A proof of the Collatz Conjecture

Jim Rock

Abstract. Collatz sequences originate from dividing an even number by two until an odd number is obtained, followed by multiplication by three and an increment of one to yield an even number. The Collatz conjecture posits that the repeated application of this process inevitably results in the number one. The Collatz conjecture holds true for every number tested, but no general method has been found to prove that it is true for all positive integers. We introduce a new methodology: the binary series. In conjunction with mathematical induction, this new methodology provides a more general method of testing positive integers for properties that cannot be established by induction alone. We partition the positive integers into distinct subsets. The binary series allows us to use geometric series that sum to one (100%) to show that all natural numbers satisfy the Collatz conjecture. This new methodology eliminates the need to test every integer and provides a general method of proof for the Collatz conjecture.

Contents

1. Introduction

Collatz sequences originate from dividing an even number by two until reaching an odd number, followed by multiplication by three and an increment of one to yield an even number. The Collatz conjecture posits that the repeated application of this process inevitably results in the number one.

An *American Mathematical Monthly* article from 2013 by Conway includes a generalization of the Collatz conjecture in his survey of easily understood arithmetic statements that are true but not provable[1]. In 1985 the *American Mathematical Monthly* published an article by Lagarias that surveyed the progress made on solving the Collatz conjecture and related problems. In that article the author states, "The $3x+1$ Conjecture is simple to state and apparently intractably hard to solve. It shares these properties with other iteration problems."[2] Mathematicians have thoroughly examined the variable length and apparent randomness of Collatz sequences, but have not made much progress in determining the truth of the Collatz conjecture. Moreover, in 2010 Lagarias added that: "It is an extraordinarily difficult problem completely out of reach of present-day mathematics."[3] In 2019 Tao wrote an article indicating the Collatz conjecture is true for almost all natural numbers.[4]. The "almost all"problem pertains to infinite proper subsets of natural numbers that are 100% dense within the set of all natural numbers. The subset of natural numbers excluding powers of two $(2^n, n > 0)$ is an example.

We solve the "almost all"problem by creating the Collatz structure that contains all natural numbers. This provides the necessary framework to prove the Collatz conjecture. The Collatz structure is treelike, containing horizontal branches of even and odd numbers connected by the Collatz algorithm and descending vertical towers composed of the even numbers. Each even number in a tower is half the size of its predecessor, as required by the Collatz algorithm.

Section 2 describes and illustrates the Collatz structure in detail. The Collatz structure (Figure 1) contains every Collatz sequence. In order to create the structure of the branches and towers, we partition the positive integers into subsets, whose elements have the same remainder when divided by 24. Section 2.1 indicates the position of various subset elements in the Collatz structure. Section 3 affirms their singular occurrence. Section 4 further describes branches and branch segments. In Section 5 we solve the "almost all"problem. In Section 6 we use the binary series along with mathematical induction and geometric series that sum to 1 (100%) to show that all natural numbers are within the Collatz structure. We show there are no circular or infinite Collatz sequences in Section 7. Appendix A explains the constraints on consecutive even integers in a branch. Appendix B shows that every possible binary series occurs in the Collatz structure. Appendix C has details regarding the linear formulas that underlie the geometric series proportion calculations, which show that 100% of the odd terms are within branches.

2. Collatz structure branches and towers

The Collatz structure, as illustrated in Figure 1 is composed of horizontal branches and vertical towers. Descending Collatz towers, represented by

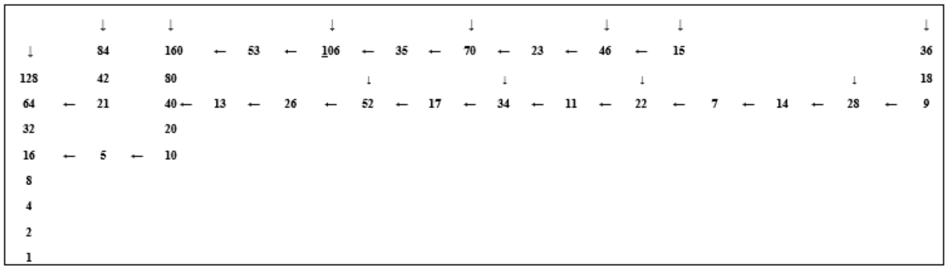

Figure 1. Collatz structure branches and towers, where $\downarrow$ indicates a descending Collatz tower and $\leftarrow$ indicates the application of the Collatz algorithm for transitioning between terms.

vertical arrows $\downarrow$, consist of terms sequentially halved. Horizontal arrows $\leftarrow$ signify the application of the Collatz algorithm for transitioning between terms within a branch. Branches have a first term $6j + 3$. The Collatz algorithm is then applied until we arrive at a last term $24k + 16$. j and k are non-negative integers.

The trunk tower positioned on the extreme left of Figure 1, is characterized by each term being a power of two. Each 4^j $(j \geq 2)$ trunk tower term is the last term in a branch. At every $a = 24m+4, 24m+10$, and $24m+22$, (m is a non-negative integer) base term in the branches of the trunk tower is a $2^j a$ $(j \geq 1)$ secondary tower. Each of the $2^{2j}a$ terms in the secondary towers is the last term in a branch. At every $a = 24m + 4, 24m + 10$, and $24m + 22$ base term in these branches is a $2^j a$ secondary tower. Each $2^{2j}a$ is the last term in a branch. This process repeats indefinitely throughout the Collatz structure. Appendix A shows that every secondary tower base term is of the form $24m + 4$, $24m + 10$, or $24m + 22$ and terms of the form $24k + 16$ are at the end of a branch.

All even terms within a branch $(24m+4 \to 12m+2, 24m+10, 24m+22)$ exhibit even factors that are limited to four or two. Consequently, we are able to define the branch binary series, which counts the number of divisions by two of the branch secondary tower base terms $24m + 4(2), 24m + 10(1)$, and $24m + 22(1)$. In Figure 1, there are two branches. One begins with 9 and ends with 40 with a binary series of $(2, 1, 1, 2)$, and the other begins with 15 and ends with 160 with a binary series of $(1, 1, 1)$.

