Dual Architecture of the Gamma Function: Complementary Extension to the Left Half-Plane and Regularization of Logarithmic Divergences

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Abstract

This work proposes a complementary extension of the factorial function to the real line, based on the separation of the bilateral integral of e^{-Ax} into two disjoint domains. For $A > 0$, the classical Gamma function $\Gamma(z)$ emerges in the right half-plane; for $A < 0$, we introduce the Symmetric Gamma Function $\bar{\Gamma}(z) = e^{-i\pi(z+1)}\Gamma(-z)$, defined in the left half-plane, with poles at the positive integers.

The duality between the integral kernels underlies a regularization mechanism for logarithmic divergences: $\Gamma_R(z) = 1/\bar{\Gamma}(z-1)$ and $\bar{\Gamma}_R(z) = 1/\Gamma(z+1)$, replacing each pole by an exact finite value. An alternative real representation $\mathcal{B}(x) = \int_0^{\infty} e^{-u}\, u^{-x} du$ (convergent for $x < 1$) is constructed, together with a trigonometric factor $C(x) = 2\sin(\pi x) + \cos(\pi x)$, defining $\mathcal{F}(x) = C(x)\mathcal{B}(x)$ for $x \leq 0$, yielding $\mathcal{F}(-n) = (-1)^n n!$, thereby unifying the complex branch, real regularization, and Laurent expansion formalisms.

The relation $\mathcal{F}(-n) \cdot \mathrm{Res}\,(\Gamma, -n) = 1$ establishes the fundamental duality, which emerges directly from the construction of the integral kernels. The method is validated in arithmetic progressions, QED, and QCD. The extension to k-loops is systematic via exponentiation: $\Gamma_R^{(k)}(-n) = [(-1)^n/n!]^k$.

Distinct physical prediction: In $d = 5$ dimensions, the vacuum energy density changes sign — a repulsive force where the MS-bar predicts attraction. This inversion has direct implications for the stability of extra dimensions in string theory and brane models.

Keywords: Gamma function; Analytic continuation; Regularization of logarithmic divergences; Integral duality; Symmetric factorial; Extra dimensions; Sign inversion in odd dimensions.

1. Introduction

1.1 The Direction of the Recurrence as a Fundamental Problem

Euler's Gamma function, defined for $\Re(z) > 0$ by the improper integral

$$\Gamma(z) = \int_0^\infty e^{-t}\, t^{z-1} dt, \Re(z) > 0, \qquad (1.1)$$

constitutes the canonical extension of the factorial to the complex plane, satisfying the functional equation

$$\Gamma(z+1) = z\Gamma(z) \qquad (1.2)$$

and the identity $\Gamma(n+1) = n!$ for $n \in \mathbb{N}$ [1,2]. Analytic continuation to the left half-plane, performed by iterated application of (1.2), establishes the existence of simple poles at the non-positive integers $z \in \{0, -1, -2, \ldots\}$.

Equation (1.2) can be rewritten as

$$\Gamma(z) = \frac{\Gamma(z+1)}{z}. \qquad (1.3)$$

This form is used to extend the function to the left, starting from known values in the right half-plane. However, the same equation admits the equivalent form

$$\Gamma(z) = z\Gamma(z-1), \qquad (1.4)$$

which moves to the right. For $z = 1$, expression (1.4) would give $\Gamma(1) = 1 \cdot \Gamma(0)$. Knowing that $\Gamma(1) = 1$, one would obtain $\Gamma(0) = 1$, which contradicts the fact that integral (1.1) diverges at $z = 0$. One concludes that the integral representation (1.1) recognizes only one direction for the application of the recurrence: that which **subtracts one unit**, moving to the left, as in (1.3). The opposite direction — which **adds one unit**, moving to the right, as in (1.4) — is not supported by the integral representation.

Tradition, in calculating $\Gamma(-1/2)$, starts from $\Gamma(1/2) = \sqrt{\pi}$ and applies the recurrence in the reverse direction (adding one unit):

$$\Gamma\left(-\frac{1}{2}\right) = \frac{\Gamma\left(\frac{1}{2}\right)}{-\frac{1}{2}} = -2\sqrt{\pi}. \qquad (1.5)$$

Integral (1.1) diverges at $z = -1/2$. The value $-2\sqrt{\pi}$ is obtained by an algebraic convention, not by a necessity imposed by the integral representation. **Tradition circumvents the divergence; the symmetric method resolves it, respecting the direction imposed by the integral.**

1.2 The Necessity of Regularizing the Divergent Side

The divergence of integral (1.1) for negative arguments signals that the classical representation has no authority over these points. To access the negative domain consistently, the following principle is proposed: **to unite the positive domain (where the integral converges) with the negative domain (where the integral diverges), it is necessary to regularize the divergent side, not to assign it a finite value by convention.**

This principle underpins the construction presented in this work. An alternative integral representation is proposed, convergent for $x < 1$,

$$\mathcal{B}(x) = \int_0^\infty e^{-u}\, u^{-x} du, x < 1, \qquad (1.6)$$

which converges directly for all negative integers and for $x \leq 0$ (except at the pole $x = 1$). Integral (1.6) does not depend on the Gamma function and is defined exclusively by its convergence on the real axis.

It is observed that the arbitrary scale μ of the MS-bar scheme, introduced by 't Hooft and Veltman in 1972, does not correspond to any physical property of the quantum vacuum [8,9]. Different choices of μ produce different intermediate values, which only cancel after renormalization. In contrast, the choice $M_0 = 1$ in the symmetric method is fixed and unique — the constant 1 is the **canonical normalization** of the integration space measure in the ultraviolet limit.

Additionally, a real regularization factor is proposed

$$C(x) = 2\sin(\pi x) + \cos(\pi x), \qquad (1.7)$$

which, multiplied by $\mathcal{B}(x)$, yields the regularized function

$$\mathcal{F}(x) = C(x) \cdot \mathcal{B}(x). \qquad (1.8)$$

For negative integer arguments, $x = -n$ with $n \in \mathbb{N}_0$, we verify

$$\mathcal{F}(-n) = (-1)^n n!. \qquad (1.9)$$

These values coincide with the symmetric factorial $!(-n)$ and establish a duality with the residues of the classical Gamma function.

1.3 Duality as a Fundamental Structure of the Gamma Function

1.3.1 Construction from the Integral Kernels

The construction of the classical Gamma function and the symmetric Gamma function begins with the separation of the bilateral integral

$$I(A) = \int_{-\infty}^{\infty} e^{-At}\, dt. \qquad (1.10)$$

The resulting integral kernels are

$$K_\Gamma(t,z) = t^{z-1}e^{-t}, t > 0, K_{\bar{\Gamma}}(t,z) = t^{z+1}e^{t}, t < 0. \qquad (1.11)$$

These kernels satisfy the relation

$$K_\Gamma(t,z) \cdot K_{\bar{\Gamma}}(t,-z) = 1. \qquad (1.12)$$

The integral defining $\Gamma(z)$ converges only for $\mathrm{Re}\,(z) > 0$. For $\mathrm{Re}\,(z) \leq 0$, the integral diverges these points are singular. The expected nature of these points, by the very definition of the integral, would be divergence.

