Photon Mass Operator in a Constant Magnetic Field

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Abstract

We define the mathematical object called photon mass operator and we evaluate it in the homogenous magnetic field using the Green function technique in the homogenous magnetic field. The physical meaning of the photon mass operator is the vacuum polarization by the external field.

1 Introduction

It is well-known that pulsars might possess a relatively intense electric field in addition to a strong magnetic field $(10^{10} - 10^{14}G)$. This is why there is an interest in studying some quantum-electrodynamic processes in the presence of both types of strong fields.

A lot of work has already been done concerning classical and quantum electrodynamics in the presence of an external homogeneous strong magnetic field, but little attention has been devoted to the inclusion of an external strong electric field. Maybe the reason for this reluctance lies in the well-known fact that a strong static electric field wi11 spontaneously break down in producing electron-positron pairs (the so-called Klein catastrophe) and in consequence is no longer static. Here we consider the strong magnetic field being constant due to the action of an external mechanism which is constantly pouring energy into the system.

2 Vacuum polarization in a constant magnetic field

We define the mathematical object called photon mass operator and we evaluate it in the homogenous magnetic field using the Green function technique in the homogenous magnetic field. The physical meaning of the photon mass operator is the vacuum polarization by the external field.

The Feynman propagator of photon involves the one-loop radiative correction to this propagator and it can be graphically represented by the Feynman diagram of the second order. The physical meaning of this diagram is the process $\gamma \rightarrow (e^- + e^+) \rightarrow \gamma$, where γ is notation for photon, and e^-, e^+ is the electron-positron pair. It means that photon can exist in the virtual intermediate state with e^+, e^- being virtual particles.

We calculate the dynamical modifications to the propagation properties of the photon due to the presence of homogeneous external electromagnetic fields. The calculation is carried out by a nonperturbative method in the external fields, but just to second order in the fermion-photon coupling constant.

The Lagrangian describing to the lowest order the vacuum polarization process is the following one (Tsai, 1974; Urrutia, 1978):

$$\mathcal{L}^{(2)} = ie^2 \int (dx')(dx'') \mathrm{Tr} \gamma A_1(x') G(x', x'') \gamma A_2(x'') G(x'', x') + C.T. \tag{1}$$

where the contact term C.T. will be determined later from the normalization conditions and gauge invariance requirement. In this text we follow the approach by Dittrich et al. (1978). This approach is the integral part of the global and general approach by Schwinger and Ditrich (Schwinger, 1950; 1969; 1970; 1973; Dittrich, 1978). The non-source approch is involved for instance in the monographs by Akhiezer et al. and Berestetskii et al. (Akhiezer et al., 1965; Berestetskii et al., 1982).

Substituting for $G(x', x'')$,

$$G(x', x') = \Phi(x', x'') \int \frac{(dp)}{(2\pi)^4} e^{ip(x'-x'')} G(p) \tag{2}$$

into eq. (1) and using

$$A(x) = \int \frac{(dp)}{(2\pi)^4} e^{ipx} A(k) \tag{3}$$

and

$$\Phi(x', x'')\Phi(x'', x') = 1 \tag{4}$$

we get

$$\mathcal{L}^{(2)} = - \int (dk) A_1^\mu(-k) A_2^\nu(k) \Pi_{\mu\nu}(k) \tag{5}$$

where

$$\Pi_{\mu\nu}(k) = -ie^2 \mathrm{tr} \langle \gamma_\mu G(p) \gamma_\nu G(p-k) \rangle + C.T. \tag{6}$$

with

$$G(p) = i \int_0^\infty ds \exp\left\{ -is \left(m^2 + p_\parallel^2 + \frac{\tan z}{z} p_\perp^2 \right) \right\} \quad \times$$
$$\frac{1}{\cos z} \left[\left(m - \gamma p_\parallel \right) e^{i\sigma_3 z} - \frac{1}{\cos z} \gamma p_\perp \right] \tag{7}$$

in a homogenous magnetic field with the notation

$$(ab)_\parallel \stackrel{d}{=} -a^0 b^0 + a_3 b_3 \quad (ab)_\perp \stackrel{d}{=} a_1 b_1 + a_2 b_2 \tag{8}$$

$$z = seH \tag{9}$$

and

$$\langle f(p)\rangle \equiv \int \frac{(dp)}{(2\pi)^4} f(p) \tag{10}$$

The mathematical object $\Pi_{\mu\nu}$ is the photon mass operator and the goal of this article is to evaluate it in the presence of the homogenous magnetic field of the intensity H in the z-direction.