The Figure 1 secondary tower bases are 10, 46, 70, and 106. The predecessor of each term is twice its size. Starting in a secondary tower from an arbitrary n-th term $24k_n + 16$, the rest of the smaller secondary tower terms alternate between $12k_j + 8$ and $24k_{j-1} + 16$ for $1 \leq j \leq k$.

$$24k_n + 16 \to 12k_n + 8 \to 6k_n + 4 = 24k_{n-1} + 16(k_n = 4k_{n-1} + 2) \ldots$$

division by two and $(k_j = 4k_{j-1} + 2)$ leads to its tower base term

$$24k_2{+}16 \to 12k_2{+}8 \to 6k_2{+}4 = 24k_1{+}16(k_2 = 4k_1{+}2) \to 12k_1{+}8 \to 6k_1{+}4.$$

$$\text{If } k_1 = 4m,\ 6k_1 + 4 = 24m + 4.$$
$$\text{If } k_1 = 4m + 1,\ 6k_1 + 4 = 24m + 10.$$
$$\text{If } k_1 = 4m + 3,\ 6k_1 + 4 = 24m + 22.$$

Figure 1 contains the following example:

$$160 \to 80 \to 40 \to 20 \to 10$$

$$(24)(6){+}16 \to (12)(6){+}8 \to (24)(1){+}16 \to (12)(1){+}8 \to (6)(1){+}4 = (24)(0){+}10.$$

The $6j + 3$ first terms in branches are all divisible by three. This pattern extends to all terms within a "first term tower", which are expressed as $(2^t)(6j{+}3)$, where t is a non-negative integer. Each term preceding the $6j{+}3$ base term in a first term tower has one of the following forms: $24k, 24k + 6, 24k + 12$, or $24k + 18$. Next, we show the precise relationship between these two types of terms.

$$24k = (8)(6j + 3), (j = k - 1)/2),$$
$$24k + 6 = (2)(6j + 3), (j = 2k),$$
$$24k + 12 = (4)(6j + 3), (j = k),$$
$$24k + 18 = (2)(6j + 3), (j = 2k + 1).$$

2.1. Position of all integer types within the Collatz structure.

(1) $24k$ first term tower
(2) $24k + 2$ successor of $24j + 4, j = 2k$
(3) $24k + 4$ secondary tower base middle of a branch
(4) $24k + 6$ first term tower
(5) $24k + 8$ trunk/secondary tower successor of $24j + 16, j = 2k$
(6) $24k + 10$ secondary tower base middle of a branch
(7) $24k + 12$ first term tower
(8) $24k + 14$ successor of $24j + 4, j = 2k + 1$
(9) $24k + 16$ trunk/secondary tower end of a branch
(10) $24k + 18$ first term tower
(11) $24k + 20$ trunk/secondary tower successor of $24j + 16, j = 2k + 1$
(12) $24k + 22$ secondary tower base middle of a branch
(13) $6j + 1$ middle of a branch
(14) $6j + 3$ beginning of a branch
(15) $6j + 5$ middle of a branch

A Collatz sequence can start from any point within the Collatz structure. Dividing even terms by two until an odd term is reached, then multiplying by three and adding one returns an even term. Repeating this process inevitably leads to a term of the form 4^j in the trunk tower. Successive divisions by two lead to the trunk tower base term 1. All natural numbers within the Collatz structure are joined together by the Collatz algorithm. We have described three different types of towers that reside within the Collatz structure. We have also introduced the branch binary series, which will prove to be a vital innovation in the process of proving the Collatz conjecture.

3. Each term appears only once in the Collatz structure

Hypothetically, starting with two duplicate terms in separate places within a branch, we would run the Collatz algorithm in reverse. Then every term in the branch prior to the two duplicates would also be duplicates. However, this is impossible. $6j + 3$ terms can only appear at the beginning of a branch. The uniqueness of all terms in the trunk tower ensures the singularity of terms in the branches connected to the trunk tower. This principle extends to all terms in secondary towers and the branches terminating in these secondary towers. Furthermore, no two distinct branches can contain the same term. If the Collatz algorithm were to be applied in reverse from the same term in different branches, it would lead to the same $6j + 3$ first term for both branches, which is an impossibility. Such a scenario would imply that the branches are identical. Consequently, we assert that no individual term appears more than once in the entire Collatz structure.

We now know that all terms within the Collatz structure are unique. This moves us a step closer to realizing our end goal of showing that all Collatz sequences are finite and reside within the Collatz structure. Next we establish further properties of the branches and branch segments within the Collatz structure.

4. Descriptions of branches and branch segments

Branches exist in a group of four with odd first terms of the form $24h + 3, 24h + 9, 24h + 15$, or $24h + 21$ and a $24k + 16$ last term. All these first terms reduce to $6j + 3$ terms (h, j and k are non-negative integers).

Branch segments arise in the middle of branches. Their first terms are odd, and they appear in two groups of four. The first terms of the first group exhibit the form $24h + 1, 24h + 7, 24h + 13$, or $24h + 19$. They all reduce to 6j+1 terms. The first terms of the second group exhibit the form $24h + 5, 24h + 11, 24h + 17$, or $24h + 23$. They all reduce to $6j + 5$ terms.

Branches and branch segments are characterized by their first term $24h + 2c - 1$ with $1 \leq c \leq 12$, a binary series of 1s and 2s (see examples below), and a last term $24k + 16$. The length r of a binary series is the number of secondary tower base terms in a branch or branch segment. The sum of r 1s and 2s in the binary series is s.

The first terms of branches $(24h+c,\ c = 3, 9, 15, 21)$ and branch segments $(24h + c,\ c = 1, 7, 13, 19$ or $5, 11, 17, 23)$ have linear formulas $h = 2^s n + a$, where $1 \leq a < 2^s$ and s and n are non-negative integers, with s being the sum of the binary series. The linear formulas describe branches and branch segments sharing the same binary series. Reducing or increasing h by 2^s will produce a formula for a branch or branch segment with the same binary series as before. Each individual value of a for $1 \leq a < 2^s$ is a part of a different group of branches that share the same binary series when h is increased by 2^s. Branches end with $24k + 16$, $k = 3^{r+1}n + b, r \geq 0$. b and n are non-negative integers, and r is the number of terms in the binary series.

The binary series $(2, 2, 2, 1, 1)$ has a first term $24h+9$, and a linear formula $h = 2^8 n + 5$. The last term is 24k+16 with a linear formula $k = 3^6 n + 15$, For $n = 0$, we get

(24)(5)+9 = 129, 388(2), 97, 292(2), 73, 220(2),

55, 166(1), 83, 250(1), 125, 376 = (24)(15)+16.

We substitute 1 for n. We get

(24)(2^8+5)+9=6273, 18820(2), 4705, 14116(2), 3529, 10588(2),

2647, 7942(1), 3971, 11914(1), 5957, 17872 = (24)(3^6+15)+16

Next we justify using $h = 2^8 + 5$ and $k = 3^6 + 15$.

(24)(2^8+5)+9=129+(24)(2^8), 388+(3)(24)(2^8), 97+(3)(24)(2^6),

292+(3^2)(24)(2^6), 73+(3^2)(24)(2^4), 220+(3^3)(24)(2^4),

55+(3^3)(24)(2^2), 166+(3^4)(24)(2^2), 83+(3^4)(24)(2),

250+(3^5)(24)(2), 125+(24)(3^5), 376+(24)(3^6) = (24)(3^6+15)+16.