1.3.2 Euler's Reflection Formula

Euler's reflection formula,

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}, \qquad (1.13)$$

is an analytic identity valid throughout the complex plane (except at poles). For non-integer negative arguments, such as $z = -1/2$, it yields a finite value:

$$\Gamma\left(-\frac{1}{2}\right) = -2\sqrt{\pi}. \qquad (1.14)$$

It is observed that Euler's integral diverges at $z = -1/2$. The finite value obtained in (1.14) follows from identity (1.13), not from a direct evaluation of the integral representation. This procedure is a legitimate analytic continuation, but it is important to note that the value thus obtained is not extracted from the divergent integral it is a consequence of the algebraic structure of the Gamma function.

1.3.3 Non-Integer Negative Values

For non-integer negative arguments, Euler's integral diverges. The reflection formula, however, produces finite values for these points. The recurrence, used in the reverse direction, transforms a divergence into an apparent finite value.

In the proposed symmetric method, the natural direction of the recurrence is maintained (always subtracting one unit). For non-integer negative arguments, it is considered that:

1. The integral diverges.
2. The points are poles (of the classical Gamma function) or regular points (of the symmetric Gamma function after regularization).
3. The regularized value is obtained by duality with $\mathcal{F}(x)$, not by analytic continuation via the reflection formula.

1.3.4 Consequences for Odd Dimensions

The difference between the two approaches manifests concretely in odd dimensions. For example, for $d = 5$ (argument $\Gamma(-5/2)$):

Method	$\Gamma(-5/2)$	Energy density $\mathcal{E}_0(5)$	Sign of the force
MS-bar (via reflection)	$-8\sqrt{\pi}/15$	$-\dfrac{m^5}{120\pi^2}$	Negative (attractive)
Symmetric method (via duality)	Pole; regularized: $-4/(3\sqrt{\pi})$	$+\dfrac{m^5}{48\pi^3}$	Positive (repulsive)

Euler's reflection formula, by connecting the two domains, leads to a negative sign for the energy density (attractive force). The symmetric method, by recognizing the pole and regularizing via duality, leads to a positive sign (repulsive force). **The two methods differ in their prediction for this observable.**

1.3.5 Synthesis

Euler's reflection formula is a valid mathematical identity. When applied to arguments where the integral representation diverges, it produces finite values that follow from the algebraic structure of the Gamma function, not from a direct evaluation of the integral.

The symmetric method proposes an alternative: to recognize the divergence, treat the points as poles, and regularize via the duality $\mathcal{F}(-n) \cdot \text{Res}\,(\Gamma, -n) = 1$.

The difference between the two methods translates into distinct physical predictions, namely the sign of the vacuum energy density in $d = 5$ dimensions.

1.4 Structure of the Paper

The paper is organized into three complementary axes. Section 2 presents the formal definitions, including the operator $\mathcal{B}(x)$, the regularization factor $C(x)$, the unified function $\mathcal{F}(x)$, and the regularization of poles via duality. Section 3 validates the method in arithmetic progressions and in applications to quantum field theory (vacuum polarization, electron self-energy, anomalous magnetic moment), including the extension to k-loops and the application to non-abelian theories. Section 4 presents the conclusions and suggestions for future work.

2. DEFINITION AND FUNDAMENTAL PROPERTIES

2.1 Classical Factorial and Symmetric Factorial

For $n \in \mathbb{N}$, the classical factorial is defined by the descending product:

$$n! = n(n-1)(n-2)\cdots 1, 0! = 1. \qquad (2.1)$$

This operator converges for all $n \geq 0$. Its extension to the complex plane is mediated by Euler's Gamma function:

$$n! = \Gamma(n+1), \Re(n+1) > 0. \qquad (2.2)$$

For $n \in \mathbb{Z}^-$, the **Symmetric Factorial** is defined by the ascending product:

$$!(n) = n(n+1)(n+2)\cdots(-1), !(0) = 1. \qquad (2.3)$$

This operator converges for all $n \leq 0$. Its extension to complex values is mediated by the Symmetric Gamma Function, to be introduced in Section 2.3.

Fundamental property. For $n = -m$ with $m \in \mathbb{N}^+$:

$$!(-m) = (-m)(-m+1)\cdots(-1) = (-1)^m m!. \qquad (2.4)$$

Recurrence. From the product definition:

$$!(n) = n \cdot !(n+1), n < 0. \qquad (2.5)$$

Duality Table of Factorials

Operator	Domain	Product Structure	Recurrence
$n!$ (classical)	$n \geq 0$	$n(n-1)\cdots 1$ (descending)	$(n+1)! = (n+1)n!$
$!(n)$ (symmetric)	$n \leq 0$	$n(n+1)\cdots(-1)$ (ascending)	$!(n-1) = n \cdot !(n)$

2.2 Bilateral Integral Representation and Branch Structure

Consider the integral:

$$I(A) = \int_{-\infty}^{\infty} e^{-At}\, dt. \qquad (2.6)$$

Separating the branches:

$$I(A) = \int_{-\infty}^{0} e^{-At}\, dt + \int_{0}^{\infty} e^{-At}\, dt. \qquad (2.7)$$

Each branch converges in opposite domains of A, reflecting a duality structure between the half-axes.

2.2.1 Branch Structure and Contour

Let γ be an admissible contour and $\theta \in \mathbb{R}$ a branch argument. Define:

$$!(\gamma, \theta)(z) = \int_{\gamma} t_{\theta}^{-z}\, e^{t} dt, \qquad (2.8)$$

where:

$$t_{\theta}^{-z} = \exp(-z \mathrm{Log}_{\theta}(t)), \mathrm{Log}_{\theta}(t) = \ln |t| + i \arg_{\theta}(t), \arg_{\theta}(t) \in (\theta, \theta + 2\pi). \qquad (2.9)$$

Branch change $\theta \to \theta + 2\pi k$:

$$!(\gamma, \theta + 2\pi k)(z) = e^{-2\pi ikz} \, !(\gamma, \theta)(z). \qquad (2.10)$$

Discrete case. For $z = -n, n \in \mathbb{N}$:

$$!(\gamma, \theta)(-n) = (-1)^n n!. \qquad (2.11)$$

Principal case. For $\theta = \pi, \gamma = (-\infty, 0]$:

$$!(z) = e^{-i\pi z}\Gamma(1 - z). \qquad (2.12)$$

2.2.2 Demonstration of Branch Dependence

Let $k \in \mathbb{Z}$. Take $\theta' = \theta + 2\pi k$. By definition:

$$\text{Log}_{(\theta+2\pi k)}(t) = \ln\ | t | + i\arg_{\theta+2\pi k}(t) = \ln\ | t | + i(\arg_\theta(t) + 2\pi k) = \text{Log}_\theta(t) + 2\pi ik. \qquad (2.13)$$

Compute the integrand:

$$t_{\theta+2\pi k}^{-z} = \exp\left(-z\text{Log}_{(\theta+2\pi k)}(t)\right) = \exp\left(-z[\text{Log}_\theta(t) + 2\pi ik]\right) = t_\theta^{-z} \cdot e^{-2\pi ikz}. \qquad (2.14)$$

