ON THE PROJECTION ANALOGUE OF THE SLICING PROBLEM

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Abstract. We give a proof for a sharp projection analogue of the slicing problem. Moreover, we show a geometric proof of the slicing problem.

Contents

1. Introduction

The isotropic conjecture or Bourgain's slicing problem asks for the existence of a universal constant c such that

Theorem 1.1. *There exists an affine hyperplane H and a universal constant c such that*

$$m_{n-1}(E \cap K) > c,$$

for convex bodies K of unit volume.

A classic reference for these questions is [13]. The isotropic constant conjecture has already been proven by Klartag and Lehec [11]. We prove that

Theorem 1.2 (Angular Thickness Condition)**.** *Let $K \subset \mathbb{R}^n$ be a bounded measurable set of measure $|K| = 1$ that is star-shaped with respect to the origin. Then there exists a measurable radial function $R(\theta) \in [0, \infty]$ on S^{n-1} such that*

$$K = \{\, r\theta : \theta \in S^{n-1},\ 0 \le r \le R(\theta) \,\}.$$

Let $R > 0$ be such that $|B(0, R)| = c$. If the set of directions where K reaches at least radius R satisfies

$$\omega\big(\{\theta \in S^{n-1} :\ R(\theta) \ge R\}\big) \ \ge\ C\,\omega(S^{n-1}), \tag{1.1}$$

then

$$|K \cap B(0, R)| \ \ge\ Cc.$$

Moreover, constant $\frac{1}{2}$ with set up $c = 1$ is sharp: if the left-hand side of (1.1) is strictly less than $\frac{1}{2}\,\omega(S^{n-1})$, the conclusion may fail.

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The entries of the covariance matrix of a convex body K are defined as

$$(a_{ij}) = \frac{\int_K x_i x_j}{|K|} - \frac{\int_K x_i}{|K|}\frac{\int_K x_j}{|K|}.$$

We define the isotropic constant of any convex body K in a scaling-invariant way using

$$L_K^{2n} := \frac{\mathrm{Det}(CovK)}{|K|^2}.$$

The isotropic position is a position when the covariance matrix is diagonal and all the diagonal entries are the same. This kind of position exists [13].

Theorem 1.3. *The isotropic constant is universally bounded for symmetric convex bodies.*

The above is equivalent to the slicing problem [13]. So is the following formulation.

Theorem 1.4. *For any symmetric convex body K of unit volume there exists a position $T(K)$ such that*

$$c \leq |B(0,R) \cap T(K)|,$$

where

$$|B(0,R)|^{1/n} \leq C,$$

and C and c are universal constants.

Our theorem for the projection analog is the following:

Theorem 1.5. *For every convex body $K \subset \mathbb{R}^n$ there exists a universal constant $c > 0$ such that*

$$\mathbb{E}_{\varphi \in S^{n-1}}\big[\, m_{n-1}(\mathrm{Proj}_\varphi K)\,\big] \;\geq\; c\,|K|^{\frac{n-1}{n}},$$

where the expectation is taken with respect to surface measure on S^{n-1}. Moreover, the inequality is sharp, with equality for Euclidean balls.

Now, if the convex body is central symmetric, then the maximal slice is always attained in the subspace, say, H_ϕ. If we project K to that subspace in general the projection is bigger since $H_\phi \cap K \subset Proj_\phi(K)$. If K is symmetric with respect to H_ϕ, then the projection and central slice are the same with respect to H_ϕ. Moreover, a fundamental relationship between projections and central slices uses the polar body

$$K^\circ = \{y \in \mathbb{R}^n : \langle x, y\rangle \leq 1 \text{ for all } x \in K\}.$$

The projections and slices are related by polarity as follows:

$$(Proj_\phi(K))^\circ = K^\circ \cap H_\phi, \qquad (K \cap H_\phi)^\circ = Proj_\phi(K^\circ).$$

2. An elementary bound for the random projections

We use Cauchy's surface area formula:

$$|\partial K| = \frac{1}{\omega_{n-1}} \int_{S^{n-1}} m_{n-1}(\mathrm{Proj}_\varphi(K))\, d\omega(\varphi),$$

where $d\omega$ is the standard surface measure on S^{n-1} and $\omega_{n-1} = |B^{n-1}|$.

