A Generalization of the Riemann Functional Equation with Three Variables

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Abstract

The discovery of a generalization of the Riemann functional equation obtained by exploiting symmetries of a new formula for the Hurwitz zeta function with respect to the shift parameter, b, has led to the search for a similar relation for the Lerch Φ function, $\Phi(e^z, k, b)$, where a new formula is used as starting point and the symmetries with respect to b are exploited again. In addition, we uncover surprising new ways of producing analytic continuations of finite summations. Above all, this paper demonstrates, with clear elementary mathematics, how a functional equation can be derived.

Summary

1 Introduction

The paper [7], *Generalized Harmonic Numbers*, introduced a new approach that enables a new formula to be derived for the generalized harmonic numbers, $H_k(n)$. That paper showed a detailed demonstration, based on power series, of why the formula holds and how it changes depending on the choice of "initial equation" (e.g., the main ones used were $\sin \pi j = 0$, $\sin 2\pi j = 0$ and $\cos 2\pi j = 1$, with j an integer).

None of the various subsequent papers on the theme, that followed, made it clear that the generalization of that approach,

$$e^{z\,j} = \sum_{q=0}^{\infty} \frac{(z\,j)^q}{q!}, \tag{1}$$

(which holds for all z and j, as the exponential function is analytic everywhere), leads to the same formula for the partial sums of the polylogarithm that was derived in paper [8] using an indirect approach, namely,

$$\sum_{j=1}^{n} \frac{e^{z\,j}}{j^k} = \frac{e^{z\,n}}{2n^k} - \frac{1}{2n^k} \sum_{j=0}^{k} \frac{(z\,n)^j}{j!} + \sum_{j=1}^{k} \frac{z^{k-j}}{(k-j)!} \sum_{q=1}^{n} \frac{1}{q^j} + \frac{z^k}{2(k-1)!} \int_0^1 (1-u)^{k-1} \left(e^{z\,n\,u} - 1\right) \coth \frac{zu}{2}\, du, \tag{2}$$

where each case k is obtained from (1) — by dividing each side of the equation successively by j and summing it over j, a recurrence is obtained (the integral is the part of the power series thus formed that does not admit a closed form).

The challenge that is now addressed is how to analytically continue the equation above to real or negative k. The current approach breaks down for the cases where k is a non-positive integer as the integral does not converge (that is, that term is no longer given by the anti-derivatives of the $\mathbb{1}_{k|n}$ function, as per paper [7], but by the derivatives). We expect that a workaround can be found.

Additionally, appendix (C) presents a review and a restatement of a newly found generalization of the Riemann functional equation (that allows the shift parameter of the Hurwitz zeta function, b, to be greater than one).

Despite references to the author's previous papers, this paper is sufficiently self-contained and does not require extensive prior reading. A great effort was put into making this manuscript organized and easy to follow, though many readers may still find it a bit challenging.

1.1 Generalization to harmonic progressions

The approach described previously can be generalized to produce formulae for the generalized harmonic progressions, a term that has been used previously to refer to the finite sum below,

$$\mathrm{HP}_k(n) = \sum_{j=1}^{n} \frac{1}{(aj+b)^k},$$

which yields the partial sums of the Hurwitz zeta function.

A few different formulae have been created for $\mathrm{HP}_k(n)$ in previous research, by changing the assumptions on the parameters of the initial equation.

It is trivial to state a generalization of the result of the previous section (just by making $j \mapsto j+b$ in (1)). It then gives the formula below for the partial sums of the Lerch Φ function,

$$\sum_{j=1}^{n} \frac{e^{z(j+b)}}{(j+b)^k} = -\frac{e^{z\,b}}{2\,b^k} + \frac{e^{z(n+b)}}{2(n+b)^k} + \frac{1}{2\,b^k}\sum_{j=0}^{k}\frac{(z\,b)^j}{j!} - \frac{1}{2(n+b)^k}\sum_{j=0}^{k}\frac{(z(n+b))^j}{j!}$$
$$+\sum_{j=1}^{k}\frac{z^{k-j}}{(k-j)!}\sum_{q=1}^{n}\frac{1}{(q+b)^j} + \frac{z^k}{2(k-1)!}\int_0^1 (1-u)^{k-1}\left(e^{z(n+b)u} - e^{zbu}\right)\coth\frac{zu}{2}\,du, \quad (3)$$

which was derived using a different approach in [8]. It holds for every positive integer k and n and every complex z and b.

2 A new approach for the Lerch Φ

There is a new approach to obtain the limit of (3). Note that with the generalization from (2), n no longer has to be integer. This allows a new formula for the Lerch Φ function to be deduced, where functions of n in (2) are replaced by their analytic continuations, as below (note n was replaced by b to keep the usual notation):

$$
\begin{aligned}
-e^{z\,b}\,\Phi(e^z,k,b)+\mathrm{Li}_k(e^z) = -\frac{e^{z\,b}}{2\,b^k}-\frac{1}{2\,b^k}\sum_{j=0}^{k}\frac{(z\,b)^j}{j!}+\sum_{j=1}^{k}\frac{z^{k-j}}{(k-j)!}H_j(b)\\
+\frac{z^k}{2(k-1)!}\int_0^1(1-u)^{k-1}\left(e^{z\,b\,u}-1\right)\coth\frac{zu}{2}\,du \qquad (4)
\end{aligned}
$$

The second summation in the formula above can be rewritten as:

$$
\begin{aligned}
\sum_{j=1}^{k}\frac{z^{k-j}}{(k-j)!}H_j(b) &= \lim\sum_{j=1}^{k}\frac{z^{k-j}}{(k-j)!}\left(\zeta(j)-\zeta(j,b+1)\right)\\
&= -\frac{z^k}{k!}+\frac{1}{b^k}\sum_{j=0}^{k}\frac{(z\,b)^j}{j!}+\frac{z^{k-1}}{(k-1)!}\left(H(b)-\frac{1}{b}\right)-\sum_{j=2}^{k}\frac{z^{k-j}}{(k-j)!}\left(\zeta(j,b)-\zeta(j)\right) \qquad (5)
\end{aligned}
$$