After substituting of eq. (7) into eq. (6) we get:

$$\Pi_{\mu\nu}(k) = ie^2 \int_0^\infty ds_1 \int_0^\infty ds_2 \langle \exp\left\{-is_1\left(m^2 + p_\parallel^2 + \frac{\tan z_1}{z_1}p_\perp^2\right) \quad - \right.$$

$$\left. -is_2\left(m^2 + (p-k)_\parallel^2 + \frac{\tan z_2}{z_2}(p-k)_\perp^2\right)\right\} \frac{1}{\cos z_1 \cos z_2} \quad \times$$

$$\mathrm{tr}\left\{\gamma_\mu\left((m - \gamma p_\parallel)e^{i\sigma^3 z_1} - \frac{\gamma p_\perp}{\cos z_1}\right)\gamma_\nu\left((m - \gamma(p-k)_\parallel)e^{i\sigma^3 z_2} - \frac{\gamma(p-k)_\perp}{\cos z_2}\right)\right\}\rangle \tag{11}$$

with

$$z_1 = eBs_1; \quad z_2 = eBs_2 \tag{12}$$

Introducing new variables of integration

$$s_1 = s\frac{1-v}{2}; \quad s_2 = s\frac{1+v}{2} \tag{13}$$

we get

$$s = s_1 + s_2; \quad v = \frac{s_2 - s_1}{s_2 + s_1} \tag{14}$$

$$z_1 = eBs_1 = eBs\frac{1-v}{2} \stackrel{d}{=} z\frac{1-v}{2} \equiv \xi \tag{15}$$

$$z_2 = eBs_2 = eBs\frac{1+v}{2} \stackrel{d}{=} z\frac{1+v}{2} \equiv \eta \tag{16}$$

and

$$\int_0^\infty ds_1 \int_0^\infty ds_2 \cdots = \int_0^\infty sds \int_{-1}^1 \frac{dv}{z} \cdots \tag{17}$$

Further we simplify the exponential form as it follows:

$$\exp\left\{-is_1\left(m^2 + p_\parallel^2 + \frac{\tan\xi}{\xi}p_\perp^2\right) - is_2\left(m^2 + (p-k)_\parallel^2 + \frac{\tan\eta}{\eta}(p-k)_\perp^2\right)\right\} =$$

$$\exp\left\{-is(\varphi_0 + \varphi_1)\right\} \tag{18}$$

where

$$\varphi_0 = m^2 + \frac{1-v^2}{4} k_\parallel^2 + \frac{\cos zv - \cos z}{2z \sin z} k_\perp^2 \tag{19}$$

and

$$\varphi_1 = \left(p_\parallel - \frac{1+v}{2} k_\parallel \right)^2 + \frac{\tan\xi + \tan\eta}{z} \left(p_\perp - \frac{\tan\eta}{\tan\xi + \tan\eta} k_\perp \right)^2 \tag{20}$$

After inserting of eq. (18) into $\Pi_{\mu\nu}$ of eq. (11) we get

$$\Pi_{\mu\nu}(k) = ie^2 \int_0^\infty s ds \int_{-1}^1 \frac{dv}{z} e^{-is\varphi_0} \frac{1}{\cos\xi \cos\eta} \quad \times$$

$$\mathrm{tr}\langle \exp -is\varphi_1 \left\{ \gamma_\mu \left((m - \gamma p_\parallel) e^{i\sigma^3 \xi} - \frac{\gamma p_\perp}{\cos\xi} \right) \right.$$

$$\left. \gamma_\nu \left((m - \gamma (p-k)_\parallel) e^{i\sigma^3 \xi} - \frac{\gamma (p-k)_\perp}{\cos\xi} \right) \right\} \rangle + C.T. \tag{21}$$