Substitute into the integral:

$$!(\gamma, \theta + 2\pi k)(z) = \int_\gamma t_{\theta+2\pi k}^{-z} e^t dt = e^{-2\pi ikz} \int_\gamma t_\theta^{-z} e^t dt = e^{-2\pi ikz} \cdot !(\gamma, \theta)(z). \qquad (2.15)$$

Relation with Euler's Gamma. In the principal branch $\theta = \pi, \gamma = (-\infty, 0]$:

$$!((-\infty, 0], \pi)(z) = \int_{-\infty}^{0} t_\pi^{-z} e^t dt = e^{-i\pi z} \int_0^\infty u^{-z} e^{-u} du = e^{-i\pi z}\Gamma(1 - z). \qquad (2.16)$$

2.3 Symmetric Gamma Function

From the definition of the symmetric factorial operator, the Symmetric Gamma Function is defined by

$$\bar{\Gamma}(z) = !(z + 1). \qquad (2.17)$$

Using the representation obtained in the principal branch ($\theta = \pi, \gamma = (-\infty, 0]$), $!(z) = e^{-i\pi z}\Gamma(1 - z)$, it follows that

$$\bar{\Gamma}(z) = e^{-i\pi(z+1)} \int_0^\infty u^{-(z+1)} e^{-u} du. \qquad (2.18)$$

Therefore,

$$\bar{\Gamma}(z) = e^{-i\pi(z+1)}\Gamma(-z), \qquad (2.19)$$

with convergence condition $\text{Re}(z) < 0$.

Since $\Gamma(-z)$ has simple poles at $z \in \{0,1,2,3, \dots\}$, it follows that the poles of the Symmetric Gamma Function are located at:

$$z \in \{0,1,2,3, \dots\}. \qquad (2.20)$$

2.3.1 Functional Equation

Proposition 2.3.1. In the principal branch ($\theta = \pi$), one verifies:

$$\bar{\Gamma}(z - 1) = z\bar{\Gamma}(z). \qquad (2.21)$$

Proof. From the definition, $\bar{\Gamma}(z) = e^{-i\pi(z+1)}\Gamma(-z)$. Substituting z by $z-1$:

$$\bar{\Gamma}(z-1) = e^{-i\pi z}\Gamma(1-z). \qquad (2.22)$$

By the functional equation of Euler's Gamma, $\Gamma(1-z) = (-z)\Gamma(-z)$. Thus:

$$\bar{\Gamma}(z-1) = -ze^{-i\pi z}\Gamma(-z). \qquad (2.23)$$

Since $e^{-i\pi z} = -e^{-i\pi(z+1)}$, it follows that

$$\bar{\Gamma}(z-1) = ze^{-i\pi(z+1)}\Gamma(-z) = z\bar{\Gamma}(z). \qquad (2.24)$$

2.3.2 Fundamental Values

Classical Gamma:

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}. \qquad (2.25)$$

Symmetric Gamma (principal branch):

$$\bar{\Gamma}\left(-\frac{1}{2}\right) = e^{-i\pi/2}\Gamma\left(\frac{1}{2}\right) = -i\sqrt{\pi}. \qquad (2.26)$$

Properties:

1. $!(\gamma,\theta)(-n) = (-1)^n n!, \forall\gamma,\theta.$
2. $!(\gamma,\theta+2\pi)(z) = e^{-2\pi i z}\cdot !(\gamma,\theta)(z).$
3. $!((-\infty,0],\pi)(z) = e^{-i\pi z}\Gamma(1-z).$

2.4 Geometric Interpretation in the Argand-Gauss Plane

The classical Gamma function $\Gamma(z)$ and the Symmetric Gamma Function $\bar{\Gamma}(z)$ can be interpreted as two complementary regimes in the complex plane $\mathbb{C}$, where the algebraic structure manifests as geometric orientation and phase rotation.

Fundamental relation between the two regimes. The Symmetric Gamma Function is defined by:

$$\bar{\Gamma}(z) = e^{-i\pi(z+1)}\Gamma(-z). \qquad (2.27)$$

Structural axes and geometric duality. The two regimes present distinct natural alignments:

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}(\text{real axis}), \bar{\Gamma}\left(-\frac{1}{2}\right) = -i\sqrt{\pi}(\text{imaginary axis}). \qquad (2.28)$$

Behavior in the complex plane. For $z = x + iy$, the phase factor can be decomposed as:

$$e^{-i\pi(z+1)} = e^{\pi y}\cdot e^{-i\pi(x+1)}. \qquad (2.29)$$

Special symmetry regimes. The function $\bar{\Gamma}(z)$ exhibits distinct behaviors depending on its position in the complex plane:

- For $z \in \mathbb{Z}$ → real values.
- For $z = n + 1/2$ → purely imaginary values.
- Near integers → the imaginary component tends to zero.
- Near half-integers → the real component tends to zero.

2.5 Real Regularization of the Branch Cut

The integral representation in the principal branch, established in Section 2.2,

$$!(z) = e^{-i\pi z}\int_0^\infty u^{-z}\, e^{-u} du = e^{-i\pi z}\Gamma(1-z), \qquad (2.30)$$

depends on the complex phase factor $e^{-i\pi z}$ to access values in the negative domain. In this section, an alternative possibility is explored: **replacing the complex phase factor with a real trigonometric regulator.**

2.5.1 Barrier Integral Operator

Definition 2.5.1 (Operator $\mathcal{B}$). For $x \in \mathbb{R}$ with $x < 1$, define

$$\mathcal{B}(x) = \int_0^\infty e^{-u}\, u^{-x} du. \qquad (2.31)$$

Proposition 2.5.0 (Relation with the classical Gamma function). For $x < 1$,

$$\mathcal{B}(x) = \Gamma(1-x). \qquad (2.32)$$

Example 2.5.1 (Elementary values). By direct integration:

$$\mathcal{B}(0) = 1, \mathcal{B}(-1) = 1, \mathcal{B}(-2) = 2, \mathcal{B}(-3) = 6. \qquad (2.33)$$

For $x = -1/2$, via the substitution $u = t^2$, $du = 2tdt$:

$$\mathcal{B}\left(-\frac{1}{2}\right) = \int_0^\infty e^{-u}\, u^{1/2} du = 2\int_0^\infty t^2\, e^{-t^2} dt = 2 \cdot \frac{\sqrt{\pi}}{4} = \frac{\sqrt{\pi}}{2}. \qquad (2.34)$$

2.5.2 Recurrence Equation

Lemma 2.5.1 (Recurrence of $\mathcal{B}$). For every $x < 1$,

$$\mathcal{B}(x+1) = -\frac{\mathcal{B}(x)}{x}. \qquad (2.35)$$

Proof. For $0 < \varepsilon < R < \infty$, integrate by parts:

$$\int_\varepsilon^R e^{-u}\, u^{-x} du = [-e^{-u} u^{-x}]_\varepsilon^R - x\int_\varepsilon^R e^{-u}\, u^{-(x+1)} du. \qquad (2.36)$$

Taking the limits $\varepsilon \to 0^+$ and $R \to \infty$, one obtains $\mathcal{B}(x) = -x\mathcal{B}(x+1)$. Isolating $\mathcal{B}(x+1)$ yields (2.35).

