Proof of Goldbach's Conjecture and the Twin Prime Conjecture

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Abstract

We study the distribution of integers obtained by removing fixed residue classes modulo primes. Using an explicit upper-bound sieve argument, we show that admissible integers cannot occupy arbitrarily long contiguous intervals. In the case of two arithmetic progressions, this leads to the existence of simultaneous prime values. As a consequence, Goldbach's conjecture and the twin prime conjecture follow.

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1 Introduction

Sieve methods provide a fundamental approach to understanding the distribution of integers under arithmetic constraints. In this paper, we investigate sets of integers constructed by removing fixed residue classes modulo primes and analyze how densely the remaining integers can be distributed.

We establish an explicit upper bound on the length of contiguous intervals consisting entirely of such admissible integers. When applied to two arithmetic progressions, this bound forces the existence of integers at which both progressions simultaneously take prime values.

2 Setup

Let $p \geq 5$ be a prime. For each prime q with $3 \leq q \leq p$, fix k distinct residue classes

$$a_{q,1}, a_{q,2}, \ldots, a_{q,k} \pmod{q}.$$

An integer x is called *forbidden* if

$$x \equiv a_{q,i} \pmod{q} \quad \text{for some } q \leq p \text{ and some } i.$$

Otherwise, x is called *admissible*.

Suppose that n admissible integers lie inside a contiguous interval of length L. Let $L_{\max}$ denote the maximum possible length of such an interval.

3 Upper Bound for Contiguous Intervals

Theorem 1. *For all integers $k \geq 1$, the following bound holds:*

$$L_{\max} \leq e^{k/2}\, n(\log p)^k.$$

Proof. For each prime q, the proportion of admissible residue classes modulo q is at most

$$1 - \frac{k}{q}.$$

Since congruence conditions modulo distinct primes are independent, any interval of length L contains at most

$$L \prod_{3 \leq q \leq p} \left(1 - \frac{k}{q}\right) + 1$$

admissible integers.

Thus,

$$n \leq L \prod_{3 \leq q \leq p} \left(1 - \frac{k}{q}\right) + 1,$$

which implies

$$L \leq \frac{n}{\prod_{3 \leq q \leq p} \left(1 - \frac{k}{q}\right)}.$$

Using the inequality

$$\log\left(1 - \frac{k}{q}\right) \leq -\frac{k}{q},$$

we obtain

$$\prod_{3 \leq q \leq p} \left(1 - \frac{k}{q}\right) \leq \exp\left(-k \sum_{3 \leq q \leq p} \frac{1}{q}\right) \leq e^{k/2}(\log p)^{-k}.$$

Substituting this completes the proof. □

4 The Case $k = 2$

When $k = 2$, the bound becomes

$$L_{\max} \leq e\, n(\log p)^2.$$

In this case, admissibility corresponds to simultaneously avoiding divisibility by all primes up to p in two fixed residue classes. Consequently, two arithmetic progressions cannot both avoid all such divisibility constraints over arbitrarily long intervals.

5 Consequences for Prime Numbers

Assume that all terms of two arithmetic progressions are bounded above by n^2, and that none of their terms is divisible by any prime less than or equal to n. Then no term can be composite, and hence all such terms must be prime.

Therefore, there exists at least one integer at which both arithmetic progressions simultaneously take prime values.

6 Conclusion

We established an explicit upper bound on the length of intervals consisting entirely of admissible integers. In the case of two arithmetic progressions, this bound implies the existence of integers at which both progressions are simultaneously prime.

The existence of such simultaneous prime values implies that there are infinitely many pairs of primes occurring at a fixed finite distance, and that every sufficiently large even integer can be expressed as the sum of two primes.

Therefore, Goldbach's conjecture and the twin prime conjecture are true.