The Dimensionality of the Geometric Product Can be Explored with Benefit to High-School Science Students

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Abstract

Professor Philippe Eenens, who has taught GA-based courses to freshman engineering students, asserts that "For GA to become mainstream, we must convince high-school teachers of its advantages for the teaching of basic topics of mathematics and science." One of those basic topics, according to prize-winning educator Edward Redish, is the "dimensionality" of the variables in science equations. Unlike the variables in most equations that students have worked with in math classes, the variables in science equations represent measurements that have fundamental units like time, length, and mass. Acting upon the advice of professors Eenens and Redish, we show here how the dimensionality of GA's geometric product might be explored at the high-school level in a way that would be of lasting benefit to students.

1 Introduction

Critics of GA sometimes object that the geometric product violates the "dimensionality" principle that only likes can be added to likes. Specifically, critics assert that in the geometric product of two vectors $\mathbf{u}$ and $\mathbf{v}$, a scalar $(\mathbf{u} \cdot \mathbf{v})$ is added to an area $(\mathbf{u} \wedge \mathbf{v})$:

$$\mathbf{uv} = \underbrace{\mathbf{u} \cdot \mathbf{v}}_{a\ scalar} + \underbrace{\mathbf{u} \wedge \mathbf{v}}_{an\ area}.$$

This objection stems from a misunderstanding of the operation that is represented by the symbol "+" ([1], [2]). Unfortunately, this misunderstanding will be a natural response from high-school students upon seeing the familiar symbol "+" used to represent an unfamiliar operation that appears to violate a principle as fundamental as "only likes can be added to likes". Therefore, we GA advocates must be prepared to address the dimensionality of the geometric product when we make our case for GA to high-school teachers.

An additional reason for addressing the dimensionality of the geometric product is found in the work of noted educational researcher Edward Redish, who made dimensionality the subject of the first article in his seven-part series on using mathematics in physics ([3]-[11]). As Redish explains, dimensionality is an important tool for making and checking inferences about relations among physical variables. Moreover, dimensionality is a key difference between "math as used in physics" and "math as taught in math classics". This difference is a known source of difficulties for students. For that reason, Redish indicates that understanding the concept of dimensionality is essential to becoming proficient in using mathematics effectively not only in physics, but in sciences generally .

The importance of understanding dimensionality gives GA advocates an opportunity to make a virtue out of a necessity. That is, we can show prospective high-school teachers that the (necessary) exploration of the geometric product's dimensionality can done in a way that will be of lasting benefit to all science students.

To that end, we present here a possible way of exploring the dimensionality of the geometric product of two vectors, in the context of solving a typical problem.

2 Exploring the Dimensionality of the Geometric Product, as Used in Solving a Typical Problem

Consider two geometric vectors, **u** and **v**. (E.g., vectors that represent displacements in space.) The dimensionality of each of these vectors is "L" (length). Now, consider the dimensionality of the product **uv**:

$$\mathbf{uv} = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \wedge \mathbf{v}$$

The geometric interpretation of $\mathbf{u} \wedge \mathbf{v}$ is as an "oriented area". Therefore, the dimensionality of $\mathbf{u} \wedge \mathbf{v}$ is L^2. What about $\mathbf{u} \cdot \mathbf{v}$? Although it is a scalar, it is a "product" of two quantities, each of dimensionality L. Therefore, $\mathbf{u} \cdot \mathbf{v}$ is a scalar of dimensionality L^2. Far from being novel, scalars that have dimensionality are ubiquitous in science. For example, the dimensionality of the gravitational constant, G, is $L^3 M^{-1} T^{-2}$ (length cubed, divided by the product of mass and the square of time).

Similarly, the dimensionality of the geometric product **uv** is L^2, because **uv** is a "product" of two quantities, each of dimensionality L.

Now, let's study the dimensionality of the geometric product in the context of finding the vector $\mathbf{u}$ if we know $\mathbf{u}\cdot\mathbf{v}$, $\mathbf{u}\wedge\mathbf{v}$, and the vector $\mathbf{v}$. Formally, the procedure is as follows:

$$\begin{aligned}\mathbf{uv} &= \mathbf{u}\cdot\mathbf{v}+\mathbf{u}\wedge\mathbf{v}\\ \mathbf{uvv}^{-1} &= [\mathbf{u}\cdot\mathbf{v}+\mathbf{u}\wedge\mathbf{v}]\,\mathbf{v}^{-1}\\ \mathbf{u} &= (\mathbf{u}\cdot\mathbf{v})\,\mathbf{v}^{-1}+(\mathbf{u}\cdot\mathbf{v})\,\mathbf{v}^{-1}\end{aligned} \tag{1}$$

For an example of the somewhat complex forms that the simple factors $\mathbf{u}\cdot\mathbf{v}$ and $\mathbf{u}\wedge\mathbf{v}$ may take in a typical problem, see [1], at timestamp 3:39.