2.5.3 Construction of the Real Regularization Factor

Objective. Find a real function $C(x)$ such that the product

$$\mathcal{F}(x) = C(x) \cdot \mathcal{B}(x) \qquad (2.37)$$

satisfies the recurrence equation **without** sign alternation:

$$\mathcal{F}(x) = x \cdot \mathcal{F}(x+1). \qquad (2.38)$$

Lemma 2.5.2 (Antiperiodicity of C). Condition (2.38) implies:

$$C(x+1) = -C(x). \qquad (2.39)$$

Proof. Substitute definition (2.37) into both sides of (2.38):

$$C(x) \cdot \mathcal{B}(x) = x \cdot C(x+1) \cdot \mathcal{B}(x+1).$$

Using relation (2.35), $\mathcal{B}(x+1) = -\mathcal{B}(x)/x$:

$$x \cdot C(x+1) \cdot \left(-\frac{\mathcal{B}(x)}{x}\right) = -C(x+1) \cdot \mathcal{B}(x).$$

Equating both sides:

$$C(x) \cdot \mathcal{B}(x) = -C(x+1) \cdot \mathcal{B}(x).$$

Canceling $\mathcal{B}(x) \neq 0$ (valid for $x < 1$, $x \notin \{1,2,3,...\}$), one obtains (2.39).

Proposition 2.5.1 (General solution). The functional equation (2.39) has, in the real domain, the general solution:

$$C(x) = A\sin(\pi x) + B\cos(\pi x). \qquad (2.40)$$

Verification. For $x + 1$:

$$C(x+1) = A\sin(\pi x + \pi) + B\cos(\pi x + \pi) = -A\sin(\pi x) - B\cos(\pi x) = -C(x).$$

2.5.4 Determination of the Constants

Condition 2.5.1 (Origin). For $x = 0$, impose $\mathcal{F}(0) = 1$. Since $\mathcal{B}(0) = 1$:

$$\mathcal{F}(0) = C(0) = B = 1. \qquad (2.41)$$

Condition 2.5.2 (Fractional anchor). For $x = -1/2$, impose $\mathcal{F}(-1/2) = -\sqrt{\pi}$. Since $\mathcal{B}(-1/2) = \sqrt{\pi}/2$:

$$\mathcal{F}\left(-\frac{1}{2}\right) = C\left(-\frac{1}{2}\right) \cdot \frac{\sqrt{\pi}}{2} = -\sqrt{\pi} \Rightarrow C\left(-\frac{1}{2}\right) = -2. \qquad (2.42)$$

Computing $C(-1/2) = A\sin(-\pi/2) + \cos(-\pi/2) = -A$. Hence $-A = -2$, so $A = 2$.

2.5.5 Real Regularization Factor

$$C(x) = 2\sin(\pi x) + \cos(\pi x). \qquad (2.43)$$

2.5.6 Complete Regularized Function

Definition 2.5.2 (Function $\mathcal{F}$). The following real analytic continuation is proposed for the symmetric factorial operator:

$$\mathcal{F}(x) = \begin{cases} \mathcal{B}(x) = \int_0^\infty e^{-u} u^{-x} du, & x > 0, \\ [2\sin(\pi x) + \cos(\pi x)] \int_0^\infty e^{-u} u^{-x} du, & x \leq 0. \end{cases} \qquad (2.44)$$

The function $\mathcal{F}(x)$ thus constructed is **unique**. Unlike the MS-bar scheme, which admits multiple variants (MS, MS-bar, PDS, etc.) with different subtraction prescriptions, the symmetric method has no free parameters. The constant 1 cannot be altered; the scale $M_0 = 1$ is fixed.

Proposition 2.5.2 (Integer values). For every $n \in \mathbb{N}_0$:

$$\mathcal{F}(-n) = (-1)^n n! = !(-n). \qquad (2.45)$$

2.5.7 Validation Table

x	Domain	$\mathcal{B}(x)$	$C(x)$	$\mathcal{F}(x)$	Reference
$1/2$	$x > 0$	$\sqrt{\pi}$	—	$\sqrt{\pi}$	—
0	$x \leq 0$	1	1	1	$!0 = 1$
$-1/2$	$x \leq 0$	$\sqrt{\pi}/2$	-2	$-\sqrt{\pi}$	—

x	Domain	$\mathcal{B}(x)$	$C(x)$	$\mathcal{F}(x)$	Reference
-1	$x \leq 0$	1	-1	-1	$!(-1) = -1$
-2	$x \leq 0$	2	1	2	$!(-2) = 2$
-3	$x \leq 0$	6	-1	-6	$!(-3) = -6$
-4	$x \leq 0$	24	1	24	$!(-4) = 24$

2.5.8 Relation with the Complex Formulation

Characteristic	Branch Formulation (Section 2.2)	Real Regularization (Section 2.5)
Base integral	$\int_\gamma t_\theta^{-z}\, e^t dt$	$\int_0^\infty e^{-u}\, u^{-x} du$
Correction factor	$e^{-i\pi z}$ (complex)	$2\sin(\pi x) + \cos(\pi x)$ (real)
Domain	$\mathbb{C}$	$\mathbb{R}$
Contour	$\gamma = (-\infty, 0]$	$[0, \infty)$ (positive real axis)
Values in $\mathbb{Z}^-$	$(-1)^{\mathrm{n}} n!$	$(-1)^{\mathrm{n}} n!$
Value at $x = -1/2$	$e^{-i\pi/2}\sqrt{\pi} = -i\sqrt{\pi}$	$-\sqrt{\pi}$

2.6 Regularization of Poles via Duality Between Γ and $\bar{\Gamma}$

2.6.1 Definition and Elementary Properties

Definition 2.6.1 (Regularized Values at Integer Poles). For each $n \in \mathbb{N}_0$:

$$\Gamma_R(-n) := \frac{1}{\bar{\Gamma}(-n-1)}, \bar{\Gamma}_R(n) := \frac{1}{\Gamma(n+1)}. \qquad (2.46)$$

Proposition 2.6.2 (Explicit form). For every $m \in \mathbb{N}_0$:

$$\Gamma_R(-m) = \frac{(-1)^m}{m!}, \bar{\Gamma}_R(m) = \frac{1}{m!}. \qquad (2.47)$$

Proposition 2.6.3 (Inversion identities). For each $n \in \mathbb{N}_0$:

$$\Gamma_R(-n) \cdot \bar{\Gamma}(-n-1) = 1, \bar{\Gamma}_R(n) \cdot \Gamma(n+1) = 1. \qquad (2.48)$$

2.6.2 Laurent Expansion of the Classical Gamma Function

Lemma 2.6.4 (Laurent expansion of Γ at non-positive integers). For each $n \in \mathbb{N}_0$:

$$\Gamma(z) = \frac{\mathrm{Res}\,(\Gamma, -n)}{z+n} + H_n(z), \mathrm{Res}\,(\Gamma, -n) = \frac{(-1)^n}{n!}. \qquad (2.49)$$

Phillips, Beane, and Cohen (1997) demonstrated that dimensional regularization can fail in non-perturbative contexts because **all power-law divergences are prescribed to zero** [20]. The symmetric method does not adopt this prescription. Each pole is treated individually by the duality $\mathcal{F}(-n) \cdot \mathrm{Res}\,(\Gamma, -n) = 1$. Relation (2.49) shows that the residue is preserved, not annulled.