4.1. Summary. We know why branches and branch segments cluster together in groups of four, and we have established many properties of the linear formula regarding how they create multiple branches with the same binary series. Next we will establish how the "almost all" problem is solved by a proportion formula which does not have an underlying linear formula that is subject to the "almost all" problem.

5. Branch binary series are of finite length with $24k$+16 last terms

In Section 6 we utilize geometric series summing to 1 to show that all odd integers are within the Collatz structure. The terms of the geometric series are based on linear formulas $h = 2^s n + a$ (details in Appendix C). They are susceptible to the "almost all" problem. We introduce a proportion formula for branch last terms $24k + 16$, whose geometric series also sums to 1. This formula is "pure" and has no underlying linear formulas. The proportion formula for $24k + 16$ shows that all branches in the Collatz structure are finite. The "almost all"problem is eliminated from the underlying linear formulas of the Section 6 proportion formulas.

Theorem 1. *The proportion of $24k + 16$ terms in branches with a binary series length r is $2^r/3^{r+1}$.*

Proof. Proof by induction.

$$24h + 21 \to 24(3h + 2) + 16$$

where h and k are non-negative integers, is the formula for a branch with binary series of length $r = 0$.

The proportion of $24k + 16$ terms $(24(3h + 2) + 16)$ in a branch with an empty binary series is $1/3$.

$$\text{For } r = 0, 1/3 = 2^0/3^{0+1} = 2^r/3^{r+1}.$$

Assume the proportion of $24k + 16$ terms in branches with a binary series of length $r \geq 0$ is $2^r/3^{r+1}$.

2^r is the number of different branches with a binary series of length r.
$r + 1$ applications of $2j + 1 \to 6j + 4$ exist.

2^{r+1} is the number of branch binary series of length $r + 1$. $r + 2$ applications of $2j + 1 \to 6j + 4$ exist.

There are twice as many binary series of length $r + 1$ as there are of length r, but the $24k+16$ terms in a binary series of length $r+1$ are on average three times larger than the $24k+16$ terms in a binary series of length r.

Thus, the proportion of $24k + 16$ terms in branches with a binary series of length $r + 1$ is $2/3$ of the proportion of branches with a binary series of length r, and the proportion of $24k + 16$ terms in branches with a binary series of length $r + 1$ is

$$(2/3)(2^r/3^{r+1}) = 2^{r+1}/3^{r+2}.$$

□

For the sum of a geometric series, we use the formula $s = a/(1 - t_r)$, where a is the first series term and t_r is the ratio between series terms.

1 = (1/3)/(1 - 2/3) is the total proportion.

1 (100%) of all the branches are of finite length with last terms of the form $24k + 16$. Next we show all odd numbers are in the Collatz structure.

6. All odd numbers are in Collatz structure branches

Starting with a first term $24h+2c-1$ with $1 \leq c \leq 12$ and a binary series, we derive the appropriate linear formula $h = 2^s n + a$ for the binary series by substituting the proper integers for c, s, n, and a. We then run the Collatz algorithm forward until we get a $24k+16$ last branch term. Substituting all non-negative integers for n into the formula $h = 2^s n + a$ gives a proportion $1/2^s$ of the set of all natural numbers. With an induction argument we prove the binary series proportion formulas. For the sum of the proportion formulas geometric series, we use the formula $s = a/(1-t_r)$, where a is the first series term and t_r is the ratio between series terms. The geometric series generated by the proportion formulas always sums up to 1 (100%), showing that all the $24h+2c-1$ terms are in Collatz structure branches.

The following theorems are very repetitive in groups of four. After studying Theorems 2 through 5 about $24h+3$ on a first reading, you may want to skip directly to 6.10.

6.1. $24h+3$ are first terms of branches with a binary series of every combination of 1s and 2s for every value of r.

Theorem 2. *The underlying linear formula $24h+3$ term binary series of length $r = 1(1)$ is $h = 2n$, and that for $r = 2(1,2)$ is $h = 2^3 n + 3$, where n is a non-negative integer.*

Proof. For $h = 2n$
$24h + 3 = 48n + 3 \to 144n + 10(1) \to 72n + 5 \to (24n)(9) + 16.$

For $h = 2^3 n + 3$
$24h + 3 = 192n + 75 \to 576n + 226(1) \to 288n + 113 \to 864n + 340(2) \to 216n + 85 \to (27n + 10)(24) + 16.$ □

Theorem 3. *The proportion of $24h+3$ terms with a binary series length of $r = 1$ is 1/2, and that for $r = 2$ is 1/8.*

Proof. By Theorem 2, the underlying linear formula for a binary series of length $r = 1(1)$ is $h = 2n$, $n \geq 0$.

Substituting all non-negative integers for n into the formula gives 1/2 of the set of all natural numbers.

By Theorem 2, the underlying linear formula for a binary series of length $r = 2(1,2)$ is $h = 2^3 n + 3$, $n \geq 0$.

Substituting all non-negative integers for n into the formula gives 1/8 of the set of all natural numbers. □

Theorem 4. *The proportion of* $24h+3$ *terms with a binary series of length* r *is* $3^{r-2}/2^{2r-1}$.

Proof. Proof by induction.
By Theorem 3, the proportion of 24h+3 terms with a branch binary series of length $r = 2$ is $1/8 = 3^{2-2}/2^{(2)(2)-1}$.

Assume that the proportion of $24h+3$ terms with a branch binary series of length r is $3^{r-2}/2^{2r-1}$.

The $r+1$ position in the branch binary series can be (1) or (2).

The proportion of $24h+3$ terms of binary series length $r+1 = (1/2)(3^{r-2}/2^{2r-1}) + (1/2^2)(3^{r-2}/2^{2r-1}) = 3^{r-1}/2^{2(r+1)-1}$.

See Appendix C for further details regarding the derivation of the formulas for length r and $r+1$. ☐

Theorem 5. *All* $24h+3$ *terms are in branches of the Collatz structure.*

Proof. The ratio between terms in the geometric series formed by the binary series is $(3^{r-2}/2^{2r-1})/(3^{r-1}/2^{2r+1}) = 3/4$.

By Theorem 3, the proportion of $24h+3$ terms with a binary series of length $r = 2$ is $1/8$.

By Theorem 3, the total proportion of the $24h+3$ terms in the Collatz structure is $r = 1(1) = 1/2$. $1/2 + (1/8)/(1-3/4) = 1$.

Hence, all $24h+3$ terms are in the branches of the Collatz structure. ☐

6.2. $24h+9$ are first terms of branches with a binary series of every combination of 1s and 2s for every value of r.