Combining the Cauchy surface area formula with the isoperimetric inequality

$$|\partial K| \;\geq\; n\,\omega_n^{1/n}\,|K|^{(n-1)/n},$$

we obtain

$$|K|^{n/(n-1)} \;\le\; \omega_n^{-1/n}\, n^{-1} \left(\frac{1}{\omega_{n-1}} \int_{S^{n-1}} m_{n-1}(\mathrm{Proj}_{\varphi}(K))\, d\omega(\varphi) \right).$$

Writing the integral in terms of expectation with respect to the normalized probability measure

$$d\sigma(\varphi) = \frac{1}{|S^{n-1}|}\, d\omega(\varphi), \qquad |S^{n-1}| = n\omega_n,$$

we obtain

$$\int_{S^{n-1}} m_{n-1}(\mathrm{Proj}_{\varphi}(K))\, d\omega(\varphi) = n\omega_n\, \mathbb{E}\big[m_{n-1}(\mathrm{Proj}_{\varphi}(K))\big].$$

Substituting this into the previous inequality yields

$$|K|^{n/(n-1)} \;\le\; \omega_n^{-1/n}\, n^{-1} \cdot \frac{n\omega_n}{\omega_{n-1}}\, \mathbb{E}\big[m_{n-1}(\mathrm{Proj}_{\varphi}(K))\big].$$

Hence,

$$\mathbb{E}\big[m_{n-1}(\mathrm{Proj}_{\varphi}(K))\big] \;\ge\; \omega_{n-1}\, \omega_n^{(1-n)/n}\, |K|^{(n-1)/n}.$$

Since equality in the isoperimetric inequality holds only for Euclidean balls, these are the extremizers for the above inequality. This ends the proof of Theorem 1.5.

3. On angular thick star-convex sets

In this section we proof the theorem 1.2.

Proof. Since K is star-shaped, polar coordinates yield

$$|K \cap B(0,R)| = \int_{S^{n-1}} \int_0^{\min(R, R(\theta))} r^{n-1}\, dr\, d\omega(\theta) = \frac{1}{n} \int_{S^{n-1}} \min\big(R^n,\; R(\theta)^n\big)\, d\omega(\theta).$$

The volume of the ball is

$$|B(0,R)| = \frac{\omega(S^{n-1})}{n}\, R^n = c \quad \text{(by assumption)}.$$

Define the density

$$f(R) \;:=\; \frac{|K \cap B(0,R)|}{|B(0,R)|} = \frac{1}{\omega(S^{n-1})} \int_{S^{n-1}} \min\Big(1, \Big(\frac{R(\theta)}{R}\Big)^n\Big)\, d\omega(\theta).$$

Let

$$A := \{\theta \in S^{n-1} :\; R(\theta) \ge R\}.$$

On A we have $(R(\theta)/R)^n \ge 1$, so the integrand equals 1. Therefore

$$f(R) \;\ge\; \frac{1}{\omega(S^{n-1})} \int_A 1\, d\omega(\theta) = \frac{\omega(A)}{\omega(S^{n-1})} \;\ge\; C \quad \text{by assumption (1.1)}.$$

Thus

$$|K \cap B(0,R)| = f(R) \cdot |B(0,R)| \ge Cc,$$

which proves the main statement.

