Comment on ‘Lorentz-invariance and gauge-invariance of the Aharonov–Bohm phase’ by A. L. Kholmetskii, O. V. Missevitch, and T. Yarman, Eur. Phys. J. Plus (2025) 140:140

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Abstract

I present an axiomatic foundation of non-integrable phases of quantum wave functions like the Aharonov–Bohm phase and show the gauge invariance of the phase difference in the Aharonov–Bohm setup in a much simpler manner than in that article by Kholmetskii *et al.*

Keywords: Aharonov-Bohm phase, Dirac phase, gauge invariance, Lorentz invariance, non-integrable phase

1 Introduction

In the recent article mentioned in the title [1], the Lorentz invariance as well as the gauge invariance of the Aharonov–Bohm phase in the strong relativistic limit have been shown using the principle of superposition of quantum phases. Here, I will add an axiomatic foundation of the case of non-integrable phases of quantum wave functions which are linear in the vector potential and analogous quantities. Following Dirac [2], Aharonov & Bohm's Lorentz-invariant formula for the phase shift ([3] p. 486 I) will be shown to be gauge invariant, too, and that in a most simple manner.

2 Axiomatic foundation of non-integrable phases of quantum wave functions

2.1 Relationships between interactions and conserved quantities according to and beyond Helmholtz

Helmholtz's explorations on the relationships between mechanical forces and conservation of energy [4][5] can be generalized as follows [6][7].

- For a *point-like body*, its momentum $\vec{p}(t)$ is a stationary-state function in the sense that it is time-*in*dependent in force-free states, in which $\vec{p}(t) = \vec{p}(0) = \text{const}$. Are there interactions (external forces) which leave the momentum unchanged? The answer is well known to be 'no'.

- Next, consider a *mechanical system* in a *stationary state* with total energy E. Are there interactions (external forces) which leave the amount of E unchanged? The answer is 'yes', given by forces of the form
$$-\nabla V(\vec{r}) + \vec{v} \times \vec{K}(t, \vec{r}, \vec{v}, \vec{a}, \ldots)\,. \tag{1}$$
Here, $\vec{K}(t, \vec{r}, \vec{v}, \vec{a}, \ldots)$ is a rather arbitrary vector function of time t, position $\vec{r}$, velocity $\vec{v} := \dot{\vec{r}}$, acceleration $\vec{a} := \ddot{\vec{r}}$, and higher-order time-derivatives of $\vec{r}$. The second term is due to Lipschitz [8]. Surprisingly enough, it is missing in all textbooks I am aware of. Its relevance reveals from this: It is compatible with canonical mechanics, if and only if $\vec{K}$ is restricted to the functional form $\vec{K}(t, \vec{r}, \vec{v}, \vec{a}, \ldots) = \nabla \times \vec{K}'(t, \vec{r})$. Eventually, this leads to the magnetic Lorentz force, where $\vec{K}' = q\vec{A}$ [7].

- For a *quantum-mechanical system* with wave function $\psi(\vec{r}, t)$ and Hamiltonian $H(\hat{\vec{p}}, \vec{r}, t) = H_0(\hat{\vec{p}}, \vec{r}) + H_{\text{int}}(\hat{\vec{p}}, \vec{r}, t)$, the expressions
$$|\psi(\vec{r}, t)|^2 \quad \text{and} \quad \langle\psi(\vec{r}, t)|H(\hat{\vec{p}}, \vec{r}, t)|\psi(\vec{r}, t)\rangle \tag{2}$$

are stationary-state functions in the sense that they are time-independent in stationary states with constant energy E, where

$$|\psi(\vec{r},t)|^2 = |\psi_E(\vec{r})|^2, \tag{3a}$$

$$\langle\psi(\vec{r},t)|H(\hat{\vec{p}},\vec{r},t)|\psi(\vec{r},t)\rangle = \langle\psi_E(\vec{r})|H_0(\hat{\vec{p}},\vec{r})|\psi_E(\vec{r})\rangle = E\,. \tag{3b}$$

Are there interactions which leave the stationary "weight function" [9] $|\psi_E(\vec{r})|^2$ and the energy E unchanged? The answer is 'yes' as will be shown in the next subsection.