Theorem 6. *The underlying linear formula for a* $24h+9$ *term binary series of length* $r = 1(2)$ *is* $h = 2^2n+3$, *where* n *is a non-negative integer.*

Proof. For $h = 2^2n+3$, $24h+9 = 96n+81 \to 288n+244(2) \to 72n+61 \to (9n+7)(24)+16$. ☐

Theorem 7. *The proportion of* $24h+9$ *terms with a binary series of length* $r = 1$ *is 1/4.*

Proof. By Theorem 6, the underlying linear formula for a binary series of length $r = 1(2)$ is $h = 2^2n+3$ for $n \geq 0$.

Substituting all non-negative integers for n into the formula gives $1/4$ of the set of all natural numbers. ☐

Theorem 8. *The proportion of* $24h+9$ *terms with a binary series of length* r *is* $3^{r-1}/2^{2r}$.

Proof. Proof by induction.
By Theorem 7, the proportion of $24h+9$ terms with a branch binary series of length $r = 1$ is $1/4 = 3^{1-1}/2^{(2)(1)}$.

Assume that the proportion of $24h+9$ terms with a branch binary series of length r is $3^{r-1}/2^{2r}$.

The $r+1$ position in the branch binary series can be (1) or (2).

Hence, the proportion of 24h+9 terms of binary series length $r+1 =$ $(1/2)(3^{r-1}/2^{2r}) + (1/2^2)(3^{r-1}/2^{2r}) = 3^r/2^{2(r+1)}$. □

Theorem 9. *All* $24h+9$ *terms are in branches of the Collatz structure.*

Proof. The ratio between terms in the geometric series formed by the binary series is $(3^{r-1}/2^{2r})/(3^r/2^{2r+2}) = 3/4$.

By Theorem 7, the proportion of $24h+9$ terms with a binary series of length $r = 1$ is $1/4$.

The total proportion of the $24h+9$ terms in the Collatz structure is
$(1/4)/(1-3/4) = 1$.

Hence, all $24h+9$ terms are in the branches of the Collatz structure. □

6.3. $24h$+15 are first terms of branches with a binary series of every combination of 1s and 2s for every value of r.

Theorem 10. *The underlying linear formula for a* $24h+15$ *term binary series of length* $r = 2(1,1)$ *is* $h = 2^2n+3$*, where* n *is a non-negative integer.*

There is no binary series of length 1. For a summary of all binary series types see Section 8 Conclusion.

Proof. For $h = 2^2n+3$, $24h+15 = 96n+87 \rightarrow 288n+262(1) \rightarrow 144n+131 \rightarrow 432n+394(1) \rightarrow 216n+197 \rightarrow (24)(27n+24)+16$. □

Theorem 11. *The proportion of* $24h+15$ *terms with a binary series of length* $r = 2$ *is 1/4.*

Proof. By Theorem 10, the underlying linear formula for a binary series of length $r = 2$ is $h = 2^2n+3$ for $n \geq 0$.

Substituting all non-negative integers for n into the formula gives $1/4$ of the set of all natural numbers. □

Theorem 12. *The proportion of* $24h + 15$ *terms with a binary series of length* r *is* $3^{r-2}/2^{2r-2}$.

Proof. Proof by induction.
By Theorem 11, the proportion of 24h+15 terms with a branch binary series of length $r = 2$ is $1/4 = 3^{2-2}/2^{(2)(2)-2}$.

Assume that the proportion of $24h + 15$ terms with a branch binary series of length r is $3^{r-2}/2^{2r-2}$.

The $r + 1$ position in the branch binary series can be (1) or (2).

Hence, the proportion of $24h + 15$ terms of binary series length $r + 1 = (1/2)(3^{r-2}/2^{2r-2}) + (1/2^2)(3^{r-2}/2^{2r-2}) = 3^{r-1}/2^{2(r+1)-2}$. □

Theorem 13. *All* $24h + 15$ *terms are in branches of the Collatz structure.*

Proof. The ratio between terms in the geometric series formed by the binary series is $(3^{r-2}/2^{2r-2})/(3^{r-1}/2^{2r}) = 3/4$.

By Theorem 11, the proportion of $24h + 15$ terms with a binary series of length $r = 2$ is 1/4.

The total proportion of the $24h + 15$ terms in the Collatz structure is $(1/4)/(1 - 3/4) = 1$.

Hence, All $24h + 15$ terms are in the branches of the Collatz structure. □

6.4. $24h$+19 are the first terms of branch segments with a binary series containing every combination of 1s and 2s for every value of r.

Theorem 14. *The underlying linear formula for a* $24h + 19$ *term binary series of length* $r = 1(1)$ *is* $h = 2n$, *and the underlying linear formula for* $r = 2(1, 2)$ *is* $h = 2^3n + 5$, *where* n *is a non-negative integer.*

Proof. For $h = 2n$, $24h + 19 = 48n + 19 \rightarrow 144n + 58(1) \rightarrow 72n + 29 \rightarrow (24)(9n + 3) + 16$.

For $h = 2^3n + 5$, $24h + 19 = 192n + 139 \rightarrow 576n + 418(1) \rightarrow 288n + 209 \rightarrow 864n + 628(2) \rightarrow 216n + 157 \rightarrow (27n + 24)(24) + 16$. □

Theorem 15. *The proportion of* $24h+19$ *term binary series of length* $r = 1$ *is 1/2, and that for* $r = 2$ *is 1/8.*

Proof. By Theorem 14, the underlying linear formula for a binary series of length $r = 1(1)$ is $h = 2n$, $n \geq 0$.

Substituting all non-negative integers for n into the formula gives 1/2 of the set of all natural numbers.

By Theorem 14, the underlying linear formula for a binary series of length $r = 2(1,2)$ is $h = 2^3n + 5$ for $n \geq 0$.

Substituting all non-negative integers for n into the formula gives 1/8 of the set of all natural numbers. □

Theorem 16. *The proportion of $24h + 19$ terms with a binary series of length r is $3^{r-2}/2^{2r-1}$.*

Proof. Proof by induction.
By Theorem 15, the proportion of $24h+19$ terms with a branch binary series length $r = 2$ is $1/8 = 3^{2-2}/2^{(2)(2)-1}$.

Assume the proportion of $24h+19$ terms with a branch binary series length r is $3^{r-2}/2^{2r-1}$.

The $r + 1$ position in the branch binary series can be (1) or (2).

Hence, the proportion of $24h + 19$ terms of binary series length $r + 1 = (1/2)(3^{r-2}/2^{2r-1}) + (1/2^2)(3^{r-2}/2^{2r-1}) = 3^{r-1}/2^{2(r+1)-1}$. □

Theorem 17. *All $24h + 19$ terms are in branches of the Collatz structure.*

Proof. The ratio between terms in the geometric series formed by the binary series is $(3^{r-2}/2^{2r-1})/(3^{r-1}/2^{2r+1}) = 3/4$.

By Theorem 15, the proportion of $24h + 19$ terms with a binary series of length $r = 2$ is 1/8.