Theorem 2.6.5 (Singular term via symmetric function). For each $n \in \mathbb{N}_0$:

$$\Gamma(z) = \frac{1}{[(z+n)\bar{\Gamma}(-n-1)]} + H_n(z). \qquad (2.50)$$

Theorem 2.6.6 (Equivalence between regularization and residue). For each $n \in \mathbb{N}_0$:

$$\Gamma_R(-n) = \mathrm{Res}\,(\Gamma, -n) = \frac{1}{\bar{\Gamma}(-n-1)} = \frac{(-1)^n}{n!}. \qquad (2.51)$$

2.6.3 Laurent Expansion of the Symmetric Gamma Function

Lemma 2.6.8 (Laurent expansion of $\bar{\Gamma}$ at non-negative integers). For each $n \in \mathbb{N}_0$:

$$\bar{\Gamma}(z) = \frac{\mathrm{Res}\,(\bar{\Gamma}, n)}{z-n} + K_n(z), \mathrm{Res}\,(\bar{\Gamma}, n) = \frac{1}{n!}. \qquad (2.52)$$

Proposition 2.6.9. For each $n \in \mathbb{N}_0$:

$$\bar{\Gamma}_R(n) = \mathrm{Res}\,(\bar{\Gamma}, n) = \frac{1}{n!}. \qquad (2.53)$$

2.6.4 The Point $z = 0$ (Common Pole)

Proposition 2.6.11. For $z = 0$:

$$\Gamma_R(0) = \bar{\Gamma}_R(0) = 1. \qquad (2.54)$$

2.6.5 Table of Regularization and Laurent Expansion

n	$\Gamma_R(-n)$	$\bar{\Gamma}_R(n)$	$\mathrm{Res}\,(\Gamma, -n)$	$\mathrm{Res}\,(\bar{\Gamma}, n)$
0	1	1	1	1
1	-1	1	-1	1
2	$1/2$	$1/2$	$1/2$	$1/2$
3	$-1/6$	$1/6$	$-1/6$	$1/6$
4	$1/24$	$1/24$	$1/24$	$1/24$
n	$(-1)^n/n!$	$1/n!$	$(-1)^n/n!$	$1/n!$

2.6.6 Connection with the Real Regularization $\mathcal{F}(x)$

Theorem 2.6.12 (Equivalence between $\mathcal{F}$ and the Regularized Values). For every $n \in \mathbb{N}_0$:

$$\mathcal{F}(-n) = \frac{1}{\Gamma_R(-n)} = \frac{1}{\text{Res}\,(\Gamma, -n)} = (-1)^n n! = !\,(-n). \qquad (2.55)$$

Corollary 2.6.13 (Real Regularization as Inversion).

$$\mathcal{F}(-n) \cdot \Gamma_R(-n) = 1, \forall n \in \mathbb{N}_0. \qquad (2.56)$$

2.6.7 Synthesis of the Unification

Formulation	Object	Value at $x = -n$	Nature
Complex Branch (Section 2.2)	$!\,(z) = e^{-i\pi z}\Gamma(1 - z)$	$(-1)^n n!$	Complex continuation
Real Regularization (Section 2.5)	$\mathcal{F}(x) = \mathcal{C}(x) \cdot \mathcal{B}(x)$	$(-1)^n n!$	Real continuation
Laurent Expansion (Section 2.6)	$\Gamma_R(-n) = 1/\bar{\Gamma}(-n - 1)$	$(-1)^n n!$	Singular coefficient

Fundamental relation:

$$\mathcal{F}(-n) = \frac{1}{\Gamma_R(-n)} = \frac{1}{\text{Res}\,(\Gamma, -n)} = \,!\,(-n). \qquad (2.57)$$

2.7 Application of the Product Formula for Arithmetic Progressions with Regularization

The product formula for arithmetic progressions is

$$P(n) = r^n \frac{\Gamma_R\left(\frac{a_1}{r} + n\right)}{\Gamma_R\left(\frac{a_1}{r}\right)}, \Gamma_R(-n) = \frac{(-1)^n}{n!}. \qquad (2.58)$$

Example 1: $a_n = 2n - 20$, 6 terms. Parameters: $r = 2$, $a_1 = -18$, $a_1/r = -9$, $n = 6$.

$$P_6 = 2^6 \frac{\Gamma_R(-3)}{\Gamma_R(-9)} = 64 \cdot \frac{(-1)^3 3!}{(-1)^9 9!} = 64 \cdot \frac{-1/6}{-1/362880} = 64 \cdot 60480 = 3870720. \qquad (2.59)$$

Example 2: $a_n = n - 10$, 9 terms. Parameters: $r = 1$, $a_1 = -9$, $a_1/r = -9$, $n = 9$.

$$P_9 = \frac{\Gamma_R(0)}{\Gamma_R(-9)} = \frac{1}{-1/9!} = -9! = -362880. \qquad (2.60)$$

The regularization

$$\Gamma_R(-n) = \frac{1}{\bar{\Gamma}(-n - 1)}$$

ensures that the product formula for arithmetic progressions remains consistent even in the presence of poles of the Gamma function, maintaining equivalence with the term-by-term calculation. Thus, poles come to represent only changes in analytic regime.

In Section 3, this structure will be applied to quantum physics, where:

- poles will be interpreted as typical divergences of quantum field theory and treated as renormalization processes;

- regularization will be seen as a formal mechanism for controlling these divergences;
- the duality between Γ and $\bar{\Gamma}$ will be associated with symmetries between different energy and phase regimes.

This establishes the basis for reinterpreting the mathematical results as physical structures in quantum systems.

3.0.2 The Separation Between Non-Physical and Physical Contributions

The proposed symmetric prescription is based on the following surgical separation:

$$\int_0^\infty \frac{dt}{t} e^{-tM^2} = \underbrace{\int_0^\infty \frac{dt}{t} e^{-tM_0^2}}_{\text{non-physical contribution}} + \underbrace{\int_0^\infty \frac{dt}{t} \left[e^{-tM^2} - e^{-tM_0^2}\right]}_{\text{physical scale-dependent contribution}} . \qquad (3.0.2)$$

The first integral is independent of M^2 and contains all the ultraviolet divergence — it is the **non-physical contribution**. The second integral is convergent and contains all the dependence on M^2 (masses, momenta, thresholds) — it is the **physical contribution**.

3.0.3 The Canonical Normalization of the Integration Space

The choice $M_0 = 1$ (in the natural units of QFT) fixes the non-physical contribution as

$$\int_0^\infty \frac{dt}{t} e^{-tM_0^2} = 1. \qquad (3.0.3)$$

In the MS-bar scheme, the same contribution is expressed as

$$\int_0^\infty \frac{dt}{t} e^{-tM_0^2} \longrightarrow \ln(\mu^2). \qquad (3.0.4)$$

The difference between the two methods is a real constant:

$$\Delta = 1 - \ln(\mu^2). \qquad (3.0.5)$$

For $\mu = 2$, $\Delta = 1 - \ln 4 \approx -0.3863$. While the MS-bar treats the non-physical contribution as dependent on a human choice (μ), the symmetric method fixes the "noise" of the vacuum in a unique and deterministic way.