First, let us perform the p-integration using the relation

$$\int_{-\infty}^\infty dx e^{\pm iax^2} = e^{\pm i\frac{\pi}{4}} \left(\frac{\pi}{a} \right)^{1/2} ; \quad a > 0 \tag{22}$$

Then,

$$\langle e^{-is\varphi_1} \rangle = \int \frac{(dp)}{(2\pi)^4} \quad \times$$

$$\exp \left\{ -is \left(p_\parallel - \frac{1+v}{2} k_\parallel \right)^2 + \frac{\tan\xi + \tan\eta}{z} \left(p_\perp - \frac{\tan\eta}{\tan\xi + \tan\eta} k_\perp \right)^2 \right\} \quad =$$

$$\frac{(-i)}{(4\pi)^2} \frac{1}{s^2} \frac{z}{\sin z} \cos\xi \cos\eta \tag{23}$$

Furthermore,

$$\langle e^{-is\varphi_1} p_\parallel \rangle = \int \frac{(dp)}{(2\pi)^4} \quad \times$$

$$\exp \left\{ -is \left(p_\parallel - \frac{1+v}{2} k_\parallel \right)^2 + \frac{\tan\xi + \tan\eta}{z} \left(p_\perp - \frac{\tan\eta}{\tan\xi + \tan\eta} k_\perp \right)^2 \right\} p_\parallel \quad =$$

$$\frac{1+v}{2} k_\parallel \langle e^{-is\varphi_1} \rangle \tag{24}$$

since the integral vanishes over an odd function.

The analogical evaluation gives

$$\langle e^{-is\varphi_1} p_\perp \rangle = \frac{\tan\eta}{\tan\xi + \tan\eta} k_\perp \langle e^{-is\varphi_1} \rangle \tag{25}$$

$$\langle e^{-is\varphi_1} p_{\parallel\mu} p_{\parallel\nu} \rangle = \langle e^{-is\varphi_1} \rangle \left(\left(\frac{1+v}{2} \right)^2 k_{\parallel\mu} k_{\parallel\nu} - \frac{1}{2s} g^{\parallel}_{\mu\nu} \right) \tag{26}$$

$$\langle e^{-is\varphi_1} \{ p_{\parallel\mu} p_{\perp\nu}, p_{\perp\mu} p_{\parallel\nu} \} \rangle = \langle e^{-is\varphi_1} \rangle \langle e^{-is\varphi_1} \rangle \frac{1+v}{2} \frac{\tan\eta}{\tan\xi + \tan\eta} \left\{ k_{\parallel\mu} k_{\perp\nu}, k_{\perp\mu} k_{\parallel\nu} \right\} \tag{27}$$

$$\langle e^{-is\varphi_1} p_{\perp\mu} p_{\perp\nu} \rangle = \langle e^{-is\varphi_1} \rangle \left(\left(\frac{\tan\eta}{\tan\xi + \tan\eta} \right)^2 k_{\perp\mu} k_{\perp\nu} - \frac{i}{2s} \frac{z}{\tan\xi + \tan\eta} g^{\perp}_{\mu\nu} \right) \tag{28}$$

where we introduced $g^{\parallel}$ and $g^{\perp}$ by

$$\left(g^{\parallel}_{\mu\nu} \right) = \left(g^{\parallel\mu\nu} \right) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{29}$$

and

$$\left(g^{\perp}_{\mu\nu} \right) = \left(g^{\perp\mu\nu} \right) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{30}$$

With the help of eq. (23) we can now eliminate the factor $(\cos\xi \cos\eta)^{-1}$, and we have

$$\Pi_{\mu\nu}(k) = \frac{\alpha}{2\pi} \int_0^\infty \frac{ds}{s} \int_{-1}^{1} \frac{dv}{z} e^{-is\varphi_0} I_{\mu\nu} \frac{z}{\sin z} + C.T. \tag{31}$$

where

$$I_{\mu\nu} = z \frac{1}{4\langle e^{-is\varphi_1} \rangle} \mathrm{tr} \langle \exp -is\varphi_1 \left\{ \gamma_\mu \left((m - \gamma p_\parallel) e^{i\sigma^3 \xi} - \frac{\gamma p_\perp}{\cos\xi} \right) \gamma_\nu \quad \times \right.$$
$$\left. \left((m - \gamma (p-k)_\parallel) e^{i\sigma^3 \xi} - \frac{\gamma (p-k)_\perp}{\cos\eta} \right) \right\} \rangle \tag{32}$$