By Theorem 15, the total proportion of the $24h + 19$ terms in the Collatz structure is $r = 1(1) = 1/2$. $1/2 + (1/8)/(1 - 3/4) = 1$.

Hence, all $24h + 19$ terms are in the branches of the Collatz structure. □

6.5. $24h$+1 are the first terms of branch segments with a binary series containing every combination of 1s and 2s for every value of r.

Theorem 18. *The underlying linear formula for a $24h + 1$ term binary series of length $r = 1$ (2) is $h = 2^2n + 2$, where n is a non-negative integer.*

Proof. For $h = 2^2n + 2$ $24h + 1 = 96n + 49 \to 288n + 148(2) \to 72n + 37 \to (24)(9n + 4) + 16$. □

Theorem 19. *The proportion of* $24h+1$ *terms with a binary series of length* $r = 1$ *is 1/4.*

Proof. By Theorem 18, the underlying linear formula for a binary series of length $r = 1$ (2) is $h = 2^2n + 2$ for $n \geq 0$.

Substituting all non-negative integers for n into the formula gives 1/4 of the set of all natural numbers. □

Theorem 20. *The proportion of* $24h+1$ *terms with a binary series of length* r *is* $3^{r-1}/2^{2r}$.

Proof. Proof by induction.
By Theorem 19, the proportion of $24h+1$ terms with a branch binary series of length $r = 1$ is $1/4 = 3^{1-1}/2^{(2)(1)}$.

Assume that the proportion of 24h+1 terms with a branch binary series of length r is $3^{r-1}/2^{2r}$.

The $r+1$ position in the branch binary series can be (1) or (2).

Hence, the proportion of $24h + 1$ terms of binary series length $r + 1 =$ $(1/2)(3^{r-1}/2^{2r}) + (1/2^2)(3^{r-1}/2^{2r}) = 3^r/2^{2(r+1)}$. □

Theorem 21. *All* $24h + 1$ *terms are in branches of the Collatz structure.*

Proof. The ratio between terms in the geometric series formed by the binary series is $(3^{r-1}/2^{2r})/(3^r/2^{2r+2}) = 3/4$.

By Theorem 19, the proportion of $24h + 1$ terms with a binary series of length $r = 1$ is 1/4.

The total proportion of the 24h+1 terms in the Collatz structure is $(1/4)/(1-3/4) = 1$.

All $24h + 1$ terms are in the branches of the Collatz structure. □

6.6. $24h+7$ are the first terms of branch segments with a binary series of every combination of 1s and 2s for every value of r.

Theorem 22. *The underlying linear formula for a* $24h + 7$ *term binary series of length* $r = 2$ *(1,1) is* $h = 2^2n + 2$, *where* n *is a non-negative integer.*

Proof. For $h = 2^2n+2$, $24h+7 = 96n+55 \to 288n+166(1) \to 144n+83 \to 432n + 250(1) \to 216n + 125 \to (24)(27n + 15) + 16$. □

Theorem 23. *The proportion of the $24h + 7$ terms containing a binary series of length $r = 2$ is 1/4.*

Proof. By Theorem 22, the underlying linear formula for a binary series of length $r = 2$ is $h = 2^2n + 2$ for $n \geq 0$.

Substituting all non-negative integers for n into the formula gives 1/4 of the set of all natural numbers. □

Theorem 24. *The proportion of $24h + 7$ term binary series of length r is $3^{r-2}/2^{2r-2}$.*

Proof. Proof by induction.
By Theorem 23, the proportion of $24h+7$ terms with a branch binary series of length $r = 2$ is $1/4 = 3^{2-2}/2^{(2)(2)-2}$.

Assume that the proportion of $24h + 7$ terms with a branch binary series of length r is $3^{r-2}/2^{2r-2}$.

The $r + 1$ position in the branch binary series can be (1) or (2).

The proportion of $24h + 7$ terms of binary series
length $r+1 = (1/2)(3^{r-2}/2^{2r-2})+(1/2^2)(3^{r-2}/2^{2r-2}) = 3^{r-1}/2^{2(r+1)-2}$. □

Theorem 25. *All $24h + 7$ terms are in branches of the Collatz structure.*

Proof. The ratio between terms in the geometric series formed by the binary series is $(3^{r-2}/2^{2r-2})/(3^{r-1}/2^{2r}) = 3/4$.

By Theorem 23, the proportion of $24h + 7$ terms with a binary series of length $r = 2$ is 1/4.

The total proportion of the $24h + 7$ terms in the Collatz structure is
$(1/4)/(1 - 3/4) = 1$.

All $24h + 7$ terms are in the branches of the Collatz structure. □

6.7. $24h+11$ are first terms of branch segments containing binary series of every combination of 1s and 2s for every value of r.

Theorem 26. *The underlying linear formula for a $24h + 11$ term binary series of length $r = 1(1)$ is h=2n+1, and the underlying linear formula for $r = 2(1, 2)$ is $h = 2^3n + 8$, where n is a non-negative integer.*

Proof. For $h = 2n+1$, $24h+11 = 48n+35 \to 144n+106(1) \to 72n+53 \to (24)(9n + 6) + 16$.

For $h = 2^3n + 8$, $24h + 11 = 192n + 203 \to 576n + 610(1) \to 288n + 305 \to 864n + 916(2) \to 216n + 229 \to (27n + 28)(24) + 16$. □

Theorem 27. *The proportion of* $24h+11$ *terms with binary series length* $r=1$ *is 1/2, and that for* $r=2$ *is 1/8.*

Proof. By Theorem 26, the underlying linear formula for the binary series of length $r=1$ (1) is $h=2n+1$, $n \geq 0$.

Substituting all non-negative integers for n into the formula gives $1/2$ of the set of all natural numbers.

By Theorem 26, the underlying linear formula for a binary series of length $r=2$ (1, 2) is $h=2^3n+8$ for $n \geq 0$.

Substituting all non-negative integers for n into the formula gives $1/8$ of the set of all natural numbers. □

Theorem 28. *The proportion of* $24h+11$ *terms containing a binary series of length* r *is* $3^{r-2}/2^{2r-1}$.

Proof. Proof by induction.
By Theorem 27, the proportion of $24h+11$ terms with a branch binary series length of $r=2$ is $1/8 = 3^{2-2}/2^{(2)(2)-1}$.

Assume that the proportion of $24h+11$ terms with a branch binary series length r is $3^{r-2}/2^{2r-1}$.

The $r+1$ position in the branch binary series can be (1) or (2).

The proportion of $24h+11$ terms of binary series of
length $r+1 = (1/2)(3^{r-2}/2^{2r-1}) + (1/2^2)(3^{r-2}/2^{2r-1}) = 3^{r-1}/2^{2(r+1)-1}$. □

Theorem 29. *All* $24h+11$ *terms are in branches of the Collatz structure.*

Proof. The ratio between terms in the geometric series formed by the binary series is $(3^{r-2}/2^{2r-1})/(3^{r-1}/2^{2r+1}) = 3/4$.