The symmetric method is **physical** in the sense that the constant "1" has a clear interpretation: it is the **canonical normalization of the integration space measure** in the ultraviolet limit. In contrast, the MS-bar is a non-physical scheme: coupling constants are defined by subtraction of poles in $1/\varepsilon$, not by physical observables [23]. Mass thresholds are treated with their correct analytic dependence in the symmetric method, as evidenced by the imaginary part of the vacuum polarization, which is identical to that of the MS-bar.

3.0.4 Invariance of the Imaginary Part

The imaginary part of the amplitudes governs pair production and unitarity. Since the difference between the two methods is a real constant, **the imaginary part is strictly independent of the choice of regularization**. The unitarity of the S-matrix is exactly preserved in the symmetric method.

3.1 Regularization Prescriptions

3.1.1 Symmetric Method (proposed)

For a divergent Schwinger integral of the form

$$\mathcal{I}(M^2) = \int_0^\infty \frac{dt}{t^{n+1}} e^{-tM^2}, n \in \mathbb{N}_0, \qquad (3.1.1)$$

the symmetric method postulates:

$$\mathcal{I}(M^2) \longrightarrow M_0^{2n}\Gamma_R(-n) + \int_0^\infty \frac{dt}{t^{n+1}}\left[e^{-tM^2} - e^{-tM_0^2}\right], \qquad (3.1.2)$$

with $M_0 = 1$ and $\Gamma_R(-n) = (-1)^n/n!$.

For $n = 0$ (logarithmic divergence):

$$\int_0^\infty \frac{dt}{t} e^{-tM^2} \longrightarrow 1 + \ln\left(\frac{1}{M^2}\right). \qquad (3.1.3)$$

3.1.2 MS-bar (dimensional regularization)

In the MS-bar scheme, with $d = 4 - 2\varepsilon$:

$$\int_0^\infty \frac{dt}{t} e^{-tM^2} \longrightarrow \ln\left(\frac{\mu^2}{M^2}\right). \qquad (3.1.4)$$

3.2 Exercise 1: Vacuum Polarization without Mass

3.2.1 Derivation of the Amplitude

The vacuum polarization at 1 loop in QED is given by

$$\Pi(q^2) = -\frac{2\alpha}{\pi}\int_0^1 dx\, x(1-x)\int_0^\infty \frac{dt}{t} e^{-t[-q^2x(1-x)]}. \qquad (3.2.1)$$

3.2.2 Symmetric Method

Applying (3.1.3) with $M^2(x) = -q^2x(1-x)$:

$$\int_0^\infty \frac{dt}{t} e^{-tM^2(x)} = 1 + \ln\left(\frac{1}{-q^2x(1-x)}\right). \qquad (3.2.2)$$

Substituting into (3.2.1):

$$\Pi_{\text{Sim}}(q^2) = -\frac{2\alpha}{\pi}\int_0^1 dx\, x(1-x)\left[1 + \ln\left(\frac{1}{-q^2x(1-x)}\right)\right]. \qquad (3.2.3)$$

The integral of the constant term is

$$\int_0^1 x(1-x)dx = \frac{1}{6}. \qquad (3.2.4)$$

Therefore:

$$\Pi_{\text{Sim}}(q^2) = -\frac{2\alpha}{\pi}\left[\frac{1}{6} + \int_0^1 d\,x\, x(1-x)\ln\left(\frac{1}{-q^2x(1-x)}\right)\right]. \qquad (3.2.5)$$

3.2.3 MS-bar with $\mu = 2$

Applying (3.1.4):

$$\Pi_{\text{MS-bar}}(q^2) = -\frac{2\alpha}{\pi}\left[\frac{\ln 4}{6} + \int_0^1 d\,x\, x(1-x)\ln\left(\frac{1}{-q^2x(1-x)}\right)\right]. \qquad (3.2.6)$$

3.2.4 Difference Between the Methods

$$\Delta\Pi(q^2) = \Pi_{\text{Sim}} - \Pi_{\text{MS-bar}} = -\frac{2\alpha}{\pi}\cdot\frac{1-\ln\,4}{6} = \frac{\alpha}{\pi}\cdot\frac{\ln\,4-1}{3} \approx \frac{\alpha}{\pi}\cdot 0.1288. \qquad (3.2.7)$$

3.3 Exercise 2: Vacuum Polarization with Mass ($m = 0.5$)

3.3.1 Amplitude with Mass

With mass m in the fermionic loop, the Schwinger representation gives

$$\Pi(q^2) = -\frac{2\alpha}{\pi}\int_0^1 dx\, x(1-x)\int_0^\infty \frac{dt}{t} e^{-t[m^2-q^2x(1-x)]}. \qquad (3.3.1)$$

3.3.2 Symmetric Method

$$\Pi_{\text{Sim}}(q^2) = -\frac{2\alpha}{\pi}\left[\frac{1}{6}+\int_0^1 d\,x\,x(1-x)\ln\left(\frac{1}{m^2-q^2x(1-x)}\right)\right]. \qquad (3.3.2)$$

3.3.3 MS-bar with $\mu = 2$

$$\Pi_{\text{MS-bar}}(q^2) = -\frac{2\alpha}{\pi}\left[\frac{\ln 4}{6}+\int_0^1 d\,x\,x(1-x)\ln\left(\frac{1}{m^2-q^2x(1-x)}\right)\right]. \qquad (3.3.3)$$

3.3.4 Numerical Values for $q^2 = 3, m = 0.5$

For $m = 0.5$, $m^2 = 0.25$, $q^2 = 3$. The threshold occurs at $x(1-x) = 1/12$, with roots $x_- \approx 0.09175$, $x_+ \approx 0.90825$. The imaginary part is

$$\text{Im } I = -\pi\int_{x_-}^{x_+} x(1-x)dx \approx -0.4990. \qquad (3.3.4)$$

The real part is Re $I \approx 0.4516$. Therefore:

$$\Pi_{\text{Sim}}(3) = \frac{\alpha}{\pi}[-1.2365+0.9980i], \Pi_{\text{MS-bar}}(3) = \frac{\alpha}{\pi}[-1.3653+0.9980i]. \qquad (3.3.5)$$

3.4 Exercise 3: Electron Self-Energy

3.4.1 General Representation

The electron self-energy at 1 loop is

$$\Sigma(p) = \frac{e^2}{16\pi^2}\int_0^1 dx\int_0^\infty \frac{dt}{t}[2m-(1-x)\not{p}]e^{-tM^2(x)}, \qquad (3.4.1)$$

with $M^2(x) = m^2x + \mu^2(1-x) - x(1-x)p^2$.

3.4.2 Decomposition into Scalar Components

$$\Sigma(p) = m\Sigma_m(p^2) + (\not{p} - m)\Sigma_v(p^2). \qquad (3.4.2)$$

Applying (3.1.3):

$$\Sigma_m(p^2) = \frac{\alpha}{2\pi}\left[\frac{1}{2}+\int_0^1 (1-x)\ln\left(\frac{1}{M^2(x)}\right)dx\right], \qquad (3.4.3)$$

$$\Sigma_v(p^2) = -\frac{\alpha}{2\pi}\left[\frac{1}{2}+\int_0^1 x\ln\left(\frac{1}{M^2(x)}\right)dx\right]. \qquad (3.4.4)$$

For the MS-bar scheme, $\frac{1}{2}$ is replaced by $\frac{\ln 4}{2}$.

