly the point where Hamilton concluded that he needed a fourth dimension.

We can avoid the fourth dimension by using the dual-point representation. Instead of introducing a new axis, we place the ∞i part on the imaginary axis, but label it as a separate type of point. In other words, the imaginary axis now carries two types of points:

- standard imaginary points xi;
- infinite imaginary points $x\infty i$.

Thus the algebra remains three-dimensional, but the result of multiplication is no longer a single point but two: the standard and the ∞i-point. This avoids an infinite coordinate and introduces the dual-point representation.

Consequently, the coordinates of a number can look like $(k, j, l + m\infty)$, where l is the standard imaginary coordinate and m is the coefficient of ∞i. Such a number is displayed by two points:

$$(k, j, l)_i, \qquad (k, j, m)_{\infty i}.$$

This is the basic idea of the dual-point representation: every number containing ∞i is displayed with two finite 3D points instead of one with an infinite coordinate.

6 Dual-Point Representation and Inversions

The dual-point representation allows us to see every number from $\mathbb{M}$ on a three-dimensional graph with finite coordinates. Note that this is not an "escape" from the fourth dimension because we have previously shown that the result of multiplying two triplets is still in $\mathbb{M}$, but with infinite coordinates, so the number would still be three-dimensional and not four-dimensional. The points would be displayed as follows:

- A standard triplet of the form $a + bs + ci$ is displayed with coordinates (a, b, c). This number is one point as seen from the imaginary axis.
- A number created by multiplying two triplets and containing the ∞i component, i.e. $a + bs + ci + d\infty i$, is displayed as two points:
  - the first point has coordinates (a, b, c) seen from the imaginary axis; thus this point is the same as the standard triplet;
  - the second point has coordinates (a, b, d) seen from the ∞i axis (the ∞i axis and the imaginary axis are the same axis, but for every triplet of the form $a + bs + ci + d\infty i$ displayed with two points it must be emphasized from which axis the coordinate was obtained, i or ∞i).
- The dual-point also applies to s and ∞; coordinates such as $a + bs + ci + d\infty i + e\infty$ are displayed following previous rules and as four point coordinates: (a, b, c), (a, b, d), (a, e, c), and (a, e, d). In basic $\mathbb{M}$ such a number does not appear because ∞ arises exclusively from division by zero. However, in the 5D extension ∞ is already a basis vector, so such elements naturally appear.

In conclusion, the real axis remains untouched, and on the other two axes we "glue" one more axis each: on the imaginary axis points of the form xi and $x\infty i$ are displayed, and on the infinite axis points of the form xs and $x\infty$.

Example of dual-point representation:

$$(2+3s+5i)\times(3+s+10i) = -47+11s+65i+5i\infty.$$

In the dual-point representation this point would be displayed as two point coordinates:

$$P_1 = (-47, 11, 65)_i, \qquad P_2 = (-47, 11, 5)_{i\infty}.$$

6.1 Closure under Addition and Multiplication

Theorem 6.1. $\mathbb{M}$ is closed under addition and multiplication if the results can be displayed as one or two points with finite coordinates (dual-point representation).

Proof. Addition is completely closed and gives a single point in 3D space without the need for infinite coordinates. For multiplication, take two triplets $x_1 = (a_1, b_1, c_1)$ and $x_2 = (a_2, b_2, c_2)$. Now use the general multiplication formula:

$$x_1 \times x_2 = (a_1a_2 - b_1b_2 - c_1c_2) + (a_1b_2 + b_1a_2)s + (a_1c_2 + c_1a_2)i + (b_1c_2 + c_1b_2)i\infty. \tag{11}$$

The numbers next to 1, s, and i give the standard point $(a_1a_2 - b_1b_2 - c_1c_2,\ a_1b_2 + b_1a_2,\ a_1c_2 + c_1a_2)_i$. As soon as one of b_1c_2 or b_2c_1 is not zero, we need to use the dual-point representation because the ∞i component also appears. Thus we obtain an additional point $(a_1a_2 - b_1b_2 - c_1c_2,\ a_1b_2 + b_1a_2,\ b_1c_2 + c_1b_2)_{i\infty}$. Both points share the same infinite and real coordinates, and differ in imaginary coordinates and type. All coordinates are finite and real, so the system remains three-dimensional without the need for infinite coordinates. □