The expression (32) needs to evaluate trace. For this operation we have with

$$\mathrm{tr}\langle ... \rangle = \sum S_i \tag{33}$$

where

$$S_1 = m^2 \mathrm{tr} \langle e^{-is\varphi_1} \gamma_\mu e^{i\sigma^3 \xi} \gamma \nu e^{i\sigma^3 \eta} \rangle \tag{34a}$$

$$S_2 = \mathrm{tr} \langle e^{-is\varphi_1} \gamma_\mu \gamma p_\parallel e^{i\sigma^3 \xi} \gamma_\nu \gamma (p-k)_\parallel e^{i\sigma^3 \eta} \rangle \tag{34b}$$

$$S_3 = \mathrm{tr}\langle e^{-is\varphi_1}\gamma_\mu\gamma_\mu\gamma p_\parallel e^{i\sigma^3\xi}\gamma_\nu\gamma(p-k)_\perp\rangle\frac{1}{\cos\eta} \tag{34c}$$

$$S_4 = \mathrm{tr}\langle e^{-is\varphi_1}\gamma_\mu\gamma p_\perp\gamma_\nu(p-k)_\parallel e^{i\sigma^3\eta}\rangle\frac{1}{\cos\xi} \tag{34d}$$

$$S_5 = \mathrm{tr}\langle e^{-is\varphi_1}\gamma_\mu\gamma p_\perp\gamma_\nu\gamma(p-k)_\perp\rangle\frac{1}{\cos\xi\cos\eta} \tag{34e}$$

Furthermore we use the trace relations

$$\frac{1}{4}\mathrm{tr}\left\{e^{i\sigma^3 z}\gamma_\mu\gamma_\nu\right\} = \frac{1}{4}\mathrm{tr}\left[(\cos z + i\sigma^3\sin z)\gamma_\mu\gamma_\nu\right] \quad = -\cos z\, g_{\mu\nu} + \left(\frac{F}{B}\right)_{\mu\nu}\sin z \tag{35}$$

where

$$\left(\frac{F}{B}\right)_{\mu\nu} = g_{1\mu}g_{2\nu} - g_{1\nu}g_{2\mu} \tag{36}$$

and

$$\frac{1}{4}\mathrm{tr}\left\{e^{i\sigma^3 z}\gamma_\mu\gamma_\nu\gamma_\lambda\gamma_\sigma\right\} = \cos z(g_{\mu\nu}g_{\lambda\sigma} - g_{\mu\lambda}g_{\nu\sigma} + g_{\mu\sigma}g_{\nu\lambda}) \quad -$$

$$\frac{1}{B}\sin z(F_{\mu\nu}g_{\lambda\sigma} - F_{\mu\lambda}g_{\nu\sigma} + F_{\mu\sigma}g_{\nu\lambda} - F_{\nu\sigma}g_{\mu\lambda} + F_{\nu\lambda}g_{\mu\sigma} + F_{\lambda\sigma}g_{\mu\nu}) \tag{37}$$

Using eq. (35) and (37) get for the individual S_1

$$\frac{\mathrm{tr}S_1}{4m^2\langle e^{-is\varphi_1}\rangle} = \frac{1}{4}\mathrm{tr}\langle e^{-is\varphi_1}\gamma_\mu e^{-i\sigma^3\xi}\gamma_\nu e^{i\sigma^3\eta}\rangle\langle e^{-is\varphi_1}\rangle^{-1} \quad =$$

$$\cos\eta\left(-\cos\xi g_{\mu\nu} + \left(\frac{F}{B}\right)_{\nu\mu}\sin\xi\right) - \sin\eta\,\frac{1}{4}\mathrm{tr}\left\{e^{i\sigma^3\xi}\gamma_\nu\gamma_1\gamma_2\gamma_\mu\right\} \tag{38}$$

where we used the cyclicity of the trace and the antisymmetry of $\gamma_1\gamma_2$ with respect to indices 1 and 2.