2.1 Transforming the original Lerch Φ

The limit of the partial Lerch Φ formula from (3) was first found in [8], and the new formula in (4) is evidence of its correctness (an actual proof is given in section (2.2)). The original formula, where z must not have both a non-negative real part and an imaginary part greater than 2π (in modulus), is given by:

$$
\begin{aligned}
e^{z\,b}\,\Phi(e^z,k,b) = \frac{e^{z\,b}}{2\,b^k}+\sum_{j=2}^{k}\frac{z^{k-j}}{(k-j)!}\left(-\frac{1}{2\,b^j}+\zeta(j,b)\right)\\
+\frac{\pi\,z^k}{2(k-1)!}\cot\pi b-\frac{z^{k-1}}{(k-1)!}\log\left(-\frac{z}{2\pi}\right)\\
-\frac{z^k}{2(k-1)!}\int_0^1(1-u)^{k-1}e^{z\,b\,u}\coth\frac{zu}{2}+\frac{2\pi}{z}\left(-1+\frac{\sin 2\pi bu}{\sin 2\pi b}\right)\cot\pi u\,du \qquad (6)
\end{aligned}
$$

In paper [8], it was shown that to find the limit of the partial Lerch Φ it was necessary to find an approximation for the harmonic progression of order one (which does not converge). That is, for sufficiently large n, one can write the interesting asymptotic expression [1] [6]:

$$
\sum_{j=1}^{n}\frac{1}{j+b}\sim H(n)-\frac{1}{b}\sum_{k=2}^{\infty}(-b)^k\,\zeta(k)=H(n)-H(b) \qquad (7)
$$

But this approximation is not a new result, it is already known in the literature. To prove it, the identity below, that relates the ordinary generating function of the zeta function at the odd integers to the harmonic numbers, can be used,

$$\sum_{j=1}^{\infty} b^{2j}\,\zeta(2j+1) = -\pi\int_0^1\left(\frac{\sin 2\pi b u}{\sin 2\pi b}-u\right)\cot\pi u\,du$$
$$= \frac{1}{2\,b} - H(b) - \frac{\pi}{2}\cot\pi b = -\frac{1}{2\,b} - H(-b) + \frac{\pi}{2}\cot\pi b, \quad (8)$$

where the rightmost formula follows from a property of odd functions, $f(x) = -f(-x)$.

With this identity, formula (6) can be turned into the neater form below, which also holds for integer or half-integer b:

$$e^{z\,b}\,\Phi(e^z,k,b) = \frac{e^{z\,b}}{2\,b^k} + \sum_{j=2}^{k}\frac{z^{k-j}}{(k-j)!}\left(-\frac{1}{2\,b^j}+\zeta(j,b)\right)$$
$$-\frac{z^{k-1}}{(k-1)!}\left(-\frac{1}{2\,b}+H(b)+\log\left(-\frac{z}{2\pi}\right)\right)$$
$$-\frac{z^k}{2(k-1)!}\int_0^1 (1-u)^{k-1}e^{z\,b\,u}\coth\frac{zu}{2}-\frac{2\pi}{z}(1-u)\cot\pi u\,du \quad (9)$$

Note that the $H(b)$ term is the alternating generating function of the zeta function at the positive integers:

$$H(b) = -\sum_{j=2}^{\infty}(-b)^{j-1}\,\zeta(j) \quad (10)$$

2.2 A proof of the Lerch Φ formula

Although not all the steps are provided, the equation (9) can be proved by taking the case $k = 1$, for which the integral has a known closed form, according to Mathematica[1]:

$$\int_0^1 e^{z\,b(1-u)}\coth\frac{zu}{2}-\frac{2\pi\,e^{z\,b}}{z}(1-u)\cot\pi u\,du$$
$$= -\frac{1}{z}\left(\Phi(e^z,1,-b)+e^z\,\Phi(e^z,1,-b+1)\right)-\frac{e^{z\,b}}{z}\left(\frac{1}{b}+2\log\left(-\frac{z}{2\pi}\right)+2\,H(-b)\right),$$

which proves the formula holds for $k = 1$.

[1] Some of the results introduce herein were obtained with the aid of software.

To prove the general case, the integral above is differentiated $k-1$ times with respect to b. The only challenging term is given below:

$$\frac{\mathrm{d}^{k-1}(e^{z\,b}\,H(-b))}{\mathrm{d}\,b^{k-1}} = e^{z\,b}\left(z^{k-1}\,H(-b) - \Gamma(k)\sum_{j=2}^{k}\frac{z^{k-j}}{(k-j)!}\zeta(j,-b+1)\right)$$

After some proper transformations, one concludes the result matches the formula, implying an equivalence between the differentiation and the recurrence methods.

2.3 A new approach for the polylog

There are a few different ways to obtain the limit of (2) as n approaches infinity than the one created in [8].

The first one is to let b go to infinity in formula (4), assuming $\Re(z) < 0$. Before doing this, however, one must eliminate $H(b)$ from the second summation, since it diverges. This is achieved by taking the formula for the particular case $k = 1$, multiplying it by $z^{k-1}/(k-1)!$ and subtracting the result from (4). This creates a formula that now holds even when $\Im(z) > 2\pi$:

$$\mathrm{Li}_k(e^z) = -\frac{z^k}{2\,k!} + \frac{z^k}{2(k-1)!} + \frac{z^{k-1}}{(k-1)!}\mathrm{Li}_1(e^z) + \sum_{j=2}^{k}\frac{z^{k-j}}{(k-j)!}\zeta(j) - \frac{z^k}{2(k-1)!}\int_0^1\left((1-u)^{k-1}-1\right)\coth\frac{zu}{2}\,du \quad (11)$$

Another way to obtain the polylog is to eliminate $H(n)$ from equation (2) in the same way that was just described. One then applies the delta operator to both sides of the resulting equation, $\Delta f(n) = f(n) - f(n-1)$. It can then be summed over the natural numbers since no terms diverge, and this leads to an alternative expression for the polylog.

2.4 A proof of the polylog formula

Although the polylog formula stems from the Lerch Φ formula, there is another clever way to prove it. For this to be a rigorous proof, note that equation (2) can be proved by applying the delta operator with respect to n to both sides, in which case the resulting integral has a closed form that confirms the identity. Hence, the equation holds for every n, not just integer n.

