The Röntgen Interaction and the 'Resolution of the Abraham–Minkowski Dilemma'

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The Röntgen interaction between a point electric dipole and the electromagnetic field does not carry over to a continuous medium that is polarized. Consequently, the kinetic momentum density of light in a medium is not, as Barnett contends in his 'Resolution of the Abraham–Minkowski Dilemma', the Abraham form (although the canonical momentum density is the Minkowski form). As a result, the resolution is flawed, albeit not fatally.

I. INTRODUCTION

In 2010 *Physical Review Letters* published a letter from Stephen Barnett [1] that claims to resolve the Abraham–Minkowski dilemma. His claim rests on the inherent assumption that the Röntgen interaction on a point electric dipole also applies to a continuous material medium that is polarized. That is not the case.

II. THE RÖNTGEN MOMENTUM

The Röntgen momentum arises in the derivation of the force on a point electric dipole from a Lagrangian. [2, s. 2]

Consider a particle of mass M that has an electric-dipole moment $\mathbf{d}'$ and a magnetic-dipole moment $\mathbf{m}'$ (in its own rest frame) and is moving at a non-relativistic velocity $\mathbf{v}$ (in the lab frame). A Lagrangian for the particle is

$$L = \tfrac{1}{2}Mv^2 + \left(\mathbf{d}' + \frac{\mathbf{v}\times\mathbf{m}'}{c^2}\right)\cdot\mathbf{E} + (\mathbf{m}' - \mathbf{v}\times\mathbf{d}')\cdot\mathbf{B} \tag{1}$$

and the corresponding Euler–Lagrange equation is

$$\begin{aligned}&\frac{d}{dt}\left(M\mathbf{v} + \frac{\mathbf{m}'\times\mathbf{E}}{c^2} - \mathbf{d}'\times\mathbf{B}\right)\\ &= \nabla\left[\left(\mathbf{d}' + \frac{\mathbf{v}\times\mathbf{m}'}{c^2}\right)\cdot\mathbf{E} + (\mathbf{m}' - \mathbf{v}\times\mathbf{d}')\cdot\mathbf{B}\right]\end{aligned} \tag{2}$$

The expression in parentheses on the top line of this equation is the canonical momentum [3], so [1, eqn 6]

$$\mathbf{p}_{can} = \mathbf{p}_{kin} + \frac{\mathbf{m}'\times\mathbf{E}}{c^2} - \mathbf{d}'\times\mathbf{B} \tag{3}$$

The second term is the hidden momentum associated with the magnetic moment of the particle; the third term is the Röntgen momentum associated with its electric moment.

Therefore, the (kinetic) force on a stationary particle that has both electric and magnetic moments is

$$\nabla(\mathbf{d}'\cdot\mathbf{E} + \mathbf{m}'\cdot\mathbf{B}) - \frac{d}{dt}\left(\frac{\mathbf{m}'\times\mathbf{E}}{c^2} - \mathbf{d}'\times\mathbf{B}\right) \tag{4}$$

which agrees with Griffiths and Hnizdo's impeccable calculation of the overt force on a point dipole [4, eqn 78], so it is undoubtedly correct.

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III. CONTINUOUS MATTER

So far, so good. However, Barnett makes the natural assumption that for a continuous medium Equation 3 translates directly to [5, eqn 4.4]

$$\mathbf{p}_{can} = \mathbf{p}_{kin} + \int \mathrm{d}V\left(\frac{\mathbf{M}\times\mathbf{E}}{c^2} - \mathbf{P}\times\mathbf{B}\right) \tag{5}$$

(Griffiths and Hnizdo effectively do the same. [4, eqn 80]) That would imply that [1, eqn 7]

$$\mathbf{p}_{can} + \int \mathrm{d}V\,\mathbf{D}\times\mathbf{B} = \mathbf{p}_{kin} + \int \mathrm{d}V\,\frac{\mathbf{E}\times\mathbf{H}}{c^2} \tag{6}$$

The integral on the left side of this equation is the Minkowski momentum and that on the right side is the Abraham momentum. On this basis Barnett identifies 'the Abraham and Minkowski momenta, respectively, with the kinetic and canonical momenta of the light.'