The term with tr in eq. (38) can be modified as follows

$$\frac{1}{4}\mathrm{tr}\left\{e^{i\sigma^3\xi}\gamma_\nu\gamma_1\gamma_2\gamma_\mu\right\} = \cos\xi(g_{\nu 1}g_{2\mu} - g_{\nu 2}g_{1\mu} + g_{\nu\mu}g_{12}) \quad -$$

$$\frac{1}{B}\sin\xi\,(F_{\nu 1}g_{2\mu} - F_{\nu 2}g_{1\mu} + F_{\nu\mu}g_{12} - F_{1\nu}g_{\nu 2} + F_{12}g_{\nu\mu} + F_{2\mu}g_{\nu 1}) \quad =$$

$$-\left(\frac{F}{B}\right)_{\mu\nu}\cos\xi - \sin\xi(-g_{\nu 2}g_{2\mu} - g_{1\nu}g_{1\mu} + g_{\nu\mu} - g_{\mu 2}g_{\nu 2} - g_{\mu 1}g_{\nu 1}) \quad =$$

$$-\left(\frac{F}{B}\right)_{\mu\nu}\cos\xi - \sin\xi g_{\mu\nu} + 2\sin\xi g^{\perp}_{\mu\nu} \tag{39}$$

We therefore have:

$$\frac{\mathrm{tr}S_1}{4m^2\langle e^{-is\varphi_1}\rangle} = -\cos z g_{\mu\nu} + \left(\frac{F}{B}\right)_{\mu\nu} \sin zv - 2\sin\xi\sin\eta g^{\perp}_{\mu\nu} \tag{40}$$

where the second term on the right-hand side is an odd function in v, which gives no contribution after substitution in $\Pi_{\mu\nu}$. Using

$$g_{\mu\nu} = g^{\parallel}_{\mu\nu} + g^{\perp}_{\mu\nu} \tag{41}$$

we get thanks to the addition theorem

$$\frac{\mathrm{tr}S_1}{4m^2\langle e^{-is\varphi_1}\rangle} = -m^2\left[\cos zv\; g^{\perp}_{\mu\nu} + \cos z\; g^{\parallel}_{\mu\nu} + \text{odd function}\right] \tag{42}$$

The next term to evaluate is S_2. For this term, it can be shown that

$$\frac{\mathrm{tr}S_2}{4m^2\langle e^{-is\varphi_1}\rangle} =$$

$$\frac{1}{4\langle e^{-is\varphi_1}\rangle}^{-1}\mathrm{tr}\langle e^{-is\varphi_1}\gamma_\mu \gamma p_{\parallel} e^{i\sigma^3\xi}\gamma_\nu\gamma(p-k)_{\parallel}e^{i\sigma^3\eta}\rangle \overset{d}{=} C^{\alpha\beta}D_{\alpha\beta} \tag{43}$$

where

$$C^{\alpha\beta} = \langle e^{-is\varphi_1}\rangle^{-1}\left\{\langle e^{-is\varphi_1}p^{\alpha}_{\parallel}p^{\beta}_{\parallel}\rangle - k^{\beta}_{\parallel}\langle e^{-is\varphi_1}p^{\alpha}_{\parallel}\rangle\right\} \quad =$$

$$-\frac{1-v^2}{4}k^{\alpha}_{\parallel}k^{\beta}_{\parallel} - \frac{i}{s}g^{\alpha\beta}_{\parallel} \tag{44}$$

and

$$D^{\alpha\beta} = \frac{1}{4}\mathrm{tr}\left\{\gamma_\mu\gamma_{\parallel\alpha}e^{i\sigma^3\xi}\gamma_\nu\gamma_{\parallel\beta}e^{i\sigma^3\eta}\right\} \tag{45}$$

Exploiting the cyclicity of the trace we then get

$$\frac{\mathrm{tr}S_2}{4\langle e^{-is\varphi_1}\rangle} \quad =$$

$$\frac{1}{4}\mathrm{tr}\left\{\left(-\frac{1-v^2}{4}\right)\gamma k_{\parallel}e^{i\sigma^3\xi}\gamma k_{\parallel}e^{i\sigma^3\eta}\gamma_\mu - \frac{i}{2s}\gamma^{\lambda}_{\parallel}e^{i\sigma^3\xi}\gamma_\nu\gamma_{\parallel\lambda}e^{i\sigma^3\eta}\gamma_\mu\right\} \tag{46}$$