The equation (11), which is proved, is a transformation of the polylog formula introduced in reference [8],

$$\operatorname{Li}_k(e^z) = -\frac{z^k}{2\,k!} - \frac{z^{k-1}}{(k-1)!}\log\left(-\frac{z}{2\pi}\right) + \sum_{j=2}^{k}\frac{z^{k-j}}{(k-j)!}\zeta(j) \\ - \frac{z^k}{2(k-1)!}\int_0^1 (1-u)^{k-1}\coth\frac{zu}{2} - \frac{2\pi}{z}(1-u)\cot\pi u\,du, \quad (12)$$

since the above can be turned into (11) in the same way used to create the former. Note the domain of the latter includes $z = 2\pi\boldsymbol{i}$, whereas the domain of the former does not.

The two formulae imply the following equality,

$$\int_0^1 z\coth\frac{zu}{2} - 2\pi(1-u)\cot\pi u\,du = -z - 2\log\left(-\frac{z}{2\pi}\right) - 2\operatorname{Li}_1(e^z),$$

which can be evaluated more easily by differentiating both sides with respect to z. The differentiated integral has a closed form (obtained via software) that matches the derivative of the function on the right-hand side. Since the original equation holds at least at a point z, the two formulae are equivalent (where both are defined) and thus this constitutes a proof of formula (12).

3 Generalization via recurrence

The motivation for this section is the question: "Do the functions defined by the summations that appear in the formulae of the Lerch Φ and the polylog functions admit analytic continuations? And if so, can one produce explicit expressions for them?"

3.1 Polylog

Starting with the polylog, it is reasonable to expect that its summation admits an analytic continuation, but obtaining a closed form for it has proved very challenging.

The analytic continuation of the summation may be the following infinite series,

$$f(z) = \sum_{j=2}^{k}\frac{z^{k-j}}{(k-j)!}\zeta(j) = \frac{1}{\Gamma(k)}\sum_{j=1}^{\infty}\left(-\frac{z^{k-1}}{j} - \frac{z^k}{k} + \frac{e^{z\,j}\,\Gamma(k+1, z\,j)}{k\,j^k}\right), \quad (13)$$

where the equivalence with the upper incomplete gamma appears in reference [1], after the definition of the zeta function is applied and the order of the summations is inverted. However, even assuming that this is the true analytic continuation being sought, this new series is not very tractable, as it has terms that diverge.

Surprisingly, a better answer lies in the summation itself. The most obvious choice would be to sum the terms to infinity, but this does not work since the resulting series diverges (e.g., for a non-integer positive k, the terms become the reciprocal of the Gamma function at large negative arguments, and this grows faster than the decay of the powers of z). The solution is to express zeta at the positive integers in terms of their values at the negative integers (since they are related by the Riemann functional equation),

$$\zeta(j)=\begin{cases}\dfrac{(2\pi\boldsymbol{i})^j}{2\,\Gamma(j)}\zeta(1-j), \text{ if } j \text{ is even},\\ \dfrac{2\,(2\pi)^{j-1}}{\Gamma(j)}\sin\dfrac{\pi j}{2}\,\zeta'(1-j), \text{ if } j \text{ is odd},\end{cases}$$

since in this new form it becomes possible to obtain a closed form for the summation, at least in the first case.

This solution exposes a divide in how one can express the function $f(z)$: it necessarily involves a separation of the summation of zeta at the even integers (for which there is a known closed from) from zeta at the odd integers (for which there is no known closed form). Nonetheless, one now has:

$$f(z)=\sum_{j=2}^{k}\frac{z^{k-j}}{(k-j)!}\zeta(j)=\frac{(2\pi\boldsymbol{i})^k}{2\,\Gamma(k)}\sum_{j=2}^{k}\binom{k-1}{j-1}\left(\frac{z}{2\pi\boldsymbol{i}}\right)^{k-j}\zeta(1-j)$$
$$+\frac{2\,(2\pi)^{k-1}}{\Gamma(k)}\sum_{j=2}^{k}\binom{k-1}{j-1}\left(\frac{z}{2\pi}\right)^{k-j}\sin\frac{\pi j}{2}\,\zeta'(1-j) \quad (14)$$

The first summation has a closed form (a negative Hurwitz zeta), while the second one does not. The best one can do is to produce a not so convoluted power series for it, which is done in section (4.4).

4 Generalization via differentiation

The generating function of the polylog formula from (12) allows a new formula for $\Phi(e^z,1,-b+1)$ to be derived:

$$\frac{1}{b}\sum_{j=1}^{\infty}b^j\,\mathrm{Li}_j(e^z)=-\frac{1}{2\,b}\sum_{j=1}^{\infty}\frac{(z\,b)^j}{j!}-\sum_{j=1}^{\infty}\frac{(z\,b)^{j-1}}{(j-1)!}\log\left(-\frac{z}{2\pi}\right)+\frac{1}{b}\sum_{j=1}^{\infty}b^j\sum_{j=2}^{j}\frac{z^{j-j}}{(j-j)!}\zeta(j)$$
$$-\frac{z}{2}\int_0^1\sum_{j=1}^{\infty}\frac{(z\,b(1-u))^{j-1}}{(j-1)!}\coth\frac{zu}{2}-\frac{2\pi}{z}\sum_{j=1}^{\infty}\frac{(z\,b)^{j-1}}{(j-1)!}(1-u)\cot\pi u\,du$$

The advantage of this formula is that it can be used to uncover a relation between the Lerch Φ at the positive and at the negative order parameters, just like the Riemann functional equation. (However, it should be noted that using $\Phi(e^z, 1, b)$ from formula (6) would work just the same for this purpose – the result does not change.) The equation above simplifies to:

$$e^z\,\Phi(e^z,1,-b+1) = \frac{1-e^{z\,b}}{2\,b} - e^{z\,b}\log\left(-\frac{z}{2\pi}\right) + \frac{e^{z\,b}}{b}\sum_{j=2}^{\infty} b^j\,\zeta(j) \\ - \frac{z}{2}\int_0^1 e^{z\,b(1-u)}\coth\frac{zu}{2} - \frac{2\pi\,e^{z\,b}}{z}(1-u)\cot\pi u\,du \tag{15}$$