Sharpness. Let $0 < \varepsilon < \frac{1}{2}$. Choose a measurable set $A \subset S^{n-1}$ with

$$\omega(A) = \Big(\tfrac{1}{2} - \varepsilon\Big)\omega(S^{n-1}).$$

Define a star-shaped set

$$K = \{\, r\theta : \theta \in A,\; 0 \le r \le L \,\},$$

where $L > 0$ is chosen so that $|K| = 1$. Then

$$|K| = \int_A \int_0^L r^{n-1} dr\, d\omega(\theta) = \frac{\omega(A)}{n} L^n = 1 \quad \Longrightarrow \quad L^n = \frac{n}{\omega(A)}.$$

Using $|B(0,R)| = 1$ gives $\frac{\omega(S^{n-1})}{n} R^n = 1$, so $R^n = \frac{n}{\omega(S^{n-1})}$. Therefore

$$\Big(\frac{L}{R}\Big)^n = \frac{n/\omega(A)}{n/\omega(S^{n-1})} = \frac{\omega(S^{n-1})}{\omega(A)} = \frac{1}{\frac{1}{2} - \varepsilon} > 1,$$

and hence $L \geq R$, so $R(\theta) \geq R$ for all $\theta \in A$ and $R(\theta) = 0$ otherwise. Thus

$$f(R) = \frac{1}{\omega(S^{n-1})} \int_{S^{n-1}} \min\Big(1, \Big(\frac{R(\theta)}{R}\Big)^n\Big) d\omega(\theta) = \frac{\omega(A)}{\omega(S^{n-1})} = \tfrac{1}{2} - \varepsilon < \tfrac{1}{2}.$$

Consequently

$$|K \cap B(0,R)| = f(R) \cdot |B(0,R)| = \tfrac{1}{2} - \varepsilon,$$

which violates the desired inequality. Since ε is arbitrary, the constant $\frac{1}{2}$ in (1.1) is a lower bound for $c = 1$. □

4. A geometric proof of the slicing problem

We start with a known theorem [8]

Theorem 4.1. *Let K be convex body. Denote*

$$L_{K,\theta} := \left(\int_K \langle \theta, x \rangle^2 \, dx \right)^{1/2}.$$

Then there exists universal constants c_1 and c_2 such that

$$\frac{c_1}{L_{K,\theta}} \leq |H_\theta \cap K| \leq \frac{c_2}{L_{K,\theta}},$$

for all $\theta \in S^{n-1}$.

We use the following symmetrization method.

Definition 4.2 (Cylinder Symmetrization)**.** Let Ω be a bounded, measurable set in $\mathbb{R}^n$. The cylinder symmetrization of Ω, with respect to direction θ, denoted by Ω^*_θ, is the closed spherical cylinder with the midpoint of the base at the origin. Moreover

$$\mathcal{L}^n(\Omega^*) = \mathcal{L}^n(\Omega), \tag{4.1}$$

and

$$\mathcal{L}^{n-1}(H_\theta \cap K^*_\theta) = \mathcal{L}^{n-1}(H_\theta \cap K) \tag{4.2}$$

where $\mathcal{L}^n$ denotes the n-dimensional Lebesgue measure (volume).

One of our main results is the following.

Theorem 4.3 (A proof on the slicing problem)**.** *Let K be an isotropic symmetric convex body. Then for any direction $\theta \in S^{n-1}$*

$$|H_\theta \cap K| \geq c,$$

where c is a universal constant and H_θ is the hyperplane through the origin perpendicular to θ.

Proof. It follows from the theorem 4.1 that in isotropic position all the central slices are essentially of the same size. So w.l.o.g we consider only the maximal measure central slice. We proceed by steps.

Step 1: Cylinder Symmetrization.

Assume that the maximal slice size is attained in a slice orthogonal to θ-axis. Apply cylinder symmetrization to K with respect to the θ-axis. This produces a body K_θ^* with the following properties:

- $|K_\theta^*| = |K| = 1$ (volume preservation),
- K_θ^* is a body of revolution (trivial),
- $|H_\theta \cap K_\theta^*| = |H_\theta \cap K|$. (central slice preservation).

