Solid angles subtended by Archimedean solids at their vertices

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In this paper, a comprehensive table of the standard values of solid angles subtended at the vertices of all thirteen Archimedean solids [1] is presented. The solid angles are analytically evaluated using the standard formula for solid angle in conjunction with the tetrahedral formulation, employing known face or apex angles of α, β and γ associated with each vertex configuration [2]. The derivations are carried out assuming the observer's eye is located at a vertex of the Archimedean solid and directed toward its geometric centre. The solid angle subtended at a vertex is evaluated using algebraic sum of solid angles subtended by elementary right triangles [3,4]. It depends solely on the local geometry of the solid, specifically the number of faces meeting at the vertex and the angular relationships between those faces. While a qualitative visual comparison of different Archimedean solids from a vertex perspective is straightforward, precise evaluation and quantitative comparison of the absolute solid angles and their relative differences require an analytical treatment. Accordingly, exact values of the solid angles for all thirteen Archimedean solids are derived and presented in tabular form for clarity and ease of reference. The tabulated results provide a valuable resource for the geometric analysis, visualization, and comparative study of Archimedean solids and may be useful in applications involving spatial perception, polyhedral geometry, and related mathematical modeling problems.

Table of solid angles subtended by all 13 Archimedean solids at their vertices

S/N	**Archimedean solid**	**Solid angle subtended by Archimedean solid at each of its identical vertices (in Ste-radian (sr))**
1	Truncated tetrahedron ($\alpha = 60^o, \beta = 120^o, \gamma = 120^o$)	$2\sin^{-1}\left(\frac{5(\sqrt{11}-\sqrt{2})}{18\sqrt{3}}\right) + 4\sin^{-1}\left(\frac{\sqrt{11}-\sqrt{2}}{6}\right) = 2\cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \approx 1.910633236\ sr$
2	Truncated hexahedron (cube) ($\alpha = 60^o, \beta = 135^o, \gamma = 135^o$)	$2\sin^{-1}\left(\frac{\sqrt{7+4\sqrt{2}}-1}{4\sqrt{3}}\right) + 4\sin^{-1}\left(\frac{\sqrt{6+\sqrt{2}}-\sqrt{2-\sqrt{2}}}{4}\right) \approx 2.801755744\ sr$
3	Truncated octahedron ($\alpha = 90^o, \beta = 120^o, \gamma = 120^o$)	$2\sin^{-1}\left(\frac{2\sqrt{10}-2}{9}\right) + 4\sin^{-1}\left(\frac{\sqrt{10}-1}{3\sqrt{2}}\right) = \pi \approx 3.141592654\ sr$
4	Truncated dodecahedron ($\alpha = 60^o, \beta = 144^o, \gamma = 144^o$)	$2\sin^{-1}\left(\frac{\sqrt{173-9\sqrt{5}}-\sqrt{58-24\sqrt{5}}}{10\sqrt{6}}\right) + 4\sin^{-1}\left(\frac{\sqrt{20(9+2\sqrt{5})}-5+\sqrt{5}}{20}\right) \approx 3.87132031\ sr$
5	Truncated icosahedron ($\alpha = 108^o, \beta = 120^o, \gamma = 120^o$)	$2\sin^{-1}\left(\frac{2\sqrt{605+184\sqrt{5}}-\sqrt{170-2\sqrt{5}}}{36\sqrt{5}}\right) + 4\sin^{-1}\left(\frac{\sqrt{58+18\sqrt{5}}-2}{12}\right) \approx 4.248741371\ sr$
6	Cuboctahedron	$4\sin^{-1}\left(\frac{1}{\sqrt{3}}\right) \approx 2.461918835\ sr$

7	Icosidodecahedron	$$4\sin^{-1}\left(\sqrt{\frac{5+2\sqrt{5}}{15}}\right) \approx 3.673752748\ sr$$
8	Small rhombicuboctahedron	$$2\sin^{-1}\left(\frac{\sqrt{29-2\sqrt{2}}-\sqrt{9-4\sqrt{2}}}{4\sqrt{6}}\right)+6\sin^{-1}\left(\frac{\sqrt{5+2\sqrt{2}}-1}{4}\right) \approx 3.481429563\ sr$$
9	Small rhombicosidodecahedron	$$2\sin^{-1}\left(\frac{3\left(\sqrt{35+9\sqrt{5}}-\sqrt{5-\sqrt{5}}\right)}{20\sqrt{2}}\right)+2\sin^{-1}\left(\frac{\sqrt{355-20\sqrt{5}}-\sqrt{105-40\sqrt{5}}}{20\sqrt{3}}\right)+4\sin^{-1}\left(\frac{\sqrt{55+20\sqrt{5}}-\sqrt{5}}{10\sqrt{2}}\right) \approx 4.446308933\ sr$$
10	Snub cube	$$2\sin^{-1}\left(\frac{\left(1-\sqrt{1-K^2}\right)\sqrt{2K^2-1}}{K^2\sqrt{2}}\right)+8\sin^{-1}\left(\frac{\left(1-\sqrt{1-K^2}\right)\sqrt{4K^2-1}}{2K^2\sqrt{3}}\right) \approx 3.589629551\ sr$$ $where, K \approx 0.928191378$
11	Snub dodecahedron	$$2\sin^{-1}\left(\frac{\left(\sqrt{5}+1\right)}{4K}\right)-2\sin^{-1}\left(\sqrt{\left(\frac{5+2\sqrt{5}}{5}\right)}\left(\frac{\sqrt{1-K^2}}{K}\right)\right)+8\sin^{-1}\left(\frac{\left(1-\sqrt{1-K^2}\right)\sqrt{4K^2-1}}{2K^2\sqrt{3}}\right) \approx 4.509685356\ sr$$ $where, K \approx 0.97273285$
12	Great rhombicuboctahedron ($\alpha = 90^o, \beta = 120^o, \gamma = 135^o$)	$$2\sin^{-1}\left(\frac{\sqrt{69+2\sqrt{2}}-\sqrt{9-4\sqrt{2}}}{12}\right)+2\sin^{-1}\left(\frac{\sqrt{26+12\sqrt{2}}-\sqrt{2}}{8}\right)+2\sin^{-1}\left(\frac{\sqrt{118+47\sqrt{2}}-\sqrt{10-\sqrt{2}}}{12\sqrt{2}}\right)=\frac{5\pi}{4} \approx 3.926990817\ sr$$
13	Great rhombicosidodecahedron ($\alpha = 90^o, \beta = 120^o, \gamma = 144^o$)	$$2\sin^{-1}\left(\frac{\sqrt{171+4\sqrt{5}}-\sqrt{21-8\sqrt{5}}}{6\sqrt{10}}\right)+2\sin^{-1}\left(\frac{\sqrt{155+60\sqrt{5}}-\sqrt{5}}{20}\right)+2\sin^{-1}\left(\frac{\sqrt{165+25\sqrt{5}}-\sqrt{15-5\sqrt{5}}}{12\sqrt{2}}\right)=\frac{3\pi}{2} \approx 4.71238898\ sr$$

It is worth noticing that the solid angle subtended by Great Rhombicosidodecahedron i.e. the largest Archimedean solid at its vertex is greatest $\approx 4.71238898\ sr$.

Note: *Above articles had been derived & illustrated by Mr H.C. Rajpoot (B Tech, Mechanical Engineering)*

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