2.2 Interactions leaving $|\psi_E(\vec{r},t)|^2$ and E unchanged. Gauge invariance

Obviously, the value of the stationary "weight function" (more accurately, weight *density*) $|\psi_E(\vec{r},t)|^2$ is not changed when $\psi_E(\vec{r},t) = \psi_E(\vec{r})e^{-Et/\hbar} =: \psi_{E,0}(\vec{r},t)$ is replaced by[1]

$$\psi_{E,\beta}(\vec{r},t) := \psi_{E,0}(\vec{r},t)e^{-i\beta(\vec{r},t)},$$
$$\beta(\vec{r},\,t) = \frac{q}{\hbar}\int_{t_0}^{t}\phi(\odot,t')\,\mathrm{d}t' - \frac{q}{\hbar}\int_{\vec{r}_0}^{\vec{r}}\vec{A}(\vec{r}',\odot)\cdot\mathrm{d}\vec{r}'\,. \tag{4}$$

Within electromagnetism, q is the electrical charge, while $\vec{A}(\vec{r},t)$ and $\phi(\vec{r},t)$ are the vector and scalar potentials, respectively. $\vec{A}(\vec{r}',\odot)$ will not be differentiated w.r.t. t and $\phi(\odot,t')$ not w.r.t. $\vec{r}$,

$$\frac{\partial\beta}{\partial t}(\vec{r},\,t) = \frac{q}{\hbar}\phi(\vec{r},t)\,,\quad \nabla\beta(\vec{r},\,t) = -\frac{q}{\hbar}\vec{A}(\vec{r},t)\,. \tag{5}$$

Then, the value of $\langle\psi_E(\vec{r},t)|H_0(\hat{\vec{p}},\,\vec{r})|\psi_E(\vec{r},t)\rangle = E$ is also not changed when at once $H_0(\hat{\vec{p}},\,\vec{r})$ is replaced with

$$H_\beta(\hat{\vec{p}},\,\vec{r}) := H_0(\hat{\vec{p}}+\hbar\,\nabla\beta(\vec{r},t),\,\vec{r}) + \hbar\,\frac{\partial\beta}{\partial t}(\vec{r},t)$$
$$= H_0(\hat{\vec{p}}-q\vec{A}(\vec{r},t),\,\vec{r}) + q\phi(\vec{r},t) \tag{6}$$

for the Schrödinger equation, similarly for the Klein-Gordon and Dirac equations. This is the well-known gauge invariance of Schrödinger wave mechanics.

[1] Dirac's [2] phase β is the negative of the Aharonov-Bohm phase [1][3].

2.3 (Ehrenberg-Siday-)Aharonov-Bohm effect

For any closed path $s_\mu = (ct, -x, -y, -z)$ in space-time, the phase β (4) can be written as (cf. [3] p. 486 I, where the action $S = \hbar\beta$, and [1] (4))

$$\beta = \frac{q}{\hbar} \oint \left(\phi(\vec{r}, t)\, \mathrm{d}t - \vec{A}(\vec{r}, t) \cdot \mathrm{d}\vec{r} \right) = \frac{q}{\hbar} \oint A^\mu \mathrm{d}s_\mu. \tag{7}$$

Here, the limitations in β (4) are lifted and the vector potential $\vec{A}$ is no longer bound to be a gradient field as in (5).[2]

In single-connected regions, $\beta \equiv 0$. In the Aharonov-Bohm setup [3], however, the coil makes the region, where the electrons fly, to be *not* single-connected. Consequently, β does not vanish but is showing up in the (Ehrenberg-Siday-)Aharonov-Bohm effect [3][10][11].

The phase β (7) is manifest Lorentz invariant but not manifest gauge invariant. The latter issue will be addressed in the next section.