Let C be an n-dimensional spherical cylinder defined as $C = B_{n-1}(R) \times [0, H]$. We assume the total volume of the cylinder is unity:

$$\mathrm{Vol}_n(C) = 1 \tag{4.3}$$

Step 2: Isotropic position.

Transform K_θ^* to its isotropic position $T(K_\theta^*)$. Let $\theta \in S^{n-1}$ be the fixed direction. Choose an orthonormal basis for $\mathbb{R}^n$ with $e_n \parallel \theta$. In this coordinate system, the isotropic transformation T can be represented as a diagonal matrix:

$$T = \mathrm{diag}(a_1, \dots, a_n), \quad \prod_{i=1}^{n} a_i = 1,$$

since the isotropic position is rotation invariant. On hyperplane $\{x_n = 0\}$, points $(x_1, \dots, x_{n-1}, 0)$ map to $(a_1x_1, \dots, a_{n-1}x_{n-1}, 0)$. The $(n-1)$-dimensional volume scales by

$$\prod_{i=1}^{n-1} a_i.$$

Now, K_θ^* is a spherical cylinder, so it's isotropic position is a spherical cylinder. So

$$a_i = a$$

for all $1 \le i \le n-1$ and

$$a_n = a^{-n+1}.$$

Step 3. Bounding the parameter a from above

Now,

$$\begin{aligned} nL_{K_\theta^*}^2 &\le \int_{K_\theta^*} \langle \theta, x\rangle^2\,dx + \sum_{i=2}^{n} \int_K \langle \theta_i, x\rangle^2\,dx \\ &\le \int_{K_\theta^*} \langle \theta, x\rangle^2\,dx + (n-1)\max_i \int_{K_\theta^*} \langle \theta_i, x\rangle^2\,dx \\ &\le c_1 |H_{\theta_i} \cap K_\theta^*|^{-2} + (n-1)\max_i \int_{K_\theta^*} \langle \theta_i, x\rangle^2\,dx \\ &\le CL_K^2 + (n-1)\max_i \int_{K_\theta^*} \langle \theta_i, x\rangle^2\,dx, \end{aligned}$$

where we used GM-AM inequality in the first inequality, and the theorem 4.1 in the third and the last inequality. So we have

$$L_{K_\theta^*}^2 \le \frac{C}{n} L_K^2 + \frac{(n-1)}{n}\max_i \int_{K_\theta^*} \langle \theta_i, x\rangle^2\,dx \le \frac{C\ln N}{\sqrt{n}} + \max_i \int_{K_\theta^*} \langle \theta_i, x\rangle^2\,dx,$$

where we used a bound from Bourgain $L_K \leq (\ln n) n^{1/4}$. Thus,

$$L^2_{K^*_\theta} \leq C' \max_i \int_{K^*_\theta} \langle \theta_i, x\rangle^2 \, dx, \tag{4.4}$$

for some universal constant C'. Moreover, since TK^*_θ is isotropic and $L_{K^*_\theta}$ is affine invariant:

$$\begin{aligned} nL^2_{K^*_\theta} &= \int_{TK^*_\theta} \langle \theta, x\rangle^2 \, dx + \sum_{i=2}^n \int_{TK^*_\theta} \langle \theta_i, x\rangle^2 \, dx \\ &= L^2_{K^*_\theta} + a^2(n-1) \max_i \int_{K^*_\theta} \langle \theta_i, x\rangle^2 \, dx \end{aligned}$$

Thus, from above and from (4.4)

$$a^2 \max_i \int_{K^*_\theta} \langle \theta_i, x\rangle^2 \, dx = L^2_{K^*_\theta} \leq C \max_i \int_{K^*_\theta} \langle \theta_i, x\rangle^2 \, dx.$$

The above implies

$$a \leq C, \tag{4.5}$$

for some universal constant C.