6.2 Inversions in $\mathbb{M}$

In $\mathbb{M}$ the inversion formula for a general triplet is not $1/m = \frac{a-bs-ci}{a^2+b^2+c^2}$, as we would expect, if b or c is not zero. It follows that the inverse of any standard triplet must be a dual-point object and contain the ∞i component. This is because the equation $m \cdot (x + ys + zi) = 1$ has no solution due to overdetermination: four equations with three unknowns. However, a solution exists if we allow the inverse to be a dual-point object, because there is a solution that satisfies two 3×3 systems. This can exist because in the dual-point representation one point can be independent of the other in this context and have three coordinates, not four. The same applies to the second point. The inverse of a general triplet as a dual-point object is given by the formula:

$$\boxed{\frac{1}{m} = \left\{ \left(\frac{a}{A}, -\frac{b}{A}, -\frac{c}{A}\right)_i, \left(\frac{a}{B}, -\frac{b}{B}, \frac{c}{B}\right)_{i\infty} \right\}}.$$

where $A = a^2 + b^2 + c^2$ and $B = a^2 + b^2 - c^2$, with $B \neq 0$. This is the closest we can get to the inverse of a general triplet in pure $\mathbb{M}$. If you try the formula, you will see that the result does not satisfy $m \times (1/m) = 1$. The imaginary and infinite parts are zero, and the real part is one, but we notice that an ∞i part remains that is not zero. If b or c is zero, then the formula $1/m = \frac{a-bs-ci}{a^2+b^2+c^2}$ is valid.

Analysis of inversion results The fact that $m \cdot m^{-1} = 1$ is not always a true statement is a direct consequence of modifying the null multiplication axiom $0 \cdot x = 0$. Since $m(1/m) = 1$ partially relies on the null multiplication axiom, which is no longer valid, it is not unusual that its implications, such as inversions, also break down. Under the inversion of x it is considered logical that x is not zero; if we allow the inverse of x to be zero, the chain of implications collapses. We can try to force $mn = 1$ to obtain the exact inverse.

Exact reciprocal number (solving $m \cdot n = 1$). If we try to find a solution to the equation $mn = 1$ in pure $\mathbb{M}$, we see that there are no two standard triplets m and n that satisfy it. If we analyze more deeply, we see that there is no dual-point solution either. Thus it can be concluded that $mn = 1$ has no solution in $\mathbb{M}$ if the coefficients of every component are nonzero. This introduces the 5D (next section). In $\mathbb{M}$ there is one more unused element, namely ∞. ∞ is not an element of any standard triplet in $\mathbb{M}$, but appears exclusively as the result of $1/0$. If we separate ∞i, i, s, and ∞, we obtain a new element expressed as $X = x_0 + x_1 s + x_2 i + x_3 \infty i + x_4 \infty$. Then a solution for $mn = 1$ exists: if we set $m = a + bs + ci$ and $n = x_0 + x_1 s + x_2 i + x_3 \infty i + x_4 \infty$, then there exists n given by the matrix

$$M \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad M = \begin{pmatrix} a & -b & -c & 0 & 0 \\ b & a & 0 & 0 & -b \\ c & 0 & a & 0 & 0 \\ 0 & c & b & a-2b & c \\ 0 & 0 & 0 & -c & a-b \end{pmatrix}.$$

This solution is the solution for n in $mn = 1$ and is an element of the 5D algebra.