However, an adaptation of Jackson's textbook derivation of the electromagnetic stress-energy tensor [6, s. 12.10] shows that this assumption is wrong.

A. Canonical Stress-Energy Tensor

An electromagnetic Langragian density for a continuous medium that is polarized and/or magnetized is

$$\begin{aligned}\mathcal{L} &= \frac{\epsilon_0 E^2}{2} - \frac{B^2}{2\mu_0} + \mathbf{P}\cdot\mathbf{E} + \mathbf{M}\cdot\mathbf{B}\\ &= -\frac{\mu_0}{4}g^{\lambda\mu}g^{\nu\sigma}(D_{\mu\sigma}D_{\lambda\nu} - P_{\mu\sigma}P_{\lambda\nu})\end{aligned} \tag{7}$$

where

$$D_{\mu\lambda} = \frac{1}{\mu_0}F_{\mu\lambda} - P_{\mu\lambda} \tag{8}$$

and $P_{\mu\lambda}$ is the polarization-magnetization tensor [7, eqn 21].

Assume that $\partial^\mu M_{\mu\lambda} = 0$, where $M_{\mu\lambda}$ is the dual of $P_{\mu\lambda}$. Then by the Poincaré lemma there exist four-potentials Φ_λ and Ψ_λ such that

$$\mu_0 D_{\mu\lambda} = \partial_\mu\Phi_\lambda - \partial_\lambda\Phi_\mu \tag{9}$$

and

$$\mu_0 P_{\mu\lambda} = \partial_\mu\Psi_\lambda - \partial_\lambda\Psi_\mu \tag{10}$$

Equation 8 implies that (ignoring differences in gauge)

$$\Phi_\lambda + \Psi_\lambda = A_\lambda \tag{11}$$

The canonical electromagnetic stress-energy tensor [8] in a continuous medium follows from treating Φ_λ and Ψ_λ as independent variables (i.e. Ψ_λ constitutes an additional degree of freedom):

$$\begin{aligned} &\frac{\partial \mathcal{L}}{\partial(\partial_\alpha \Phi^\lambda)}\partial^\beta \Phi^\lambda + \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \Psi^\lambda)}\partial^\beta \Psi^\lambda - g^{\alpha\beta}\mathcal{L} \\ =& - g^{\alpha\mu} D_{\mu\lambda}\partial^\beta \Phi^\lambda + g^{\alpha\mu} P_{\mu\lambda}\partial^\beta \Psi^\lambda - g^{\alpha\beta}\mathcal{L} \end{aligned} \tag{12}$$

B. Symmetric Stress-Energy Tensor

The first term on the bottom line of Equation 12 is equal to

$$\mu_0 g^{\alpha\mu} D_{\mu\lambda} D^{\lambda\beta} + \partial_\lambda (D^{\lambda\alpha}\Phi^\beta) - \Phi^\beta \partial_\lambda D^{\lambda\alpha} \tag{13}$$

The four-divergence of the second term of this expression vanishes, and the Euler–Lagrange equation

$$\partial_\lambda \frac{\partial \mathcal{L}}{\partial(\partial_\lambda \Phi_\alpha)} = \frac{\partial \mathcal{L}}{\partial \Phi_\alpha} \tag{14}$$

implies that the last term equals zero (as $\mathcal{L}$ does not depend on Φ_α). The same reasoning applies to the second term on the bottom line of Equation 12. Consequently, the symmetric electromagnetic stress-energy tensor [9] in a continuous medium is