The justification for the above is the transformation below, where the expression on the right-hand side only holds for $|b| < 1$:

$$\sum_{k=1}^{\infty} b^k \sum_{j=2}^{k} \frac{z^{k-j}}{(k-j)!}\zeta(j) = e^{z\,b}\sum_{j=2}^{\infty} b^j\,\zeta(j)$$

4.1 Exploiting the symmetries

If the signs of both variables in (15) are flipped, the equation becomes:

$$e^{-z}\,\Phi(e^{-z},1,b+1) = -\frac{1-e^{z\,b}}{2\,b} - e^{z\,b}\log\left(\frac{z}{2\pi}\right) - \frac{e^{z\,b}}{b}\sum_{j=2}^{\infty}(-b)^j\,\zeta(j) \\ - \frac{z}{2}\int_0^1 e^{z\,b(1-u)}\coth\frac{zu}{2} - \frac{2\pi\,e^{z\,b}}{z}(1-u)\cot\pi u\,du$$

Note the integral does not change, so if this new equation is subtracted from (15) it vanishes:

$$e^z\,\Phi(e^z,1,-b+1) - e^{-z}\,\Phi(e^{-z},1,b+1) \\ = \frac{1-e^{z\,b}}{b} - e^{z\,b}\left(\log\left(-\frac{z}{2\pi}\right) - \log\left(\frac{z}{2\pi}\right)\right) + \frac{2\,e^{z\,b}}{b}\sum_{j=1}^{\infty} b^{2j}\,\zeta(2j)$$

The ordinary generating function of the Riemann zeta function at the even integers is a well-known result from the literature [2] [6]:

$$\sum_{j=1}^{\infty} b^{2j}\,\zeta(2j) = \frac{1-\pi b\cot\pi b}{2}$$

When this closed form is replaced into the formula, after a few possible simplifications it yields the two-variable equation:

$$e^{z}\,\Phi(e^{z},1,-b+1)-e^{-z}\,\Phi(e^{-z},1,b+1)\\ =\frac{1}{b}-e^{z\,b}\left(\log\left(-z\right)-\log\left(z\right)\right)-\pi\,e^{z\,b}\cot\pi b \quad (16)$$

4.2 The Leibniz rule for derivatives

To make the order parameter of the Lerch Φ function go from one to k in (16), both sides of the equation are differentiated $k-1$ times with respect to b, which can be achieved with the Leibniz rule.

The Leibniz rule[1] is a binomial formula used to find the k-th derivative of the product of two functions:

$$\frac{\mathrm{d}^{k}(f(x)\cdot g(x))}{\mathrm{d}\,x^{k}}=\sum_{j=0}^{k}\binom{k}{j}f^{(j)}(x)g^{(k-j)}(x) \quad (17)$$

Here the only term that is not trivial to differentiate $k-1$ times is $e^{z\,b}\cot\pi b$. In order to employ the Leibniz rule, one can rely on this identity from reference [9]:

$$\frac{\mathrm{d}^{k}(\cot a\,x)}{\mathrm{d}\,x^{k}}=-\boldsymbol{i}\,\delta_{0\,k}-2\,\boldsymbol{i}(2\,\boldsymbol{i}\,a)^{k}\,\mathrm{Li}_{-k}\left(e^{2\,\boldsymbol{i}\,a\,x}\right) \quad (18)$$

Therefore, differentiating the only term of (16) that is not trivial $k-1$ times with respect to b gives:

$$\frac{\mathrm{d}^{k-1}(e^{z\,b}\cot\pi b)}{\mathrm{d}\,b^{k-1}}=e^{z\,b}\sum_{j=0}^{k-1}\binom{k-1}{j}\left(-\boldsymbol{i}\,\delta_{0\,j}-2\,\boldsymbol{i}(2\pi\boldsymbol{i})^{j}\,\mathrm{Li}_{-j}\left(e^{2\pi\boldsymbol{i}\,b}\right)\right)z^{k-1-j}$$

An analytic continuation for this summation can be produced with the identity below, again from reference [9]:

$$\Phi(e^{z},-k,b+1)=e^{-z}\sum_{j=0}^{k}\binom{k}{j}\mathrm{Li}_{-j}\left(e^{z}\right)b^{k-j}, \quad (19)$$

which gives:

$$\frac{\mathrm{d}^{k-1}(e^{z\,b}\cot\pi b)}{\mathrm{d}\,b^{k-1}}=-\boldsymbol{i}\,e^{z\,b}\left(z^{k-1}+2\left(2\pi\boldsymbol{i}\right)^{k-1}e^{2\pi\boldsymbol{i}\,b}\,\Phi\left(e^{2\pi\boldsymbol{i}\,b},-k+1,1+\frac{z}{2\pi\boldsymbol{i}}\right)\right), \quad (20)$$

or since:

$$\Phi\left(e^{2\pi\boldsymbol{i}\,b},-k+1,1+\frac{z}{2\pi\boldsymbol{i}}\right)=e^{-2\pi\boldsymbol{i}\,b}\left(-\left(\frac{z}{2\pi\boldsymbol{i}}\right)^{k-1}+\Phi\left(e^{2\pi\boldsymbol{i}\,b},-k+1,\frac{z}{2\pi\boldsymbol{i}}\right)\right), \quad (21)$$

one also has the alternative formulation below:

$$\frac{\mathrm{d}^{k-1}(e^{z\,b}\cot \pi b)}{\mathrm{d}\,b^{k-1}} = \boldsymbol{i}\,e^{z\,b}\left(z^{k-1} - 2\,(2\pi\boldsymbol{i})^{k-1}\,\Phi\left(e^{2\pi\boldsymbol{i}\,b}, -k+1, \frac{z}{2\pi\boldsymbol{i}}\right)\right) \tag{22}$$

4.3 The generalized functional equation

To conclude the reasoning, the following formulae are needed:

$$\frac{\mathrm{d}^{k-1}\,\Phi(e^{z}, 1, -b+1)}{\mathrm{d}\,b^{k-1}} = \Gamma(k)\,\Phi(e^{z}, k, -b+1), \tag{23}$$

and,

$$\frac{\mathrm{d}^{k-1}\,\Phi(e^{-z}, 1, b+1)}{\mathrm{d}\,b^{k-1}} = -(-1)^{k}\,\Gamma(k)\,\Phi(e^{-z}, k, b+1) \tag{24}$$

Since differentiating the remaining terms of (16) is relatively simple, they are skipped.

