PROVING THE INEQUALITY ABOUT EULER'S TOTIENT FUNCTION

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ABSTRACT. This inequality has implication for the Riemann Hypothesis.

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Let me denote $L_0 := e^\gamma \ln\ln(N_k/p_k)$, $L_1 := e^\gamma \ln\ln N_k$, $\beta_c := p_k - 1$. Nicolas has shown [1] that if

$$G(k) = \frac{N_k}{\varphi(N_k)} - L_1 > 0 \tag{1}$$

is not violated for all k, the Riemann Hypothesis is true; if the Riemann Hypothesis is false, the sign of $G(k)$ changes infinitely many times. Notably, a failed attempt to prove it found is in Wang's paper [2] (an elementary problem occurs in the transiton from his Eq. (9) to Eq. (10)). Below is my attempt.

In inequality (1), the primorial of order k is given by

$$N_k = \prod_{i=1}^{k} p_i, \tag{2}$$

where p_i are the first k prime numbers, starting from $p_1 = 2$. The constant $\gamma \approx 0.577216$ is the Euler–Mascheroni constant, and $\varphi(N_k)$ is Euler's totient function for the primorial,

$$\varphi(N_k) = N_k \prod_{i=1}^{k} \left(1 - \frac{1}{p_i}\right). \tag{3}$$

Therefore,

$$G(k) = G_0(k) - L_1, \tag{4}$$

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where

$$G_0(k) = \prod_{i=1}^{k} \frac{p_i}{p_i - 1}. \tag{5}$$

As the sign of $G(k)$ fluctuates in case the Riemann Hypothesis is false, there is a k with

$$G(k) = G_0(k) - L1 < 0, \tag{6}$$

but

$$G(k-1) = G_0(k)\frac{\beta_c}{p_k} - L_0 > 0. \tag{7}$$

Therefore, for a certain $\epsilon = \epsilon(k)$ for this k with $0 < \epsilon < 1$, one has

$$\epsilon\, G(k) + (1-\epsilon)\, G(k-1) = 0. \tag{8}$$

This can be rewritten as $\epsilon d + G(k-1) = 0$, where

$$d = L_0 - L_1 + \frac{G_0(k)}{p_k}. \tag{9}$$

As $G(k-1) > 0$ and $\epsilon > 0$, one has $d < 0$. This means that

$$\frac{G_0(k)}{p_k} < L1 - L_0 =: A. \tag{10}$$

On the other hand, from the inequality (7) itself one obtains

$$\frac{G_0(k)}{p_k} > \frac{L_0}{\beta_c} =: B. \tag{11}$$

For counter-examples to Nicolas inequality to exist, one needs $A > B$, in particular $A \neq B$; notably, $A = B$ implies $G_0(k)/p_k = A = B$, $d = 0$, and $G(k-1) = G(k) = 0$.

Another inequality is $(-d)\,\beta + G(k-1) > 0$ with positive free parameter $\beta > 0$; $-d > 0$, $G(k-1) > 0$. Expressing $G(k-1)$ by $G_0(k)$ and extracting $G_0(k)/p_k$, one has

$$\frac{G_0(k)}{p_k} > \frac{1}{\beta_c - \beta}\,((1+\beta)\, L_0 - \beta\, L_1) =: K, \tag{12}$$

where $\beta_c - \beta > 0$. If $\beta_c - \beta < 0$, one has $G_0(k)/p_k < K$. This means that there is a critical $\beta = \beta_c$, such that $G_0(k)/p_k = K$ with

$$(1+\beta_c)\, L_0 - \beta_c\, L_1 = 0. \tag{13}$$

However, this is not true because then $A = B$; hence, there are no counter-examples to the Nicolas inequality.

Alternative proof

The formulas $A > B$ and $A > K$ are equivalent because both are giving the same inequality $\beta_c\, L_1 > p_k\, L_0$. On the other hand, $K - B = S$ with

$$S = \frac{\beta}{\beta_c - \beta} \left(\frac{p_k\, L_0}{\beta_c} - L_1 \right) < 0, \tag{14}$$

where $\beta < \beta_c$. $A > K$ is equivalent to $A - S > K - S = B$ and $A > B + S < B$. Hence, from $A > K$ we cannot conclude that $A > B$. However, $A > K$ and $A > B$ have to be equivalent. With this, one comes to a contradiction (unless $S = 0$, which implies $A = B$, which is in contradiction to $A \neq B$).

Moreover,

$$A - K = \frac{\beta_c}{\beta_c - \beta} \left(L_1 - \frac{p_k\, L_0}{\beta_c} \right). \tag{15}$$

Formula $A - K = -S$ looks like $\beta_c\, F = \beta\, F$ with a function F on both sides. This formula becomes $\beta_c = \beta$. Because of $K = B + S$, one has $A = B$. Notably, $A = B$ implies $L_1 - p_k\, L_0/\beta_c = 0$. This is in contradiction to $A \neq B$.

Conclusion

In this publication, I have given proofs for the Riemann Hypothesis.

References

[1] Nicolas, J.-L. (1983) "Petites valeurs de la fonction d'Euler." *J. Number Theory* **17** 3, 375–388.
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[2] Wang, C.-T. (2025) "On an inequality about Euler's totient function." *Notes on Number Theory and Discrete Mathematics* **31** 2, 299–304.
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