7 The Fifth Dimension in $\mathbb{M}$

$\mathbb{M}$ naturally passes into 5D because of the basis vectors 1, s, and i, as well as ∞i, which was resolved by the dual-point, and ∞, which was omitted. In 5D we can admit all five, but then new undefined products such as $s \times \infty$ and $\infty \times \infty$ appear. We can define them using previously defined properties, specifically the behavior of s like i under exponents. We know that powers of s rotate and that there are two types of infinity, ∞ and s. From this we can define ∞s as -1, because both are infinities, and $s^2 = -1$ is defined. However, care must be taken about how similar we make ∞ and s. A better solution is to define ∞s as two types of -1 along the infinite axis (∞ and s). This definition makes sense because of the behavior of ∞ under rotation. Somewhere a line must be drawn and products must be defined as linear combinations so that we do not end up in the eighth dimension like the octonions. Because of rotations in 5D, products of combinations of ∞ and s will be defined as:

$$s \cdot \infty = -s - \infty, \qquad \infty \cdot \infty = -s - \infty, \qquad \infty i \cdot \infty i = s + \infty.$$

Thus the combination of basis vectors becomes:

$$\{1,\ s,\ i,\ \infty i,\ \infty\}, \tag{12}$$

and we obtain 5D for five basis vectors. We also need to define the multiplication table of the new basis vectors:

$\times$	1	s	i	∞i	∞
1	1	s	i	∞i	∞
s	s	-1	∞i	$-2\,\infty i$	$-s-\infty$
i	i	∞i	-1	$-\infty$	∞i
∞i	∞i	$-2\,\infty i$	$-\infty$	$s+\infty$	$-2\,\infty i$
∞	∞	$-s-\infty$	∞i	$-2\,\infty i$	$-s-\infty$

Table 2: Multiplication table of basis vectors in the 5D extension of $\mathbb{M}$

This 5D algebra is completely commutative, but even more non-associative. This is expected because the original 3D system was not completely associative, so associativity decreases with the increase of the system. Although previously associativity did not hold only in the specific combination of ∞ and 0, here it does not hold even in the basis vectors, as shown:

$$(i \cdot \infty i) \cdot s = s + \infty, \qquad i \cdot (\infty i \cdot s) = 2\infty,$$
$$(s \cdot \infty i) \cdot i = 2\infty, \qquad s \cdot (\infty i \cdot i) = s + \infty.$$

The classical element of the 5D algebra $X = x_0 + x_1 s + x_2 i + x_3 \infty i + x_4 \infty$ can also be displayed in the original 3D algebra as a set of four points:

$$(x_0, x_1, x_2)_i, \quad (x_0, x_1, x_3)_{i\infty}, \quad (x_0, x_4, x_2)_{\infty,\, i}, \quad (x_0, x_4, x_3)_{\infty,\, i\infty}.$$

Thus it is possible to construct not only a valid 3D number set, but also a 5D number system if we accept products of basis vectors as linear combinations of basis vectors. Near the end of the paper we define a new number f that will be the result of ∞s, so that ∞s is not a linear combination of basis vectors. Before continuing, note that the multiplication result of ∞i and i is $-\infty$, not $-s$. This is defined in this way because of the exponential functions we are about to define.

8 Exponents and Exponential Functions in $\mathbb{M}$

All powers in the 3D and 5D algebras are defined by classical power series:

$$\exp(m) = \sum_{n=0}^{\infty} \frac{m^n}{n!}.$$

From the 5D vector table m^n can be computed, although convergence is not studied.

8.1 Examples of Powers of a Classical Triplet in $\mathbb{M}$

Any power of a classical triplet with nonzero coefficients will be displayed as a dual-point object if it is a square, and as an element of the 5D algebra if it is a cube or higher. Let us compute the powers of $4 - 2s + i$:

$$\begin{aligned}
m^2 &= 11 - 16s + 8i - 4\,\infty i, \\
m^3 &= 4 - 86s + 43i - 64\,\infty i + 4\,\infty, \\
m^4 &= -199 - 344s + 176i - 680\,\infty i + 88\,\infty, \\
m^5 &= -1692 - 478s + 505i - 2616\,\infty i + 868\,\infty, \\
m^6 &= -9517 + 3884s + 328i - 8116\,\infty i + 6380\,\infty.
\end{aligned}$$

All powers from the cube onward can be displayed with finite coordinates in the 5D system. Note that if the definition of $\infty i \cdot i$ were s, powers of a general triplet would not be elements of the 5D algebra, but dual-point objects in classical $\mathbb{M}$. There would be no clear bridge between 3D and 5D.