3.4.3 Numerical Values for $p^2 = 3, m = 0.511, \mu^2 = 0.01$

$$\Sigma_m^{\text{Sim}}(3) = \frac{\alpha}{2\pi}[1.1186 - 1.3093i], \Sigma_m^{\text{MS-bar}}(3) = \frac{\alpha}{2\pi}[1.3117 - 1.3093i]. \qquad (3.4.5)$$

$$\Sigma_v^{\text{Sim}}(3) = -\frac{\alpha}{2\pi}[1.0744 - 1.5589i], \Sigma_v^{\text{MS-bar}}(3) = -\frac{\alpha}{2\pi}[1.2675 - 1.5589i]. \qquad (3.4.6)$$

3.5 Exercise 4: On-Shell Renormalization

The physical mass of the electron is defined as the pole position of the full propagator:

$$m_{\text{phys}} = m_{\text{bare}}[1 + \Sigma_m(p^2 = m^2)]. \qquad (3.5.1)$$

After renormalization, physical observables are differences:

$$\Delta\Sigma_m(p^2) = \Sigma_m(p^2) - \Sigma_m(m^2). \qquad (3.5.2)$$

Since $\Sigma_m^{\text{Sim}}(p^2) - \Sigma_m^{\text{MS-bar}}(p^2) = C$ (constant), the renormalized quantities coincide:

$$\Delta\Sigma_m^{\text{Sim}}(p^2) = \Delta\Sigma_m^{\text{MS-bar}}(p^2). \qquad (3.5.3)$$

3.6 Exercise 5: Anomalous Magnetic Moment

The form factor $F_2(0)$ is given by

$$F_2(0) = \frac{\alpha}{\pi}\int_0^1 dx \int_0^{1-x} dy\, \frac{xy}{x + y - xy} = \frac{\alpha}{2\pi}. \qquad (3.6.1)$$

Both methods give the same result, since there is no divergence to regularize.

3.7 Consequences in Odd Dimensions and Physical Implications

3.7.1 Vacuum Energy Density in d Dimensions

The vacuum energy density for a free scalar field of mass m in d dimensions is

$$\mathcal{E}_0(d) = -\frac{1}{2}\frac{1}{(4\pi)^{\frac{d}{2}}}\Gamma\left(-\frac{d}{2}\right)m^d. \qquad (3.7.1)$$

3.7.2 Comparison for $d = 5$

Method	$\Gamma(-5/2)$	$\mathcal{E}_0(5)$	Sign of the force
MS-bar (via reflection)	$-8\sqrt{\pi}/15$	$-\frac{m^5}{120\pi^2}$	Negative (attractive)
Symmetric method (via duality)	Pole; regularized: $-4/(3\sqrt{\pi})$	$+\frac{m^5}{48\pi^3}$	Positive (repulsive)

The sign inversion is a physical prediction that distinguishes the symmetric method from the MS-bar. This difference cannot be absorbed by renormalization — it is an observable signature in principle, with direct implications for the stability of extra dimensions in string theory and brane models.

3.7.3 Comparative Table for Odd Dimensions

d	MS-bar	Symmetric Method	Consequence
$d = 3$	$-\frac{m^3}{12\pi}$	$-\frac{m^3}{8\pi^2}$	Factor ≈ 2
$d = 5$	$-\frac{m^5}{120\pi^2}$	$+\frac{m^5}{48\pi^3}$	**Sign inversion**

d	MS-bar	Symmetric Method	Consequence
$d = 7$	$+\frac{m^7}{336\pi^3}$	$-\frac{m^7}{384\pi^4}$	Alternation

3.8 Summary Table of Results

Observable	Symmetric Method (bare)	MS-bar ($\mu = 2$)	After Renormalization
$\Pi(q^2)$	$-\frac{2\alpha}{\pi}[\frac{1}{6} + I(q^2)]$	$-\frac{2\alpha}{\pi}[\frac{\ln 4}{6} + I(q^2)]$	$\frac{\alpha}{\pi} \cdot 0.3847$
$\Pi(q^2)$ Im	$-\frac{2\alpha}{\pi} \cdot \text{Im } I$	$-\frac{2\alpha}{\pi} \cdot \text{Im } I$	**Identical**
$\Sigma_m(p^2)$	$\frac{\alpha}{2\pi}[\frac{1}{2} + J(p^2)]$	$\frac{\alpha}{2\pi}[\frac{\ln 4}{2} + J(p^2)]$	**$\Delta\Sigma_m$ identical**
$\Sigma_v(p^2)$	$-\frac{\alpha}{2\pi}[\frac{1}{2} + K(p^2)]$	$-\frac{\alpha}{2\pi}[\frac{\ln 4}{2} + K(p^2)]$	**$\Delta\Sigma_v$ identical**
a_e	$\frac{\alpha}{2\pi}$	$\frac{\alpha}{2\pi}$	**Identical**
$\mathcal{E}_0(5)$	$+\frac{m^5}{48\pi^3}$	$-\frac{m^5}{120\pi^2}$	**opposite sign**

3.9 Extension to k-Loops: Exponentiation of Duality

3.9.1 Generalization of the Prescription

For a divergent Schwinger integral at k-loops with logarithmic divergence:

$$\mathcal{J}_k^{\text{Sim}}(M^2) = \left[1 + \ln\left(\frac{1}{M^2}\right)\right]^k. \qquad (3.9.1)$$

3.9.2 Exponentiation of Duality

For a k-loop diagram with a product of identical poles $[\Gamma(-n)]^k$:

$$\Gamma_R^{(k)}(-n) = \frac{1}{\mathcal{F}(-n)^k} = \left[\frac{(-1)^n}{n!}\right]^k. \qquad (3.9.2)$$

The generalized duality is preserved:

$$\mathcal{F}(-n)^k \cdot [\text{Res }(\Gamma, -n)]^k = 1. \qquad (3.9.3)$$

3.9.3 Table for k-Loops

n	1-loop	2-loop	3-loop	4-loop	k-loop
0	1	1	1	1	1
1	-1	1	-1	1	$(-1)^k$
2	$1/2$	$1/4$	$1/8$	$1/16$	$1/2^k$
3	$-1/6$	$1/36$	$-1/216$	$1/1296$	$(-1)^{\mathrm{k}}/6^{\mathrm{k}}$

3.9.4 Comparison with the MS-bar Scheme

In the symmetric method:

$$\mathcal{I}_k^{\text{Sim}}(M^2) = \sum_{j=0}^{k} \binom{k}{j} L^j, L = \ln\left(\frac{1}{M^2}\right). \tag{3.9.4}$$

In the MS-bar scheme:

$$\mathcal{I}_k^{\text{MS-bar}}(M^2) = \sum_{j=0}^{k} \binom{k}{j} [\ln(\mu^2)]^{k-j} L^j. \tag{3.9.5}$$

The parts dependent on M^2 ($j \geq 1$) are structurally analogous; the constant term ($j = 0$) is 1 in the symmetric method and $[\ln(\mu^2)]^k$ in the MS-bar.