The new generalized functional equation is then:

$$\begin{aligned}\Gamma(k)\left(e^{z}\,\Phi(e^{z}, k, -b+1) + (-1)^{k}\,e^{-z}\,\Phi(e^{-z}, k, b+1)\right)&\\ = \frac{(-1)^{k-1}\,\Gamma(k)}{b^{k}} &+ (2\pi\boldsymbol{i})^{k}\,e^{z\,b}\,\Phi\left(e^{2\pi\boldsymbol{i}\,b}, -k+1, \frac{z}{2\pi\boldsymbol{i}}\right)\\ &- z^{k-1}\,e^{z\,b}\,(\pi\boldsymbol{i} + \log(-z) - \log(z))\,,\\ &\text{where } (-1)^{k} = \exp(k\pi\boldsymbol{i}),\text{ if } k \notin \mathbb{Z}\end{aligned} \tag{25}$$

From a relation similar to (21),

$$\Phi\left(e^{z}, k, b+1\right) = e^{-z}\left(-\frac{1}{b^{k}} + \Phi\left(e^{z}, k, b\right)\right), \tag{26}$$

the formula can be simplified to:

$$\begin{aligned}\Gamma(k)\left(\Phi(e^{z}, k, -b) + (-1)^{k}\,\Phi(e^{-z}, k, b)\right)&\\ = \frac{\Gamma(k)}{(-b)^{k}} &+ (2\pi\boldsymbol{i})^{k}\,e^{z\,b}\,\Phi\left(e^{2\pi\boldsymbol{i}\,b}, -k+1, \frac{z}{2\pi\boldsymbol{i}}\right)\\ &- z^{k-1}\,e^{z\,b}\,(\pi\boldsymbol{i} + \log(-z) - \log(z))\,,\\ &\text{where } (-1)^{k} = \exp(k\pi\boldsymbol{i}),\text{ if } k \notin \mathbb{Z}\end{aligned} \tag{27}$$

Before trying to specify the domain of these relations, it is necessary to delve into some of the details. Branch is a choice of a single-valued version of a multi-valued function. For any consistent choice of principal branch of the log, the expression below,

$$\pi \boldsymbol{i} + \log(-z) - \log(z) = \begin{cases} 0 & \text{, if } z < 0 \text{ or } 0 < \Im(z) \le 2\pi, \\ 2\pi \boldsymbol{i} & \text{, if } z > 0 \text{ or } -2\pi \le \Im(z) < 0. \end{cases}$$

is always a pure branch (monodromy) term, it can only take the values zero or $2\pi\boldsymbol{i}$. Since the term is needed for purely real $z > 0$ (and other z-domains), it can not be dropped from the equation.

Throughout this manuscript, the principal branch of the complex logarithm defined by,

$$\log z = \log|z| + \boldsymbol{i}\arg(z) \iff z = |z| \cdot \exp(\boldsymbol{i}\arg(z)),\ \arg(z) \in (-\pi, \pi], \tag{28}$$

is used. All expressions involving logarithms are evaluated under this convention, and branch cuts are inherited from Mathematica's implementation, which was also the software used for testing.

It is hard to specify the exact domain of the newly found relations, but a naive formulation seems to be:

$$\begin{cases} k \notin \mathbb{Z}_- \cup \{0\}, \\ -2\pi \le \Im(z) \le 2\pi \text{ and } z \ne 0, \\ \Im(b) > 0, \\ \text{or } b < 0, \\ \text{or } 0 < \Re(b) < 1 \text{ and } -2\pi \le \Im(b) \le 0, \\ \text{or } -1 < \Re(b) < 0 \text{ and } 0 \le \Im(b) \le 2\pi. \end{cases}$$

The sets given above each contain at least one point for which the equation holds – they are not supposed to be combined into a Cartesian product.

Though improper, non-positive k holds at the limit. The relations must hold for $\Im(z) = 2\pi$, if $\Re(k) > 0$. For integer k (even negative), the equation may hold outside of the presumed domain. These relations have been tested numerically for these assumptions to be made, but they are neither certain nor proven, given how convoluted a three-variable function can be.

4.4 Invariance relative to the initial equation

The generalized relation that is obtained by differentiating the initial equation a certain number of times does not depend on it. For example, when formula (9) is used, starting from $k = q$ and differentiating it $k - q$ times with respect to b yields the same result as starting from $k = 1$. This is easier to see by looking at the case of the term $\Phi(e^z, q, b)$:

$$\frac{\partial^{k-q}}{\partial b^{k-q}} \Phi(e^z, q, b) = (-1)^{k-q} \frac{\Gamma(k)}{\Gamma(q)} \Phi(e^z, k, b)$$

But not all the terms of the formula have this property on their own, some must be combined.

This fact can be stated as follows. For integer q with $1 \leq q \leq k$:

$$\frac{\partial^{k-q}}{\partial b^{k-q}} \left(e^{-z\,b} \left(\Gamma(q) \sum_{j=2}^{q} \frac{z^{q-j}}{(q-j)!} \left(-\frac{1}{2\,b^j} + \zeta(j,b) \right) - z^{q-1} \left(-\frac{1}{2\,b} + H(b) \right) \right) \right)$$
$$= (-1)^{k-q}\, e^{-z\,b} \left(\Gamma(k) \sum_{j=2}^{k} \frac{z^{k-j}}{(k-j)!} \left(-\frac{1}{2\,b^j} + \zeta(j,b) \right) - z^{k-1} \left(-\frac{1}{2\,b} + H(b) \right) \right) \quad (29)$$

This finding finally allows one to obtain the analytic continuation of the summation from section (3), although by means of a power series. One just has to set $q = 1$ and make use of the power series in (10).