8.2 Exponential Function with s as Exponent and Powers of s

First we compute the inverse of a pure infinite element bs, $b \in \mathbb{R}, b \neq 0$:

$$\frac{1}{bs} = \frac{1}{b} \cdot \frac{1}{s} = \frac{1}{b}(-s) = -\frac{1}{b}s. \tag{13}$$

Since we have previously defined how s behaves under powers, no additional clarification is needed here; briefly, from $s^2 = -1$ it follows that $1/s = -s$.

For every real number $a > 0$, $a^s = \exp(s \ln a)$. Since s is defined like i, namely $s^2 = -1, s^3 = -s, s^4 = 1$, we see that this exponential function is identical to a^i, except that s stands in place of i. Therefore:

$$a^s = \sum_{k=0}^{\infty} \frac{(\ln a)^k}{k!} s^k = \cos(\ln a) + s \sin(\ln a).$$

Everything is identical to i, except that s is in place of i. Here are a few examples:

$$1^s = 1, \qquad 2^s \approx 0.769 + 0.639\, s, \qquad e^s = \cos 1 + s \sin 1 \approx 0.540 + 0.841\, s.$$

Since both elements, i and s, satisfy $x^2 = -1$, they behave identically under exponentiation. Let us also look at Euler's identity:

$$e^{\pi s} = \cos\pi + s\sin\pi = -1.$$

This proves that s generates rotations on its axis.

8.3 Powers of a General Triplet

For a general classical triplet $m = p + qs + ri$ we have

$$\exp(m) = \sum_{n=0}^{\infty} \frac{m^n}{n!}.$$

Because of non-associativity of multiplication in $\mathbb{M}$, generally

$$e^{A+B} = e^A e^B$$

does not hold. Therefore $e^{p+qs+ri}$ cannot be decomposed into a product $e^p e^{qs} e^{ri}$. Only the power series is correct.

On the other hand, the product $e^p e^{qs} e^{ri}$ can be computed directly:

$$e^p e^{qs} e^{ri} = e^p \big(\cos q + s\sin q\big)\big(\cos r + i\sin r\big)$$

$$e^p e^{qs} e^{ri} = e^p \big(\cos q\cos r + s\sin q\cos r + i\cos q\sin r + \infty i\sin q\sin r\big).$$

But this is not equal to $e^{p+qs+ri}$. Special cases in which the decomposition into a product of exponentials is possible are those in which only one non-real basis vector appears. For example:

$$e^{p+qs} = e^p(\cos q + s\sin q),$$

$$e^{p+ri} = e^p(\cos r + i\sin r).$$

Here are several interesting values related to the pure axis s:

- $\ln s$: solving $e^{x+ys} = s$ gives $\ln s = \frac{\pi}{2}s \pmod{2\pi}$.
- $s^s = \exp(s\ln s) = \exp(s\cdot\frac{\pi}{2}s) = e^{-\pi/2} \approx 0.2079$; this is a real number and identical to i^i, which further proves that s generates rotations.
- $\sqrt{s} = s^{1/2} = \exp(\frac{1}{2}\ln s) = \exp(\frac{\pi}{4}s) = \cos\frac{\pi}{4} + s\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}s$.
- For every real a, $s^a = \exp(a\ln s) = \exp(a\frac{\pi}{2}s) = \cos\frac{a\pi}{2} + s\sin\frac{a\pi}{2}$.