2.4 Non-integrable phase and gauge invariance

Before doing so, let us notice the following. The phase β (4) is non-integrable, if

$$\frac{\partial^2 \beta}{\partial x \partial y} - \frac{\partial^2 \beta}{\partial y \partial x} \propto -\frac{\partial A_y}{\partial x} + \frac{\partial A_x}{\partial y} = -B_z \neq 0 \quad \text{etc.} \tag{8}$$

Dirac ([2] p. 66) notes,

> "The connection between non-integrability of phase and the electromagnetic field given in this section is not new, being essentially just Weyl's Principle of Gauge Invariance in its modern form."[3]

That gauge invariance has been dealt with in Subsection 2.2.

[2] In an Aharonov-Bohm setup [3], outside the coil, along the paths of the electrons, the vector potential $\vec{A}$ *is* a gradient field.

[3] Dirac refers to [12]. On p. 331, Weyl writes, "Es scheint mir darum dieses nicht aus der Spekulation, sondern aus der Erfahrung stammende neue Prinzip der Eichinvarianz zwingend darauf hinzuweisen, daß das elektrische Feld ein notwendiges Begleitphänomen nicht des Gravitationsfeldes, sondern des materiellen, durch $\bar{\Psi}$ dargestellten Wellenfeldes ist." En.: Therefore, this new principle of gauge invariance, which does not come from speculation, but from experience, seems to me to indicate compellingly that the electric field is a necessary accompanying phenomenon not of the gravitational field, but of the material wave field represented by $\bar{\Psi}$. Weyl's $\bar{\Psi}$ is not our ψ.

3 Gauge invariance of the phase β (7)

According to Dirac (cf. [2] (4)), the expression (7) for the phase β can be transformed as

$$\beta = \frac{q}{\hbar} \oint_{\partial S} A^\mu \, \mathrm{d}s_\mu = \frac{q}{\hbar} \iint_S (\partial^\mu A^\nu - \partial^\nu A^\mu) \, \mathrm{d}S_{\mu\nu} = \frac{q}{\hbar} \iint_S F^{\mu\nu} \mathrm{d}S_{\mu\nu} \tag{9}$$

($F^{\mu\nu}$ being the Faraday tensor). This way, the gauge invariance is obvious.

In case that the path lies completely in the $3d$ position space, $s_\mu = (0, -\vec{s})$ ([2] p. 67), formula (9) simplifies to

$$\beta = \frac{q}{\hbar} \oint_{\partial S} \vec{A} \cdot \mathrm{d}\vec{s} = \frac{q}{\hbar} \iint_S (\nabla \times \vec{A}) \cdot \mathrm{d}\vec{S} = \frac{q}{\hbar} \iint_S \vec{B} \cdot \mathrm{d}\vec{S} = \frac{q}{\hbar} \Phi \,, \tag{10}$$

Φ being the magnetic flux through the surface S [3].

4 Summary and conclusions

Generalizing Helmholtz's explorations of the relation between forces and energies [4][5], I have presented an axiomatic foundation of the Aharonov-Bohm phase [3] and related it to Dirac's *non*-integrable phase [2]. The phase factors (*not* the phases β themselves) *uniquely* determine the electromagnetic field [13].

Using results of Dirac's 1931 pioneering work on non-integrable phases and magnetic monopoles [2], I have shown that Aharonov & Bohm's formula (7) for the phase shift is not only Lorentz invariant but also gauge invariant, and that in a much simpler manner than in [1].

Admittedly, in this treatment, the Aharonov-Bohm phase ϕ_{AB} is a *semi-classical*, *non-relativistic*, and *linear* (low-field limit) functional of the scalar and vector potentials as given in Aharonov & Bohm's original article [3]. Within Schrödinger wave mechanics, the more general expression

$$\phi_{\mathrm{AB}} = \frac{1}{\hbar} \int_s H_{\mathrm{int}}(\hat{\vec{p}}, \vec{r}, t) \, \mathrm{d}t \tag{11}$$

(after [1] (2), where $\hbar = 1$) is *non*-linear in the vector potential. However, a non-linear and fully quantised description of the (Ehrenberg-Siday-)Aharonov-Bohm effect as well as its description for non-closed paths (for references, see [1]) are far beyond the scope of this comment.

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