Returning to Eq. (1), the dimensionality of $\mathbf{u}\cdot\mathbf{v}$ and $\mathbf{u}\wedge\mathbf{v}$ is L^2, but what is the dimensionality of $\mathbf{v}^{-1}$? To find out, let's examine the definition of $\mathbf{v}^{-1}$:

$$\mathbf{v}^{-1}=\frac{\mathbf{v}}{v^2}$$

Here, $\mathbf{v}$ (dimensionality L) is divided by v^2 (dimensionality L^2), so the dimensionality of $\mathbf{v}^{-1}$ is L^{-1}—not L, as it would be for a vector. But $\mathbf{v}^{-1}$ does "act like" a vector when multiplied by scalars and bivectors in Eq. 1.

Or does it? To see what's really going on here, we'll rewrite

$$\mathbf{u}=(\mathbf{u}\cdot\mathbf{v})\,\mathbf{v}^{-1}+(\mathbf{u}\cdot\mathbf{v})\,\mathbf{v}^{-1}$$

as

$$\begin{aligned}\mathbf{u} &= [\mathbf{u}\cdot\mathbf{v}]\left[\frac{\mathbf{v}}{v^2}\right]+[\mathbf{u}\wedge\mathbf{v}]\left[\frac{\mathbf{v}}{v^2}\right],\\ &= \left[\mathbf{u}\cdot\left(\frac{\mathbf{v}}{v}\right)\right]\left[\frac{\mathbf{v}}{v}\right]+\left[\mathbf{u}\wedge\left(\frac{\mathbf{v}}{v}\right)\right]\left[\frac{\mathbf{v}}{v}\right]\\ &= [\mathbf{u}\cdot\hat{\mathbf{v}}]\,\hat{\mathbf{v}}+[\mathbf{u}\wedge\hat{\mathbf{v}}]\,\hat{\mathbf{v}}.\end{aligned} \tag{2}$$

The unit vector $\hat{\mathbf{v}}$ is dimensionless $\left(\frac{L}{L}\right)$: it represents only a direction. Thus, the interpretation of Eq. (2) is as shown in Fig. 1.

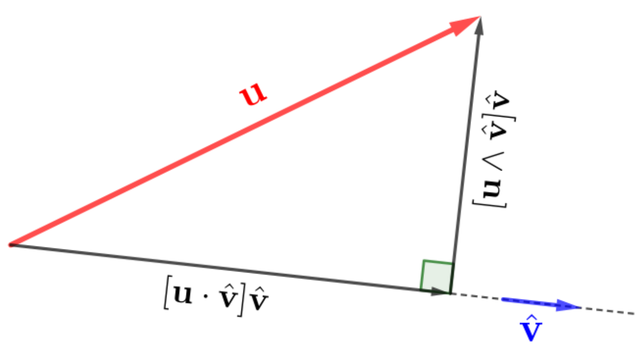

Figure 1: The vector $[\mathbf{u}\cdot\hat{\mathbf{v}}]\,\hat{\mathbf{v}}$ is the component of vector $\mathbf{u}$ upon the direction of the known vector $\mathbf{v}$. In contrast, the vector $[\mathbf{u}\wedge\hat{\mathbf{v}}]\,\hat{\mathbf{v}}$ is the component of $\mathbf{u}$ perpendicular to $\mathbf{v}$. The factor $[\mathbf{u}\wedge\hat{\mathbf{v}}]\,\hat{\mathbf{v}}$ is an operator that rotates $\hat{\mathbf{v}}$ by 90^o in the direction of the rotation from $\hat{\mathbf{v}}$ to $\mathbf{u}$, then multiplies that rotated vector by the scalar factor $\|\mathbf{u}\wedge\hat{\mathbf{v}}\|$.

3 Summary

Because the "+" operation in the geometric product can be readily misunderstood as violating the principle that "only likes can be added to likes", GA proponents should be prepared to explain the dimensionality of the geometric product to teachers. The need for teachers, in turn, to explain this to students can be turned into a virtue, by providing an opportunity for teachers to help students better understand such a fundamental concept as dimensionality.

References

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