8.4 Powers and Exponential Function of ∞ and ∞i

For defining powers of ∞, the 5D multiplication table is key. If we start computing, we see a pattern:

$$\infty^2 = -s - \infty, \qquad \infty^3 = -(\infty\cdot s + \infty^2) = -(-s-\infty) - (-s-\infty) = 2s + 2\infty,$$

and so on. In general, powers of ∞ generate only linear combinations of ∞ and s. Furthermore, by substituting ∞ into the power series and solving, we obtain the general formula for x^∞:

$$\boxed{e^{t\infty} = 1 + \frac{1 - e^{-2t} - 2t}{4}\,s + \left(t + \frac{1 - e^{-2t} - 2t}{4}\right)\infty, \qquad t \in \mathbb{R}.}$$

Here t is a real parameter; for x^∞ we set $t = \ln x$. We also obtain the following interesting results:

$$1^\infty = 1,$$

$$e^\infty = 1 + \frac{1 - e^{-2} - 2}{4} s + \frac{1 - e^{-2} + 2}{4} \infty \approx 1 + 0.283\, s + 1.283\, \infty.$$

This further confirms the importance of the result $\infty i \cdot i = -\infty$. As with ∞, so with ∞i there is a pattern. Let $U = \infty i$. From the multiplication table:

$$U^2 = s + \infty, \qquad U^3 = -4U.$$

From this we obtain the closed formula for a real parameter t:

$$\boxed{e^{t\,\infty i} = 1 + \frac{\sin^2 t}{2}(s + \infty) + \frac{\sin 2t}{2} \infty i}.$$

Additionally, if along the pure axis ∞ or ∞i there is also a real part p, the following formulas hold:

$$e^{p+t\,\infty} = e^p \left[1 + \frac{1 - e^{-2t} - 2t}{4} s + \left(t + \frac{1 - e^{-2t} - 2t}{4}\right) \infty\right],$$

$$e^{p+t\,\infty i} = e^p \left[1 + \frac{\sin^2 t}{2}(s + \infty) + \frac{\sin 2t}{2} \infty i\right].$$

Other exponential functions are not computable in closed form because any combination of two basis vectors generates all five components. The six listed formulas were feasible because the corresponding powers behave cyclically. Other values e^m can be computed exclusively by power series, but not as a ready-made formula.

8.5 Potential Operations with Zero

Using the formula for exponentials of ∞, we can compute the zeroth root:

$$x^\infty = x^{1/0} = \exp((\ln x)\infty) = 1 + \frac{1 - x^{-2} - 2\ln x}{4} s + \left(\ln x + \frac{1 - x^{-2} - 2\ln x}{4}\right) \infty.$$

We see that the coordinates are finite and that the zeroth root is an ordinary triplet in 5D. Furthermore, we can define another undefined number in this algebraic framework using the axioms, namely 0^0:

$$0^0 = 0^{1-1} = 0 \cdot 0^{-1} = 0 \cdot \frac{1}{0} = 0 \cdot \infty = 1.$$

Furthermore, for every negative real number n, $0^n = \infty$, which follows directly from the definition $\frac{1}{0} = \infty$.

9 Implications of $\mathbb{M}$ and Continuity in Calculus

9.1 Preservation of Continuity in Calculus

Many functions in mathematics are undefined at zero, for example x/x. For this example we will analyze the functions x/x and $1/x$, with emphasis on x/x. In $\mathbb{M}$, ∞ serves as a bridge that will carry these points.

- **Solving** x/x This function is the division identity function. In classical mathematics it is perfectly flat all the time at $y = 1$, except at $x = 0$, where the function is undefined. In $\mathbb{M}$ it is perfectly flat over the entire domain of real numbers:

Proof. Let $f(x) = \frac{x}{x}$, $x \in \mathbb{M}$. Let us compute the value of the function at $x = 0$:

$$f(0) = 0 \times \frac{1}{0}. \tag{14}$$

Substitute $\frac{1}{0} = \infty$ according to the established axioms:

$$f(0) = 0 \times \infty. \tag{15}$$

By further substituting $0 \times \infty = 1$ we obtain:

$$f(0) = 1. \tag{16}$$

This function is perfectly flat in $\mathbb{M}$, has no discontinuities, and is completely continuous. □

- **Solving** $1/x$ In a similar way, it is possible to solve $1/x$ at $x = 0$. To display the solution, three dimensions will be needed, similar to the complete display of an irrational function.