4.5 Boundary cases

One may now ask the question: "What does the general functional equation become when each of the three variables is zero?" Or more precisely, "What is the limit at zero, if the expression is not defined at zero?" Herein, to make it directly comparable to the generalized Riemann functional equation from the literature, the transformation $z \mapsto 2\pi\boldsymbol{i}\,z$ is applied:

$$\Gamma(k) \left(e^{2\pi\boldsymbol{i}\,z}\, \Phi(e^{2\pi\boldsymbol{i}\,z}, k, -b+1) + (-1)^k\, e^{-2\pi\boldsymbol{i}\,z}\, \Phi(e^{-2\pi\boldsymbol{i}\,z}, k, b+1) \right)$$
$$= -\frac{(-1)^k\,\Gamma(k)}{b^k} + (2\pi\boldsymbol{i})^k\, e^{2\pi\boldsymbol{i}\,z\,b}\, \Phi\left(e^{2\pi\boldsymbol{i}\,b}, -k+1, z \right)$$
$$- (2\pi\boldsymbol{i}\,z)^{k-1}\, e^{2\pi\boldsymbol{i}\,z\,b}\, (\pi\boldsymbol{i} + \log(-\boldsymbol{i}\,z) - \log(\boldsymbol{i}\,z)) \quad (30)$$

Note that with this transformation, the z-domain (and only z) has changed to $0 \leq \Re(z) \leq 1$. Also, the rightmost term is always constant (either zero, if $0 \leq \Re(z) \leq 1$, or $2\pi\boldsymbol{i}$). Like the general equation, the domain of these particular cases is not entirely known.

4.6 When k is a non-positive integer

When k is a non-positive integer, the left-hand side is undefined (the Gamma function diverges but the expression in brackets goes to zero).

In their book on complex analysis[5], Stein and Shakarchi explain that this limit can be solved using the Gamma function's residue property. Let n be a non-negative integer.

Then, provided that f is differentiable and analytic at $k = -n$ and $f(-n) = 0$, then:

$$\lim_{k \to -n} \Gamma(k)\, f(k) = \frac{(-1)^n}{n!}\, f'(-n) \tag{31}$$

Therefore, the marginal functional equation at the non-positive integers $k = -n$ is:

$$\begin{aligned}(-1)^n \left.\frac{\partial\, \Phi(e^{2\pi \boldsymbol{i}\, z}, k, -b)}{\partial k}\right|_{k=-n} &+ \left.\frac{\partial\, \Phi(e^{-2\pi \boldsymbol{i}\, z}, k, b)}{\partial k}\right|_{k=-n} \\ &= -b^n \log(-b) - \pi \boldsymbol{i}\, \Phi\left(e^{-2\pi \boldsymbol{i}\, z}, -n, b\right) + \frac{n!\, e^{2\pi \boldsymbol{i}\, z\, b}}{(2\pi \boldsymbol{i})^n}\, \Phi\left(e^{2\pi \boldsymbol{i}\, b}, n+1, z\right) \\ &\qquad - \frac{n!\, e^{2\pi \boldsymbol{i}\, z\, b}}{(2\pi \boldsymbol{i}\, z)^{n+1}} \left(\pi \boldsymbol{i} + \log(-\boldsymbol{i}\, z) - \log(\boldsymbol{i}\, z)\right)\end{aligned} \tag{32}$$

Although it is hard to specify the exact domain of this equation, it seems to include $-1 < \Re(z) < 1$ and $\Re(b) \leq 1$.

4.7 When z is zero

This next particular case, $z = 0$, is straightforward to derive. For this one, formula (20), instead of (22), is used. If k is not a non-positive integer:

$$\Gamma(k)\left(\zeta(k, -b+1) + (-1)^k\, \zeta(k, b+1)\right) = -\frac{(-1)^k\, \Gamma(k)}{b^k} + (2\pi \boldsymbol{i})^k\, \mathrm{Li}_{-k+1}\left(e^{2\pi \boldsymbol{i}\, b}\right) \tag{33}$$

And by means of an identity analogous to (21),

$$\zeta(k, b+1) = -\frac{1}{b^k} + \zeta(k, b), \tag{34}$$

an equivalent expression is obtained,

$$\Gamma(k)\left(\zeta(k, -b) + (-1)^k\, \zeta(k, b)\right) = \frac{\Gamma(k)}{(-b)^k} + (2\pi \boldsymbol{i})^k\, \mathrm{Li}_{-k+1}\left(e^{2\pi \boldsymbol{i}\, b}\right),$$

with the same aforementioned domain.

To find the limit of the above at the non-positive integers $k = -n$, the result from Stein and Shakarchi [5] is applied again, giving:

$$\begin{aligned}(-1)^n \left.\frac{\partial\, \zeta(k, -b)}{\partial k}\right|_{k=-n} &+ \left.\frac{\partial\, \zeta(k, b)}{\partial k}\right|_{k=-n} \\ &= -b^n \log(-b) - \pi \boldsymbol{i}\, \zeta(-n, b) + \frac{n!}{(2\pi \boldsymbol{i})^n}\, \mathrm{Li}_{n+1}\left(e^{2\pi \boldsymbol{i}\, b}\right)\end{aligned} \tag{35}$$

4.8 A generalization of the Riemann equation

The case $b = 0$ is the one that proves that this new functional equation is a generalization of the Riemann functional equation. It is either trivial, if one assumes $k < 0$, or challenging, otherwise. But both cases lead to the same solution, the b term vanishes and the Lerch Φ converges to the Hurwitz zeta function, on the right-hand side.

The assumption that $k > 0$ implies that the limit below must hold:

$$\lim_{b\to 0} -\frac{(-1)^k\,\Gamma(k)}{b^k} + (2\pi\boldsymbol{i})^k\, e^{2\pi\boldsymbol{i}\,z\,b}\,\Phi\left(e^{2\pi\boldsymbol{i}\,b}, -k+1, z\right) = (2\pi\boldsymbol{i})^k\,\zeta(-k+1,z) \tag{36}$$

Therefore, if k is not a non-positive integer, one has:

$$\begin{aligned}\Gamma(k)\left(\mathrm{Li}_k\left(e^{2\pi\boldsymbol{i}\,z}\right) + (-1)^k\,\mathrm{Li}_k\left(e^{-2\pi\boldsymbol{i}\,z}\right)\right)&\\ &= (2\pi\boldsymbol{i})^k\,\zeta(-k+1,z)\\ &\quad - (2\pi\boldsymbol{i}\,z)^{k-1}\,\left(\pi\boldsymbol{i} + \log\left(-\boldsymbol{i}\,z\right) - \log\left(\boldsymbol{i}\,z\right)\right),\end{aligned} \tag{37}$$

which is just a transformation of the generalized Riemann functional equation from the literature (which holds if $0 \le \Re(z) \le 1$)[2], except for the extra term on the right-hand side. The extra term enables the equation to hold for purely imaginary z with $\Im(z) < 0$. And for integer k when $-1 \le \Re(z) \le 0$.