Step 4: Bounding the slice size

We split by cases.

Case 1: $a_n > 1$.

Then $a < 1$, and T shrinks the central slice. By theorem 4.1, there exist universal constants c_1 and c_2 such that

$$c_1 L^{-1}_{T(K)^*_\theta} \leq |H_\theta \cap T(K)^*_\theta| \leq c_2 L^{-1}_{T(K)^*_\theta}. \tag{4.6}$$

Now, $L_{T(K)^*_\theta}$ is well known to be universally bounded. Moreover, we have

$$c \leq |H_\theta \cap T(K)^*_\theta| = |T(H_\theta \cap K^*_\theta)| = a^{n-1} |H_\theta \cap K^*_\theta| \leq |H_\theta \cap K^*_\theta|,$$

So the claim follows from above in this case.

Case 2: $a_n \leq 1$.

Let $\theta_i \in S^{n-1}$ be any unit vector orthogonal to θ. Denote

$$L_{K^*_\theta} := \left(\int_{K^*_\theta} \langle \theta, x\rangle^2 \, dx \right)^{1/2}.$$

Now,

$$L^n_{T(K^*_\theta)} = (L_{K^*_{\theta_i}})^{(n-1)} L_{K^*_\theta}.$$

Then

$$\begin{aligned} L_{T(K)^*_\theta} &= \left(\int_{K^*_\theta} \langle \theta_i, Tx\rangle^2 \, dx \right)^{1/2} \\ &= a \left(\int_{K^*_\theta} \langle \theta_i, x\rangle^2 \, dx \right)^{1/2} \\ &\leq a c_2 |H_{\theta_i} \cap K^*_\theta|^{-1} \\ &\leq C |H_{\theta_i} \cap K^*_\theta|^{-1}, \end{aligned}$$

where we used the bound for a (4.5) from the previous step. Thus, we have an universal bound

$$|H_{\theta_i} \cap K_\theta^*| \leq C'. \tag{4.7}$$

We aim to find a relationship between the volume of the base slice, S_{base}, and the volume of the axial slice, S_{axial}.

The geometric definitions for the relevant volumes are:

$$\text{Total Volume: } V_n = S_{\text{base}} \cdot H \tag{4.8}$$

$$\text{Base Slice } ((n-1)\text{-ball})\text{: } S_{\text{base}} = B_{n-1}(0, R^{n-1}) \tag{4.9}$$

$$\text{Axial Slice } ((n-2)\text{-ball } \times \text{ Height})\text{: } S_{\text{axial}} = B_{n-2}(0, R^{n-2}) \cdot H \tag{4.10}$$

From the unit volume constraint (4.8):

$$1 = S_{\text{base}} \cdot H \implies H = S_{\text{base}}^{-1} \tag{4.11}$$

From the base slice formula (4.9), we solve for the radius R:

$$R^{n-1} = S_{\text{base}} \cdot |B_{n-1}(0,1)|^{-1}$$
$$R = (S_{\text{base}} \cdot |B_{n-1}(0,1)|)^{-1/(n-1)}$$

We substitute H and R into equation (4.10):

$$S_{\text{axial}} = (S_{\text{base}} \cdot |B_{n-1}(0,1)|)^{-(n-2)/(n-1)} \, |B_{n-2}(0,1)| \cdot S_{\text{base}}^{-1}.$$

Thus,

$$S_{\text{axial}} = (|B_{n-1}(0,1)|)^{-1+1/(n-1)} \, |B_{n-2}(0,1)| \cdot S_{\text{base}}^{-2+1/(n-1)} \geq c S_{\text{base}}^{-2+1/(n-1)},$$

where c is an universal constant. Now, we have a bound for the S_{axial}. Thus, using the above and the bound (4.7) we have that

$$c \leq S_{\text{base}}^{2-1/(n-1)},$$

which gives us the claim after a little algebra in this case.

This ends the proof.

□

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