By Theorem 27, the proportion of $24h+11$ terms with a binary series of length $r=2$ is $1/8$.

By Theorem 27, the total proportion of the $24h+11$ terms in the Collatz structure is $r=1(1)=1/2$. $1/2+(1/8)/(1-3/4)=1$.

Hence, all $24h+11$ terms are in the branches of the Collatz structure. □

6.8. $24h+17$ are the first terms of branch segments with the binary series of every combination of 1s and 2s for every value of r.

Theorem 30. *The underlying linear formula for a* $24h+17$ *term binary series of length* $r=1$ *(2) is* $h=2^2n+4$, *where* n *is a non-negative integer.*

Proof. For $h = 2^2n+4$ $24h+17 = 96n+113 \to 288n+340(2) \to 72n+85 \to (24)(9n+10)+16$. ☐

Theorem 31. *The proportion of $24h+17$ terms with a binary series of length $r = 1$ is 1/4.*

Proof. By Theorem 30, the underlying linear formula for a binary series of length $r = 1$ (2) is $h = 2^2n + 4$ for $n \geq 0$.

Substituting all non-negative integers for n into the formula gives 1/4 of the set of all natural numbers. ☐

Theorem 32. *The proportion of $24h+17$ terms with a binary series of length r is $3^{r-1}/2^{2r}$.*

Proof. Proof by induction.
By Theorem 31, the proportion of $24h+17$ terms with a branch binary series of length $r = 1$ is $1/4 = 3^{1-1}/2^{(2)(1)}$.

Assume that the proportion of $24h+17$ terms with a branch binary series of length r is $3^{r-1}/2^{2r}$.

The $r+1$ position in the branch binary series can be (1) or (2).

The proportion of $24h+17$ terms of binary series length
$r+1 = (1/2)(3^{r-1}/2^{2r}) + (1/2^2)(3^{r-1}/2^{2r}) = 3^r/2^{2(r+1)}$. ☐

Theorem 33. *All $24h+17$ terms are in branches of the Collatz structure.*

Proof. The ratio between terms in the geometric series formed by the binary series is $(3^{r-1}/2^{2r})/(3^r/2^{2r+2}) = 3/4$.

By Theorem 31, the proportion of $24h+17$ terms with a binary series of length $r = 1$ is 1/4.

The total proportion of the $24h+17$ terms in the Collatz structure is $(1/4)/(1-3/4) = 1$.

Hence, all $24h+17$ terms are in the branches of the Collatz structure. ☐

6.9. $24h+23$ are the first terms of branch segments with the binary series of every combination of 1s and 2s for every value of r.

Theorem 34. *The underlying linear formula of a $24h+23$ term binary series of length $r = 2$ (1,1) is $h = 2^2n+4$, where n is a non-negative integer.*

Proof. For $h = 2^2n+4$, $24h+23 = 96n+119 \to 288n+358(1) \to 144n+179 \to 432n+538(1) \to 216n+269 \to (24)(27n+33)+16$. ☐

Theorem 35. *The proportion of* $24h + 23$ *terms with a binary series of length* $r = 2$ *is 1/4.*

Proof. By Theorem 34, the underlying linear formula for a binary series of length $r = 2$ is $h = 2^2n + 4$ for $n \geq 0$.

Substituting all non-negative integers for n into the formula gives $1/4$ of the set of all natural numbers. □

Theorem 36. *The proportion of* $24h + 23$ *terms with a binary series of length* r *is* $3^{r-2}/2^{2r-2}$.

Proof. Proof by induction.
By Theorem 35, the proportion of $24h+23$ terms with a branch binary series of length $r = 2$ is $1/4 = 3^{2-2}/2^{(2)(2)-2}$.

Assume that the proportion of $24h + 23$ terms with a branch binary series of length r is $3^{r-2}/2^{2r-2}$.

The $r + 1$ position in the branch binary series can be (1) or (2).

The proportion of $24h + 23$ terms of binary series
length $r+1 = (1/2)(3^{r-2}/2^{2r-2}) + (1/2^2)(3^{r-2}/2^{2r-2}) = 3^{r-1}/2^{2(r+1)-2}$. □

Theorem 37. *All* $24h + 23$ *terms are in branches of the Collatz structure.*

Proof. The ratio between terms in the geometric series formed by the binary series is $(3^{r-2}/2^{2r-2})/(3^{r-1}/2^{2r}) = 3/4$.

By Theorem 35, the proportion of $24h + 23$ terms with a binary series of length $r = 2$ is $1/4$.

The total proportion of the $24h + 23$ terms in the Collatz structure is $(1/4)/(1 - 3/4) = 1$.

All $24h + 23$ terms are in the branches of the Collatz structure. □

6.10. All even terms are present in the branches and/or towers.

All odd numbers are in branches.

Therefore, all $(2n + 1 \rightarrow 6n + 4)$
$24m + 4$ $(n = 4m) \rightarrow 12m + 2$ $(24j + 2,\ m = 2j,\ 24j + 14,\ m = 2j + 1)$,
$24m + 10$ $(n = 4m + 1)$,
$24m + 16$ $(n = 4m + 2)$,
$24m + 22$ $(n = 4m + 3)$
are present in branches.

All $(2s)(6j + 3)$, $24k$, $24k + 6$, $24k + 12$, and $24k + 18$

are present in the first term towers.

All $24k + 16 \rightarrow 12k + 8$ $(24j + 8,\ k = 2j,\ 24j + 20,\ k = 2j + 1)$
are present in the secondary and trunk towers.

Thus, all even terms are present in the branches and/or towers.

6.11. Summary. In Section 6 we have conclusively shown that all natural numbers are within the Collatz structure. This implies that there are no circular or unending Collatz sequences. However, in Section 7 we will explain why in detail.

7. No circular or unending Collatz sequences exist

A circular Collatz sequence, if it were to exist, would be unable to incorporate any terms of the form $6j + 3$. The sole predecessors of $6j + 3$ terms are of the form $(2^s)(6j + 3)$, and such terms cannot be part of a circular sequence due to their self-referential nature in terms of predecessors. Furthermore, terms of the forms $6j + 1$ or $6j + 5$ are also precluded from being part of a circular Collatz sequence, as shown in theorem 1 they invariably reside within finite branches that contain $6j + 3$ terms.

Additionally, all even terms are situated either in branches or towers of the Collatz structure. The intrinsic structure of branches, which are defined by their sequential and terminating nature, rules out the possibility of their inclusion in any circular sequence. Towers are directly connected to branches. Consequently, this structural analysis leads to the conclusion that circular Collatz sequences do not exist within the framework of the Collatz structure.