  Proof. Let $f(x) = \frac{1}{x}$, $x \in \mathbb{M}$. Compute the value of the function at $x = 0$; then the function reads $f(0) = \frac{1}{0}$. According to the axiom defining ∞, this number is ∞. Thus the function at that point is equal to ∞ and will be displayed as $(0, 1, 0)_\infty$ in the third dimension. □

- **Universal property of subtraction** In $\mathbb{M}$, for every $x \in \mathbb{M}$,

  $$x - x = 0.$$

  This follows directly from the definition of addition and makes expressions such as $\infty - \infty$ uniquely defined, unlike in classical analysis where such expressions are undefined.

9.2 Implications of $\mathbb{M}$

$\mathbb{M}$ would mainly be used in computing for programs that crash at zero. A great advantage of $\mathbb{M}$ is that it easily handles zeros (that is, undefined numbers containing an operation with zero, and there are many such) without too many sacrifices, unlike Wheel Theory and division rings. $\mathbb{M}$ has an extensive network and can also be used as a 5D space. Furthermore, $\mathbb{M}$ would be useful in robotics for handling singularities and in physical simulations.

10 Further Extension of the Algebra and Addition of Arbitrary Dimension

The previous 5D construction is not the final limit. It illustrates a general principle: when the product of two basis elements does not belong to the existing basis, we can declare it a new basis vector and thus extend the dimension of the algebra. This directly shows that it is possible to construct algebras whose dimensions are not powers of two, at the cost of further weakening associativity.

10.1 7D Extension with $f = \infty s$ and $fi = s \cdot \infty i$

In this subsection we introduce a new, separate algebra with seven basis elements. It does not directly extend the 5D algebra, but changes some multiplication rules, so it should be viewed as a new structure. We start from the 5D basis

$$\{1,\ s,\ i,\ \infty i,\ \infty\}.$$

We define two new basis elements

$$f = \infty \cdot s, \qquad fi = s \cdot (\infty i).$$

They cannot be expressed as linear combinations of the previous basis elements, so we introduce them as new independent vectors. The new basis is

$$\{1,\ s,\ i,\ \infty i,\ \infty,\ f,\ fi\}.$$

In this 7D algebra we define $i \cdot \infty i = -s$. This is a change from the 5D algebra, where this product was $-\infty$. Therefore the exponential formulas from the previous section do not transfer directly to the 7D algebra. Multiplication of basis elements is given by the following table, which follows from the stated definitions and the requirement of commutativity:

$\times$	1	s	i	∞i	∞	f	fi
1	1	s	i	∞i	∞	f	fi
s	s	-1	∞i	fi	f	$-\infty$	$-\infty i$
i	i	∞i	-1	$-s$	∞i	fi	$-f$
∞i	∞i	fi	$-s$	$-f$	fi	$-\infty i$	∞
∞	∞	f	∞i	fi	f	$-\infty$	$-\infty i$
f	f	$-\infty$	fi	$-\infty i$	$-\infty$	$-f$	$-fi$
fi	fi	$-\infty i$	$-f$	∞	$-\infty i$	$-fi$	f

This table is symmetric, which confirms that the algebra is commutative. Associativity does not hold even for simple triples, which is expected because the 5D algebra was already non-associative. What is interesting about this table is that it is completely closed in dimensions, i.e., there is no dual-point and no product of basis vectors defined as a linear combination of basis vectors. This clearly shows that it is possible to have a commutative algebra of dimension 7, but not an associative division algebra, in accordance with Frobenius' theorem. The table is possible thanks to the definition $s \cdot i = \infty i$. If we declared si a separate element, we would need an eighth dimension, i.e., we would obtain a structure equivalent to the octonions. This shows that the definition $si = \infty i$ enables closure of the algebra in seven dimensions, but is not a necessity, but a choice.

10.2 Note on Extension with an Infinitesimal ε

It is also possible to consider an extension with a new element ε, for which

$$\varepsilon^2 = 0,$$

and we define

$$s = \varepsilon^{-1}, \qquad \infty = \varepsilon^{-2}.$$

In other words, s and ∞ are powers of the infinitesimal ε. Thus we obtain the basis

$$\{1,\ s,\ i,\ \infty i,\ \infty,\ \varepsilon,\ \varepsilon i\}.$$

Since $\varepsilon^2 = 0$, all higher powers of ε are also zero. Therefore for every $n \geq 2$,

$$\frac{1}{\varepsilon^n} = \frac{1}{0} = \infty,$$

so the element ∞ appears frequently in the table.