$$\begin{aligned} V^{\alpha\beta} &= \mu_0 g^{\alpha\mu} D_{\mu\lambda} D^{\lambda\beta} - \mu_0 g^{\alpha\mu} P_{\mu\lambda} P^{\lambda\beta} - g^{\alpha\beta}\mathcal{L} \\ &= \Theta^{\alpha\beta} - g^{\alpha\mu} P_{\mu\lambda} F^{\lambda\beta} - g^{\alpha\mu} M_{\mu\lambda} G^{\lambda\beta} \end{aligned} \tag{15}$$

where $\Theta^{\alpha\beta}$ is the (symmetric) electromagnetic stress-energy tensor in free space and $G^{\lambda\beta}$ is the dual of $F^{\lambda\beta}$.

The corresponding force density on the medium is $-\partial_\alpha V^{\alpha\beta}$, so Newton's second law becomes [10]

$$\begin{aligned} \partial_\alpha T^{\alpha\beta} &= -\partial_\alpha (\Theta^{\alpha\beta} - g^{\alpha\mu} P_{\mu\lambda} F^{\lambda\beta} - g^{\alpha\mu} M_{\mu\lambda} G^{\lambda\beta}) \\ &= -\mu_0 J_\lambda D^{\lambda\beta} + \mu_0 (\partial^\mu P_{\mu\lambda}) P^{\lambda\beta} + (\partial^\mu M_{\mu\lambda}) G^{\lambda\beta} \end{aligned} \tag{16}$$

Clearly, the stress-energy tensor is not conserved in the presence of the currents J_λ, $\partial^\mu P_{\mu\lambda}$, or $\partial^\mu M_{\mu\lambda}$ [11], which is why they must be set to zero for the above procedure to work.

C. Comparison with Point Dipoles

In the absence of free charges and currents the four-divergence of $\Theta^{\alpha\beta}$ is [6, s. 12.10.C]

$$\begin{aligned} \partial_\alpha \Theta^{\alpha\beta} &= \frac{1}{\mu_0}(\partial^\mu F_{\mu\lambda}) F^{\lambda\beta} \\ &= (\partial^\mu D_{\mu\lambda} + \partial^\mu P_{\mu\lambda}) F^{\lambda\beta} = (\partial^\mu P_{\mu\lambda}) F^{\lambda\beta} \end{aligned} \tag{17}$$

Therefore,

$$\begin{aligned} \partial_\alpha (T^{\alpha\beta} + g^{\alpha\mu} M_{\lambda\mu} G^{\lambda\beta}) &= -\partial_\alpha (\Theta^{\alpha\beta} + g^{\alpha\mu} P_{\lambda\mu} F^{\lambda\beta}) \\ &= -P_{\lambda\mu}\partial^\mu F^{\lambda\beta} \end{aligned} \tag{18}$$

The expression in parentheses on the right side of the top line of this equation is a stress-energy tensor due to Medina and Stephany [12, eqn 14]. The spatial component of the term on the bottom line is

$$\begin{aligned} &(\mathbf{P}\cdot\nabla)\mathbf{E} + \mathbf{P}\times(\nabla\times\mathbf{E}) \\ &+ (\mathbf{M}\cdot\nabla)\mathbf{B} + \mathbf{M}\times(\nabla\times\mathbf{B}) \end{aligned} \tag{19}$$

which mirrors the canonical force on a point dipole,

$$\frac{d\mathbf{p}_{can}}{dt} = \nabla(\mathbf{d}\cdot\mathbf{E} + \mathbf{m}\cdot\mathbf{B}) \tag{20}$$

Thus, Equation 18 is the analogue in a continuous medium of Equation 20, and the equivalent of the canonical momentum is the sum of the volume integrals of the momentum densities from the two stress-energy tensors on its left side:

$$\begin{aligned} \mathbf{p}_{can} &= \frac{1}{c}\int \mathrm{d}V (T^{0j} + g^{0\mu} M_{\lambda\mu} G^{\lambda j}) \\ &= \mathbf{p}_{kin} + \int \mathrm{d}V\, \frac{\mathbf{M}\times\mathbf{E}}{c^2} \end{aligned} \tag{21}$$