Proving circular sequences do not exist eliminates the last item lying outside the Collatz structure. The only remaining obstacle is the possibility of an unending Collatz sequence within the Collatz structure. We dispatch that problem next.

Collatz sequences can start from any position within the Collatz structure and invariably converge toward the trunk tower at terms of the form 4^{j+1}, where j is a positive integer, ultimately culminating at the base term 1. The non-existence of unending Collatz sequences is rigorously established and elucidated in Theorem 38, where a detailed proof is presented to support this assertion.

The usage factor for secondary tower base terms is the combined proportion of all three secondary base terms used in Collatz sequences within the Collatz structure.

Theorem 38. *The usage factor for secondary tower base terms in Collatz sequences with a binary series of length r is $3^r/4^r$.*

Proof. We will prove Theorem 7.1 by induction.

There are two binary series of length one (1), (2) for a Collatz sequence with one secondary tower base term.

The usage factor is $1/2^1 + 1/2^2 = 3/4$, verifying the formula for $r = 1$.

Assume that the usage factor for all binary series of length r is $3^r/4^r$.

The $r+1$ secondary tower base term of a Collatz sequence of that length contains (1) one or (2) two divisions by two.

The length $r+1$ usage factor is $(1/2)(3^r/4^r)+(1/4)(3^r/4^r) = 3^{r+1}/4^{r+1}$. □

The usage factor proportion ratio of Collatz sequence binary series is

$$(3^r/4^r)/(3^{r+1}/4^{r+1}) = 3/4.$$

The sum of the geometric series of the binary series usage factors is

$$(3/4)/(1 - 3/4) = 3.$$

100% of all three secondary tower base terms appear in Collatz sequences. Each Collatz sequence binary series is of finite length. However, neither a longest binary series nor a longest Collatz sequence exists.

7.1. Summary. In Section 7 We have conclusively shown that there are no circular or unending Collatz sequences. This result was not unexpected, and the Usage Factor was the key to proving that there are no unending Collatz sequences.

100% of all three secondary tower base terms appear in Collatz sequences. Each Collatz sequence binary series is of finite length. However, neither a longest binary series nor a longest Collatz sequence exists.

8. Conclusion

All positive integers are in the branches or towers of the Collatz structure

The terms of the form $24h + 7, 24h + 15$, and $24h + 23$ exhibit proportion formulas $3^{r-2}/2^{2r-2}$ and are the first terms in the branch or branch segments with a binary series of $(1, 1)$ and $(1, 1, \ldots)$, where the binary series value can be 1 or 2 from the third position and beyond.

The terms of the form $24h + 1, 24h + 9$, and $24h + 17$ exhibit proportion formulas $3^{r-1}/2^{2r}$ and are the first terms in the branch or branch segments with a binary series of (2) and $(2, \ldots)$, where the binary series value can be 1 or 2 from the second position and beyond.

The terms of the form $24h + 3, 24h + 11$, and $24h + 19$ exhibit proportion formulas $3^{r-2}/2^{2r-1}$ and are the first terms in the branch or branch segments with a binary series of (1), (1,2), and (1,2,. . .), where the binary series value can be 1 or 2 from the third position and beyond.

The proportion formulas create geometric series that all sum to 1 (100%). All odd terms with binary series of length $r > 0$ are in the branches. Appendix B shows that there are branches with every possible binary series of finite length.

Branch segments of the form $24h + 5 \to 24(3h) + 16$ and $24h + 13 \to 24(3h+1) + 16$ and branches $24h+21 \to 24(3h+2)+16$ have a zero length binary series.

The Collatz structure, with its towers and branches, provides a standardized framework with which to study the Collatz conjecture. It allows for the creation of the binary series from the branches. This provides the necessary basis for showing that the Collatz structure contains all positive integers, by using the proportion formulas to create geometric series that all sum to one. Consequently, the problem of checking that the Collatz sequences are finite for all individual integers is solved. This also extends previous work studying the length and seemingly random nature of Collatz sequences, and the "almost all" problem.

The Collatz structure, Collatz sequences, and their associated binary series allow for the development of proportion formulas. From proportion formulas, we build geometric series. If a structure contains the kind of integer sequences similar to Collatz, we may be able to generate the Collatz proof process. We would need a unique property contained within the structure of the sequences (mirroring the binary series) and a means analogous to the geometric series to measure the unique property.

Appendix A. The placement of even terms within a branch

Starting with $6n + 1$, $6n + 3$, and $6n + 5$, we run the Collatz algorithm forward until we come to another odd term. There are never more than two consecutive even terms between the odd terms. The secondary tower base terms $24m + 4$, $24m + 10$ and $24m + 22$ are in the middle of a branch, and $24m + 16$ is always at the end of a branch.

$\boldsymbol{6n + 1 \to 18n + 4}$
If $n = 4j$, $18n + 4 = 72j + 4$ ($24m + 4$, $m = 3j$) $\to 36j + 2 \to 18j + 1$.
If $n = 4j + 1$, $18n + 4 = 72j + 22$ ($24m + 22$, $m = 3j$) $\to 36j + 11$.
If $n = 4j + 2$, $18n + 4 = 72j + 40$ ($24m + 16$, $m = 3j + 1$) Last term in the branch.
If $n = 4j + 3$, $18n + 4 = 72j + 58$ ($24m + 10$, $m = 3j + 2$) $\to 36j + 29$.

$\boldsymbol{6n + 3 \to 18n + 10}$
If $n = 4j$, $18n + 10 = 72j + 10$ ($24m + 10$, $m = 3j$) $\to 36j + 5$.
If $n = 4j + 1$, $18n + 10 = 72j + 28$ ($24m + 4$, $m = 3j + 1$) $\to 36j + 14 \to 18j + 7$.
If $n = 4j + 2$, $18n + 10 = 72j + 46$ ($24m + 22$, $m = 3j + 1$) $\to 36j + 23$.
If $n = 4j + 3$, $18n + 10 = 72j + 64$ ($24m + 16$, $m = 3j + 2$) Last term in

the branch.

$6n + 5 \to 18n + 16$
If $n = 4j$, $18n + 16 = 72j + 16$ ($24m + 16$, $m = 3j$) Last term in the branch.
If $n = 4j + 1$, $18n + 16 = 72j + 34$ ($24m + 10$, $m = 3j + 1$) $\to 36j + 17$.
If $n = 4j + 2$, $18n + 16 = 72j + 52$ ($24m + 4$, $m = 3j + 2$) $\to 36j + 26 \to 18j + 13$.
If $n = 4j + 3$, $18n + 16 = 72j + 70$ ($24m + 22$, $m = 3j + 2$) $\to 36j + 35$.

Appendix B. Every possible sized binary series is realized

The terms of the form $24h + 7$, $24h + 15$, and $24h + 23$ are the first terms in the branch or branch segments with a binary series of $(1, 1)$ and $(1, 1, \ldots)$, where the binary series value is 1 or 2 from the third position and beyond.