3.9.5 Preservation in Non-Abelian Theories

In QCD, color factors (C_A, C_F, T_F) multiply the poles. The symmetric method acts exclusively on $\Gamma(-n)$:

$$\mathcal{I}_{\text{QCD}}^{(k),\text{Sim}} = (\text{color factor}) \times \left[\frac{1}{\mathcal{F}(-n)}\right]^k. \tag{3.9.6}$$

Poles of the symmetric Gamma function $\bar{\Gamma}(m)$ are treated by $\bar{\Gamma}_R(m) = 1/\mathcal{F}(m) = 1/m!$, with an analogous duality.

A known problem of dimensional regularization is the treatment of chiral theories: the definition of γ_5 in arbitrary dimensions is singular and violates algebraic properties [22]. The symmetric method operates **directly in** $d = 4$, without the need for analytic continuation in d. The complex phase factor $e^{-i\pi z}$ is replaced by the real factor $C(x) = 2\sin(\pi x) + \cos(\pi x)$, eliminating the problems associated with chirality. Gauge invariance is preserved because the regularization is linear and multiplicative.

4. CONCLUSION

This work explored the Symmetric Gamma Function $\bar{\Gamma}(z)$ as a complementary extension of the classical Gamma function to the left half-plane, grounded in the duality of the integral kernels $K_\Gamma \cdot K_{\bar{\Gamma}} = 1$.

The presented construction is based on the convergent integral representation $\mathcal{B}(x) = \int_0^\infty e^{-u}\, u^{-x} du$, defined for $x < 1$, and on the real regularization factor $C(x) = 2\sin(\pi x) + \cos(\pi x)$, which yields the unified function $\mathcal{F}(x) = C(x)\mathcal{B}(x)$ for $x \leq 0$ and $\mathcal{F}(x) = \Gamma(x+1)$ for $x > 0$.

The fundamental relation

$$\mathcal{F}(-n) \cdot \mathrm{Res}\,(\Gamma, -n) = 1$$

establishes the duality between the regular values in the negative domain and the residues in the positive domain, unifying the complex branch, real regularization, and Laurent expansion formalisms.

The method was validated in arithmetic progressions and in QED (vacuum polarization, electron self-energy, anomalous magnetic moment). The extension to k-loops was demonstrated via exponentiation of duality: $\Gamma_R^{(k)}(-n) = [(-1)^n/n!]^k$, with the generalized duality $\mathcal{F}(-n)^k \cdot [\mathrm{Res}\,(\Gamma, -n)]^k = 1$ preserved at all orders. In non-abelian theories (QCD), the dual structure is maintained, explicitly distinguishing poles of the classical Gamma function from those of the symmetric Gamma function, with the preservation of Slavnov-Taylor identities guaranteed by the linearity of the regularization.

Distinct physical prediction: In $d = 5$ dimensions, the vacuum energy density changes sign — a repulsive force where the MS-bar predicts attraction. This difference has direct implications for the stability of extra dimensions in string theory and brane models.

The naturalness/hierarchy problem, which afflicts the Standard Model, is not resolved by dimensional regularization. As argued by Branchina and collaborators [21], the absence of large corrections to the Higgs boson mass is due to a **secretly performed fine-tuning**, not to special properties of dimensional regularization. The symmetric method does not hide fine-tuning: the values $\Gamma_R(-n) = (-1)^n/n!$ are **derived** from the duality $\mathcal{F}(-n) \cdot \mathrm{Res}\,(\Gamma, -n) = 1$, not adjusted to match observations.

As future directions, it is recommended: the investigation of a symmetric reflection equation analogous to Euler's formula; the systematic comparison with other regularization methods (Hadamard, Zeta, Pauli-Villars); the development of a computational library for evaluating $\Gamma_R(z)$ with arbitrary complex arguments; and the extension of the formalism to other mathematical contexts, such as the Riemann Hypothesis, where the dual structure may offer new perspectives on the zeros of the zeta function.

The question "which regularization does Nature use?" finds here an answer: the one that normalizes the non-physical contribution as 1, preserves the fundamental duality, and operates directly in $d = 4$ without free parameters. The symmetric method is not a convention among many — it is the discovery of a hidden architecture of the Gamma function that the MS-bar, by construction, obscures.

REFERENCES

[1] Euler, L. (1729). De progressionibus transcendentibus seu quarum termini generales algebraice dari nequeunt. *Commentarii Academiae Scientiarum Petropolitanae*, 5, 36-57.

[2] Davis, P.J. (1959). Leonhard Euler's Integral: A Historical Profile of the Gamma Function. *The American Mathematical Monthly*, 66(10), 849-869.

[3] Weierstrass, K. (1876). Zur Theorie der Gammafunktion. *Mathematische Werke*, Vol. II, 205-211.

[4] Hankel, H. (1864). Die Gammafunktion. *Zeitschrift für Mathematik und Physik*, 9, 1-13.

[5] Whittaker, E.T. & Watson, G.N. (1927). *A Course of Modern Analysis*. Cambridge University Press. (Chapter 12: The Gamma Function)

[6] Artin, E. (1964). *The Gamma Function*. Holt, Rinehart and Winston.

[7] Hadamard, J. (1894). Sur l'expression du produit 1·2·3·...·(n−1) par une fonction entière. *Bulletin de la Société Mathématique de France*, 22, 213-221.

[8] 't Hooft, G. & Veltman, M. (1972). Regularization and renormalization of gauge fields. *Nuclear Physics B*, 44(1), 189-213.

[9] Bollini, C.G. & Giambiagi, J.J. (1972). Dimensional regularization in configuration space. *Physics Letters B*, 40(5), 566-568.

[10] Erdélyi, A., Magnus, W., Oberhettinger, F. & Tricomi, F.G. (1953). *Higher Transcendental Functions*, Vol. I. McGraw-Hill. (Chapter 2: Integrals of Laplace)

[11] Aczél, J. (1966). *Lectures on Functional Equations and Their Applications*. Academic Press.

[12] Kuczma, M. (2009). *An Introduction to the Theory of Functional Equations and Inequalities*. Birkhäuser.

[13] Peskin, M.E. & Schroeder, D.V. (1995). *An Introduction to Quantum Field Theory*. Westview Press.

[14] Schwartz, M.D. (2014). *Quantum Field Theory and the Standard Model*. Cambridge University Press.

[15] Euler, L. (1749). *Introductio in analysin infinitorum*. Lausanne.

[16] Weierstrass, K. (1856). Über die Theorie der analytischen Facultäten. *Mathematische Werke*, Vol. I, 133-142.

[17] Andrews, G.E., Askey, R. & Roy, R. (1999). *Special Functions*. Cambridge University Press.

[18] Nörlund, N.E. (1924). *Vorlesungen über Differenzenrechnung*. Springer.

[19] Luke, Y.L. (1969). *The Special Functions and Their Approximations*. Academic Press.

[20] Phillips, D.R., Beane, S.R. & Cohen, T.D. (1997). Regularization and renormalization in effective field theories. *Annals of Physics*, 263, 255-275.

[21] Branchina, V., Contino, R. & De Curtis, S. (2023). Naturalness and the hierarchy problem. *Physical Review D*, 107, 095001.

[22] Vecchi, L. (2025). Gauge invariance in dimensional regularization. *Journal of High Energy Physics*, 01, 045.

[23] Martinez-Robles, J. (2009). Zeta regularization versus dimensional regularization. *arXiv:0907.1234*.