Comparing this equation with Equation 5 shows that the Röntgen momentum density is missing, a casualty of the Lorentz covariance of Equation 18. However, it's not absent entirely, as it appears on the other side of Equation 18:

$$\begin{aligned} \mathbf{p}_{can} &= -\frac{1}{c}\int \mathrm{d}V (\Theta^{0j} + g^{0\mu} P_{\lambda\mu} F^{\lambda j}) \\ &= -\int \mathrm{d}V (\epsilon_0 \mathbf{E}\times\mathbf{B} + \mathbf{P}\times\mathbf{B}) \end{aligned} \tag{22}$$

The expression in parentheses on the bottom line is the Minkowski momentum density. If the Röntgen momentum density were present in Equation 21 (as it is in Equation 5), it would cancel out the second term.

Substituting Equation 22 into Equation 21 yields

$$\mathbf{p}_{kin} = -\int \mathrm{d}V \left(\mathbf{D}\times\mathbf{B} + \frac{\mathbf{M}\times\mathbf{E}}{c^2}\right) \tag{23}$$

The expression in parentheses is the sum of the Minkowski and the hidden momentum densities, which differs from the Abraham momentum density, $\mathbf{E}\times\mathbf{H}/c^2$.

IV. CONCLUSION

Despite the flaw in Barnett's argument, his overall conclusion 'that there are two distinct electromagnetic momenta, the kinetic momentum and the canonical momentum' is sound. However, his identification of the Abraham momentum as the kinetic momentum of the light is not.

[1] S. M. Barnett, Resolution of the Abraham–Minkowski dilemma, Phys. Rev. Lett. **104**, 070401 (2010).

[2] M. Sonnleitner and S. M. Barnett, The Röntgen interaction and forces on dipoles in time-modulated optical fields, Eur. Phys. J. D **71**, 336 (2017), arXiv:1704.01835 [physics.atom-ph].

[3] The canonical momentum depends on the Lagrangian, which is not unique. The Lagrangian given by Equation 1, which is gauge invariant, yields Barnett's canonical momentum.

[4] D. J. Griffiths and V. Hnizdo, What's the use of bound charge?, arXiv:1506.02590 [physics.class-ph] (2015).

[5] S. M. Barnett and R. Loudon, The enigma of optical momentum in a medium, Phil. Trans. R. Soc. A **368**, 927 (2010).

[6] J. D. Jackson, *Classical Electrodynamics*, 3rd ed. (John Wiley & Sons, Inc., Hoboken, NJ, 1999).

[7] D. J. Griffiths and V. Hnizdo, Mansuripur's paradox, Am. J. Phys. **81**, 570 (2013), arXiv:1303.0732 [physics.class-ph].

[8] This note uses the same metric sign convention as Jackson, $(+---)$.

[9] That the symmetric stress-energy tensor is equivalent to the Hilbert stress-energy tensor,

$$2\frac{\partial \mathcal{L}}{\partial g^{\alpha\beta}} - g_{\alpha\beta}\mathcal{L} \tag{24}$$

is easily demonstrated.

[10] Moving from the first to the second line of Equation 16 requires the relation

$$-P_{\lambda\mu}\partial^{\mu}F^{\lambda\beta} - M_{\lambda\mu}\partial^{\mu}G^{\lambda\beta} = \mu_0(J_{\lambda} + \partial^{\mu}P_{\mu\lambda})P^{\lambda\beta} \tag{25}$$

which can be shown by calculating the temporal and spatial components and applying Maxwell's equations as appropriate.

[11] $\partial^{\mu}M_{\mu\lambda}$ behaves like a magnetic-current density.

[12] R. Medina and J. Stephany, The force density and the kinetic energy-momentum tensor of electromagnetic fields in matter, arXiv:1404.5250 [physics.class-ph] (2014).