Their sets of binary series are as follows:

$\{(1, 1)\}$, $\{(1, 1, 1)\}$, $\{(1, 1, 2)(1, 1, 1, 1)\}$, $\{(1, 1, 1, 2)(1, 1, 2, 1)(1, 1, 1, 1, 1)\}$, $\{(1, 1, 2, 2)(1, 1, 1, 1, 2)(1, 1, 1, 2, 1)(1, 1, 2, 1, 1)(1, 1, 1, 1, 1, 1)\}, \ldots$

The terms of the form $24h + 1$, $24h + 9$, and $24h + 17$ are the first terms in the branch or branch segments with a binary series of (2) and $(2, \ldots)$, where the binary series value is 1 or 2 from the second position and beyond.

Their sets of binary series are as follows:

$\{(2)\}$, $\{(2, 1)\}$, $\{(2, 2)(2, 1, 1)\}$, $\{(2, 1, 2)(2, 2, 1)(2, 1, 1, 1)\}$, $\{(2, 2, 2)$ $(2, 1, 1, 2)(2, 1, 2, 1)(2, 2, 1, 1)(2, 1, 1, 1, 1)\}, \ldots$

The binary series sum is the same for each binary series within each set. Starting with the first two sets in each group, we form each additional binary series set. We add (2) to each binary series in the set that is two behind in the progression of sets and (1) to each binary series in the previous set. The number of binary series in each set forms a Fibonacci sequence that starts with $1, 1$ with every additional sequence number being the sum of the previous two numbers. The count of the number of binary series in these two groups of sets are $1, 1, 2, 3, 5, \ldots$

Each binary series is associated with a linear formula $h = 2^s n + a$. The proportion of each formula in the above sum is $1/2^s$. The denominators are 2^s, where s is a natural number greater than one that represents each different binary series sum. Each numerator is the total number of binary series with that particular binary series sum in the exponent of the denominator. The proportion of the branch/branch segments of each of these six term types with a binary series sum of $s = 2, 3, 4, 5, 6, \ldots$ is $1/2^2 + 1/2^3 + 2/2^4 + 3/2^5 + 5/2^6 + \ldots$.

To evaluate the above sum we subtract each term of $1/2^2 + 1/2^3 + 2/2^4 + 3/2^5 + 5/2^6 + \ldots$ from the sequence sum $1 = 2/2^1$.

$2/2^1 \ -1/2^2 = 3/2^2 - 1/2^3 = 5/2^3 - 2/2^4 = 8/2^4 - 3/2^5 = 13/2^5 \ldots$

$2/2^1,\ 3/2^2,\ 5/2^3,\ 8/2^4,\ 13/2^5 \ldots.$

This approach is akin to subtracting the numerator of the third previous term in this sequence from twice the numerator of the previous term in the sequence, which is another way of generating a Fibonacci sequence: compare $(a, b, a+b, a+2b)$ with $(a, b, a+b, 2a+2b-a)$. The denominators increase exponentially, and the numerators increase linearly. This sequence has a limit of zero as s approaches infinity, proving that the original sum whose numerators are a Fibonacci sequence has a limit of one as s approaches infinity. Every possible binary series is realized because their total proportion is 1 (100%).

The terms of the form $24h+3, 24h+11$, and $24h+19$ are the first terms in the branch or branch segments with a binary series of $(1), (1,2)$ and $(1, 2, \ldots)$, where the binary series value can be 1 or 2 from the third position and beyond. The number of binary series set elements whose sum is the same form a Fibonacci sequence.

Their sets of binary series are as follows:

$\{(1,2)\},\ \{(1,2,1)\},\ \{(1,2,2)(1,2,1,1)\},\ \{(1,2,1,2)(1,2,2,1)(1,2,1,1,1)\},$
$\{(1,2,2,2)(1,2,1,1,2)(1,2,1,2,1)(1,2,2,1,1)(1,2,1,1,1,1)\}, \ldots$

The exponent of each denominator is one greater than the previous sequence so this sequence sum will be half that of the previous sequence. The proportion of the branch/branch segments of each of these three term types with binary series sums of $s = 3, 4, 5, \ldots$ is

$$1/2^3 + 1/2^4 + 2/2^5 + 3/2^6 + 5/2^7 + \ldots = 1/2$$

plus 1/2 for $s = 1$. Every possible binary series is realized because their total proportion is 1 (100%).

Appendix C. Calculating the proportion of branch/branch segments

The proportion formula for $24h+9$ is $3^{r-1}/2^{2r}$. For the total proportion of $24h+9$ terms: the $r = 1$ binary series (2) accounts for $1/4, (3^{1-1}/2^{(2)(1)})$ of the positive integers, which is generated from $h = 2^2n+3$. (See Theorems 6 and 8 for further details.)

$h = 2^3n+6$ generates the $r = 2$ binary series (2, 1), which accounts for 1/8 of the positive integers: $(24)(2^3n+6)+9 = 192n+153 \to 576n+460(2) \to 144n+115 \to 432n+346(1) \to 216n+173 \to 648n+520 = (27n+21)(24)+16.$

$h = 2^4n+13$ generates the $r = 2$ binary series (2, 2), which accounts for 1/16 of the positive integers: $(24)(2^4n+13)+9 = 384n+321 \to 1152n+964(2) \to 288n+241 \to 864n+724(2) \to 216n+181 \to 648n+544 = (27n+22)(24)+16$.

Together, they account for $1/8 + 1/16 = 3/16, (3^{2-1}/2^{(2)(2)}, r = 2)$ of all the positive integers. Similar underlying linear formulas exist that sum to $3^{r-1}/2^{2r}$ for all r value proportions.

If the total number of underlying linear formulas for $3^{r-1}/2^{2r}$ is $h_j = 2^{s_j}n + a_j$, where j ranges from 1 to m (the total number of different binary series of length r),

then $3^{r-1}/2^{2r} = 1/2^{s_1} + 1/2^{s_2} + 1/2^{s_3} + \ldots + 1/2^{s_m}$.

For length $r+1, 3^r/2^{2(r+1)}$ of each of the existing binary series sums will be increased by 1 and 2:

$3^r/2^{2(r+1)} = 1/2^{s1+1} + 1/2^{s2+1} + 1/2^{s3+1} + \ldots + 1/2^{sm+1} + 1/2^{s1+2}+1/2^{s2+2}+1/2^{s3+2}+\ldots+1/2^{sm+2} = (1/2)(3^{r-1}/2^{2r})+(1/2^2)(3^{r-1}/2^{2r})$.

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Collatz Conjecture, 3x+1 problem, the Syracuse problem, Hasse's algorithm, binary series, Ulam's problem

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You can contact me at collatz3106@gmail.com