Further we use the easily derived formulas:

$$\gamma k_{\parallel}e^{i\sigma^3\xi} = e^{i\sigma^3\xi}\gamma k_{\parallel} \tag{47}$$

$$\gamma k_{\parallel}\gamma_\nu = -2k_{\parallel\nu} - \gamma_\nu\gamma k_{\parallel} \tag{48}$$

$$\gamma^{\lambda}_{\parallel}e^{i\sigma^3\xi} = e^{i\sigma^3\xi}\gamma^{\lambda}_{\parallel} \tag{49}$$

$$\gamma^{\lambda}_{\parallel}\gamma_\nu = -\gamma_\nu\gamma^{\lambda}_{\parallel} - 2g^{\lambda}_{\parallel\nu} \tag{50}$$

which, after substitution into above equation enables to write

$$\frac{\mathrm{tr} S_2}{4\langle e^{-is\varphi_1}\rangle} =$$

$$\left(-\frac{1-v^2}{4}\right)\frac{1}{4}\mathrm{tr}\left\{e^{i\sigma^3\xi}(-\gamma_\nu\gamma k_\parallel - 2k_{\parallel\nu})\gamma k_\parallel e^{i\sigma^3\eta}\gamma_\mu\right\} \quad -$$

$$\frac{i}{2s}\frac{1}{4}\mathrm{tr}\left\{e^{i\sigma^3\xi}(-\gamma_\nu\gamma_\parallel^\lambda - 2g_{\parallel\nu}^\lambda)\gamma_{\parallel\lambda} e^{i\sigma^3\eta}\gamma_\mu\right\} \tag{51}$$

Using

$$\gamma k_\parallel \gamma k_\parallel = -k_\parallel^2; \quad \gamma_\parallel^\lambda\gamma_{\parallel\lambda} = -2 \tag{52}$$

we have instead of eq. (51)

$$\frac{\mathrm{tr} S_2}{4\langle e^{-is\varphi_1}\rangle} =$$

$$\left(-\frac{1-v^2}{4}\right)\frac{1}{4}\mathrm{tr}\left\{e^{i\sigma^3\xi}(\gamma_\nu k_\parallel - 2k_{\parallel\nu}k_\parallel) e^{i\sigma^3\eta}\right\} \quad -$$

$$-\frac{i}{2s}\frac{1}{4}\mathrm{tr}\left\{e^{i\sigma^3\xi}2\gamma_{\perp\nu} e^{i\sigma^3\eta}\gamma_\mu\right\} \stackrel{d}{=} B_1 + B_2 + B_3 \tag{53}$$

where for B_i we have the following formulas:

$$B_1 \equiv \left(-\frac{1-v^2}{4}\right) k_\parallel^2 \frac{1}{4}\mathrm{tr}\left\{e^{i\sigma^3\xi}\gamma_\nu e^{i\sigma^3\eta}\gamma_\mu\right\} \quad =$$

$$\frac{1-v^2}{4}k_\parallel^2\left(\cos zv\; g_{\mu\nu}^\perp + \cos z\; g_{\mu\nu}^\parallel\right) + \text{odd functions} \tag{54}$$

$$B_2 \equiv -z\left(-\frac{1-v^2}{4}\right) k_{\parallel\nu} k_\parallel^\alpha \frac{1}{4}\mathrm{tr}\left\{e^{i\sigma^3\xi}\gamma_\alpha e^{i\sigma^3\eta}\gamma_\mu\right\} \quad =$$

$$z\left(-\frac{1-v^2}{4}\right) k_{\parallel\mu} k_{\parallel\nu}\cos z + \text{odd functions} \tag{55}$$

$$B_3 \equiv -\frac{i}{s}\frac{1}{4}\mathrm{tr}\left\{e^{i\sigma^3\xi}\gamma_{\perp\nu} e^{i\sigma^3\eta}\gamma_\mu\right\} \quad =$$