The multiplication table of basis vectors is completely closed, like the previous 7D table with elements f and fi. The key difference here is not the definition $si = \infty i$, but the fact that ∞ and s are expressed through ε:

$\times$	1	s	i	∞i	∞	ε	εi
1	1	s	i	∞i	∞	ε	εi
s	s	∞	∞i	∞i	∞	1	i
i	i	∞i	-1	$-\infty$	∞i	εi	$-\varepsilon$
∞i	∞i	∞i	$-\infty$	$-\infty$	∞i	∞i	$-s$
∞	∞	∞	∞i	∞i	∞	s	∞i
ε	ε	1	εi	∞i	s	0	0
εi	εi	i	$-\varepsilon$	$-s$	∞i	0	0

We see that this table is also completely closed.

As an illustration, let us compute a few powers and exponential expressions. Since $s^2 = \infty$, all higher powers satisfy $s^n = \infty$ for $n \geq 2$. Therefore

$$e^{ts} = 1 + ts + \left(\sum_{n=2}^{\infty} \frac{t^n}{n!}\right)\infty = 1 + ts + (e^t - 1 - t)\infty.$$

Furthermore, $\infty^2 = \infty$, so

$$e^{t\infty} = 1 + (e^t - 1)\infty.$$

Due to the nilpotency $\varepsilon^2 = 0$, we have

$$e^{t\varepsilon} = 1 + t\varepsilon,$$

$$e^{t\varepsilon i} = 1 + t\varepsilon i.$$

As an example of squaring,

$$(1 + s + i)^2 = 1 + 2s + 2i + s^2 + i^2 + 2si.$$

Using the multiplication table,

$$s^2 = \infty, \qquad i^2 = -1, \qquad si = \infty i,$$

we obtain

$$(1 + s + i)^2 = 2s + 2i + \infty + 2\infty i.$$

10.3 Conclusion on Dimensions

This shows that it is possible to construct algebras whose dimensions are not limited to powers of two, if we weaken algebraic properties. After the sedenions, the number systems accepted by mainstream mathematics are not particularly interesting because they lose all properties and are very difficult to analyze, while not gaining any new interesting properties. This paper opens the way for a redefinition of those number systems, each of which would add something new, that is, define undefined numbers for which new systems are introduced. It would be very interesting to see a number system that connects quaternions with dual and virtual numbers, and possibly with Wheel Theory. Such systems certainly would not be meaningless like the current 128-dimensional number systems.

11 Review, Summary, and Further Research Directions

11.1 Summary

$\mathbb{M}$ is a partial algebra with a modified null multiplication axiom and an extension into 5D, different from other division rings and wheels. $\mathbb{M}$ solves the problem of division by zero and introduces infinity as a scalar algebraic object. $\mathbb{M}$ occupies a position between $\mathbb{C}$ and $\mathbb{H}$ by dimension (3D), and extended to the fifth dimension it forms a natural higher-dimensional structure. $\mathbb{M}$ shows that it is possible to construct algebras whose dimensions are not powers of two. Commutativity is preserved in $\mathbb{M}$, unlike in quaternions where it is lost. Associativity in $\mathbb{M}$ holds except in the specific interactions of ∞ and 0. It is an open question whether a similar construction can be carried out within quaternions or their modification.

11.2 Further Directions and References

The 5D extension of $\mathbb{M}$ opens avenues for further research of this algebraic framework:

- study of convergence and exponential series;
- extension of the exponential function from a general triplet to a general element of the algebra (here one must be careful because $m^{a+b} \neq m^a \times m^b$);
- investigation of further dimension extensions, for example by introducing a new basis element for products such as ∞s;
- definition of $\ln 0$ and $\ln \infty$;
- study of cardinalities of sets (so that $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$ are not of the same measure, redefining $\aleph_0$ as s).

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