This marginal functional equation at the non-positive integers $k = -n$ holds for $|\Re(z)| < 1$ (except $n = 0$, though the general equation has a limit when $k = 0$ and $b = 0$):

$$\begin{aligned}(-1)^n\,\frac{\partial\,\mathrm{Li}_k(e^{2\pi\boldsymbol{i}\,z})}{\partial k}\bigg|_{k=-n} + \frac{\partial\,\mathrm{Li}_k(e^{-2\pi\boldsymbol{i}\,z})}{\partial k}\bigg|_{k=-n}&\\ &= -\pi\boldsymbol{i}\,\mathrm{Li}_{-n}\left(e^{-2\pi\boldsymbol{i}\,z}\right) + \frac{n!}{(2\pi\boldsymbol{i})^n}\,\zeta(n+1,z)\\ &\quad - \frac{n!}{(2\pi\boldsymbol{i}\,z)^{n+1}}\left(\pi\boldsymbol{i} + \log\left(-\boldsymbol{i}\,z\right) - \log\left(\boldsymbol{i}\,z\right)\right)\end{aligned} \tag{38}$$

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A Analytic continuation approaches

An equation is only useful if it expresses the same quantity in different ways, otherwise it just proves the formula. The following are some approaches that were attempted to produce the analytic continuation of the function $f(z)$ from section (3).

A.1 Differentiation approach

This approach consists in using the equivalence below,

$$\sum_{k=2}^{\infty} x^k \sum_{j=2}^{k} \frac{z^{k-j}}{(k-j)!}\zeta(j) = e^{z\,x} \sum_{j=2}^{\infty} x^j\,\zeta(j),$$

which holds if $|x| < 1$, and then differentiating it k times with respect to the parameter x and taking the result at $x = 0$, that is:

$$\frac{1}{k!}\frac{\mathrm{d}^k}{\mathrm{d}\,x^k}\left(e^{z\,x}\sum_{j=2}^{\infty} x^j\,\zeta(j)\right)\Bigg|_{x=0} = \sum_{j=2}^{k} \frac{z^{k-j}}{(k-j)!}\zeta(j)$$

If the signs of both x and z are flipped and the resulting equations are added, a closed form is obtained:

$$e^{z\,x}\left(\sum_{j=2}^{\infty} x^j\,\zeta(j) + \sum_{j=2}^{\infty}(-x)^j\,\zeta(j)\right) = e^{z\,x}\,(1 - \pi\,x\cot\pi x)$$

To make it easier to calculate the k-th derivative of the function above, the identity below, that stems from the Leibniz rule, is used:

$$\frac{\mathrm{d}^k}{\mathrm{d}\,x^k}\,(x\cdot f(x)) = x\,f^{(k)}(x) + k\,f^{(k-1)}(x)$$

Without repeating all the steps that have been detailed in section (4), the final result is:

$$\begin{aligned}\frac{\mathrm{d}^k e^{z\,x}\,(1-\pi\,x\cot\pi x)}{\mathrm{d}\,x^k} = e^{z\,x}\Big(z^k - \pi\boldsymbol{i}\,x\,z^k - \pi\boldsymbol{i}\,k\,z^{k-1} + x\,(2\pi\boldsymbol{i})^{k+1}\,\Phi\left(e^{2\pi\boldsymbol{i}\,x}, -k, \frac{z}{2\pi\boldsymbol{i}}\right)\\ + k\,(2\pi\boldsymbol{i})^k\,\Phi\left(e^{2\pi\boldsymbol{i}\,x}, -k+1, \frac{z}{2\pi\boldsymbol{i}}\right)\Big)\end{aligned}$$

and therefore the below is obtained:

$$\begin{aligned}&\sum_{j=2}^{k}\frac{z^{k-j}}{(k-j)!}\zeta(j) + (-1)^k\sum_{j=2}^{k}\frac{(-z)^{k-j}}{(k-j)!}\zeta(j)\\ &\qquad = \frac{1}{k!}\left(z^k - \pi\boldsymbol{i}\,k\,z^{k-1} + k\,(2\pi\boldsymbol{i})^k\,\zeta\left(-k+1, \frac{z}{2\pi\boldsymbol{i}}\right)\right)\end{aligned} \tag{39}$$

At the positive integers k, this equation holds even when $\Im(z) > 2\pi$. Unfortunately, this analytic continuation does not hold for both z and $-z$ at the same time, which would allow a system of equations to be formed to solve for each summation as a variable (if the system thus formed were not singular). A new equation is needed. It seems that it is possible to have a closed form for this particular equation only because the values of the zeta function at the odd integers cancel out. Obtaining the same for the odd values of the zeta is either impossible or simply not known.

A.2 Taylor series approach

Another way to express the function $f(z)$ from equation (13) in a useful way is to expand the integral of formula (11) with Taylor series.

From the Taylor series expansion of the cotangent [1] [3],

$$\coth\frac{zu}{2} = -\frac{2\boldsymbol{i}}{\pi}\sum_{j=0}^{\infty}\zeta(2j)\left(-\frac{zu}{2\pi\boldsymbol{i}}\right)^{2j-1},$$

it follows that

$$\left((1-u)^{k-1}-1\right)\coth\frac{zu}{2} = -\frac{2\boldsymbol{i}}{\pi}\sum_{j=0}^{\infty}\zeta(2j)\left(-\frac{zu}{2\pi\boldsymbol{i}}\right)^{2j-1}\left((1-u)^{k-1}-1\right),$$

and from

$$\int_0^1\left((1-u)^{k-1}-1\right)u^{2j-1}\,du = \begin{cases}\dfrac{\Gamma(k)\,\Gamma(2j)}{\Gamma(2j+k)}-\dfrac{1}{2j}, \text{ if } j\geq 1,\\ -H(k-1), \text{ if } j=0 \text{ and } \Re(k)>0\end{cases}$$

it follows that:

$$\int_0^1\left((1-u)^{k-1}-1\right)\coth\frac{zu}{2}\,du$$
$$= -\frac{2\,H(k-1)}{z}-\frac{4\,\Gamma(k)}{z}\sum_{j=1}^{\infty}\zeta(2j)\left(-\frac{z}{2\pi\boldsymbol{i}}\right)^{2j}\frac{\Gamma(2j)}{\Gamma(k+2j)}$$

B A polylog relation that holds always

For the polylog, a new relation that holds always (except $\Im(z) > 2\pi$) can be obtained by exploiting the symmetry with respect to z, since the integral that appears in the equations for both z and $-z$ is the same:

$$\sum_{j=2}^{k} \frac{z^{k-j}}{(k-j)!}\zeta(j) + \frac{z^k}{(-z)^k} \sum_{j=2}^{k} \frac{(-z)^{k-j}}{(k-j)!}\zeta(j)$$
$$= \frac{z^k}{k!} + \mathrm{Li}_k(e^z) + \frac{z^{k-1}}{\Gamma(k)} \log\left(-\frac{z}{2\pi}\right) + \frac{z^k}{(-z)^k}\left(\mathrm{Li}_k(e^{-z}) + \frac{(-z)^{k-1}}{\Gamma(k)} \log\left(\frac{z}{2\pi}\right)\right)$$

Since this relation deals with complex numbers, it is presented in the safest possible form, to minimize potential issues with symbolic manipulation, though simplications may apply in some cases.

Now let $g(z,k)$ be the function on the right-hand side. Then the following system of equations can be formed:

$$\begin{cases} f(z) + \dfrac{z^k}{(-z)^k}\, f(-z) = g(z,k) \\ \dfrac{(-z)^k}{z^k}\, f(z) + f(-z) = g(-z,k) \end{cases}$$

This system is singular for all k. Another equation is necessary for the system to be solvable. For example, in appendix (C), a way to express the polylog in terms of $\zeta(k,b)$ and $\zeta(k,-b)$ is found by adding a new equation that can be deduced from the properties of $\zeta(k,-b)$. Now that the summation is less mysterious, it may not be impossible to achieve that.

C Restated Riemann functional equation

In the paper [10], using the symmetries of the Hurwitz zeta formula with respect to the shift parameter b, we proved that if $k \neq 1$ is complex and b is a positive real number then:

$$\zeta(k,b) = -\sum_{j=1}^{\lfloor b \rfloor} \frac{1}{(b-j)^k}$$
$$- \boldsymbol{i}\,(2\pi)^{k-1}\,\Gamma(1-k)\left(e^{k\,\boldsymbol{i}\,\pi/2}\,\mathrm{Li}_{-k+1}\left(e^{2\pi \boldsymbol{i}\, b}\right) - e^{-k\,\boldsymbol{i}\,\pi/2}\,\mathrm{Li}_{-k+1}\left(e^{-2\pi \boldsymbol{i}\, b}\right)\right) \quad (40)$$

Notice this equation has singularities at the positive integers that apparently cannot be removed all at once, though it still holds at the limit (except $k=1$).

Now, to invert this formula and express the polylog as a function of Hurwitz zeta functions, one can exploit the symmetries with respect to the parameter b.

The first step is to deduce an expression for $\zeta(k,-b)$. Since,

$$\zeta(k,-b) = \sum_{j=0}^{\lfloor b \rfloor} \frac{1}{(j-b)^k} + \zeta(k, \lfloor b \rfloor - b + 1), \tag{41}$$

and since the formula of the Hurwitz zeta with shift $0 < \lfloor b \rfloor - b + 1 \leq 1$ can be written through equation (40), one concludes that:

$$\begin{aligned}\zeta(k,-b) &= \sum_{j=0}^{\lfloor b \rfloor} \frac{1}{(j-b)^k} \\ &\quad + \boldsymbol{i}\,(2\pi)^{k-1}\,\Gamma(1-k)\left(e^{-k\,\boldsymbol{i}\,\pi/2}\,\mathrm{Li}_{-k+1}\left(e^{2\pi\boldsymbol{i}\,b}\right) - e^{k\,\boldsymbol{i}\,\pi/2}\,\mathrm{Li}_{-k+1}\left(e^{-2\pi\boldsymbol{i}\,b}\right)\right)\end{aligned} \tag{42}$$

From these two formulae, a system of equations can be formed, where the singularity of the original equation is removed by Euler's reflection formula. If $b > 0$, then for b one has:

$$\begin{aligned}\frac{(2\pi)^k}{\Gamma(k)}\,\mathrm{Li}_{-k+1}\left(e^{2\pi\boldsymbol{i}\,b}\right) = e^{\boldsymbol{i}\,\pi\,k/2}&\left(\zeta(k,b) + \sum_{j=1}^{\lfloor b \rfloor} \frac{1}{(b-j)^k}\right) \\ &+ e^{-\boldsymbol{i}\,\pi\,k/2}\left(\zeta(k,-b) - \sum_{j=0}^{\lfloor b \rfloor} \frac{1}{(j-b)^k}\right),\end{aligned} \tag{43}$$

and for $-b$:

$$\begin{aligned}\frac{(2\pi)^k}{\Gamma(k)}\,\mathrm{Li}_{-k+1}\left(e^{-2\pi\boldsymbol{i}\,b}\right) = e^{-\boldsymbol{i}\,\pi\,k/2}&\left(\zeta(k,b) + \sum_{j=1}^{\lfloor b \rfloor} \frac{1}{(b-j)^k}\right) \\ &+ e^{\boldsymbol{i}\,\pi\,k/2}\left(\zeta(k,-b) - \sum_{j=0}^{\lfloor b \rfloor} \frac{1}{(j-b)^k}\right)\end{aligned} \tag{44}$$

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