$$\frac{i}{s}\cos zv\; g_{\mu\nu}^\perp + \text{odd functions} \tag{56}$$

The evaluation of S_3, S_4, S_5 is similar to the proceeding calculation and the synthesis of them into $I_{\mu\nu}$ is as follows:

$$I_{\mu\nu} = 2\sum_{i=1}^{5}\frac{\mathrm{tr} S_i}{4\langle e^{-is\varphi_1}\rangle} = \left(-2m^2 + \frac{1-v^2}{4}k_\parallel^2\right)\left(\cos zv g_{\mu\nu}^\perp + \cos z g_{\mu\nu}^\parallel\right) -$$

$$-(1-v^2)\cos z k_{\parallel\mu}k_{\parallel\nu} + \frac{2i}{s}\left(\cos zv g_{\mu\nu}^\perp + \frac{z}{\sin z}g_{\mu\nu}^\parallel\right) \quad -$$

$$(\cos zv - v\cot z\sin zv)(k_\mu k_\nu - k_{\perp\mu}k_{\perp\nu} - k_{\parallel\mu}k_{\parallel\nu}) \quad +$$

$$\frac{\cos zv - \cos z}{\sin^2 z}\left(g_{\mu\nu}k_\perp^2 - 2k_{\perp\mu}k_{\perp\nu}\right) \tag{57}$$

Further simplifications of $\Pi_{\mu\nu}$ can be achieved by integration by parts. We have:

$$\int_0^\infty ds\frac{i}{s^2}e^{-i\varphi_0}\frac{z}{\sin z}(\cos zv\ g^\perp_{\mu\nu} + \cos z\ g^\parallel_{\mu\nu}) \quad =$$

$$BT + i\int_0^\infty \frac{ds}{s}\left[\frac{z}{\sin z}(\cos zv\ g^\perp_{\mu\nu} + \cos z\ g^\parallel_{\mu\nu})\frac{d}{ds}\left(e^{-is\varphi_0}\right) \quad +\right.$$

$$\left. e^{-is\varphi_0}(\cos zv\ g^\perp_{\mu\nu} + \cos z\ g^\parallel_{\mu\nu})\frac{d}{ds}\left(\frac{z}{\sin z}\right)\right] = BT + i\int_0^\infty \frac{ds}{s}(F_1 + F_2 + F_3) \tag{58}$$

where BT are boundary terms. Differentiations gives

$$F_1 = \frac{z}{\sin z}(\cos zv\ g^\perp_{\mu\nu} + \cos z\ g^\parallel_{\mu\nu})(-i) \quad \times$$

$$\left[m^2 + \frac{1-v^2}{4}k_\parallel^2 - \left(\frac{v\sin zv + \cot z\cos zv}{2\sin z} - \frac{1}{z\sin^2 z}\right)k_\perp^2\right] \tag{59}$$

$$F_2 = -\frac{1}{s}e^{-is\varphi_0}\frac{z}{\sin z}(vz\sin zv\ g^\perp_{\mu\nu} + 2\sin z\ g^\parallel_{\mu\nu}) \tag{60}$$

$$F_3 = e^{-is\varphi_0}(\cos zv\ g^\perp_{\mu\nu} + \cos z\ g^\parallel_{\mu\nu})\frac{1}{s}\frac{z}{\sin z}\left[1 - z\frac{\cos z}{\sin z}\right] \tag{61}$$

After short calculation we then write:

$$\int_0^\infty \frac{ds}{z}\frac{z}{\sin z}e^{-is\varphi_0}\frac{i}{s}(\cos zv\ g^\perp_{\mu\nu} + \cos z\ g^\parallel_{\mu\nu}) = BT + \int_0^\infty \frac{ds}{z}\frac{z}{\sin z}e^{-is\varphi_0} \quad \times$$

$$\left\{\left[m^2 + \frac{1-v^2}{4}k_\parallel^2 - \left(\frac{v\sin zv + \cot z\cos zv}{2\sin z} - \frac{1}{z\sin^2 z}\right)k_\perp^2\right] \quad \times\right.$$

$$(\cos zv\ g^\perp_{\mu\nu} + \cos z\ g^\parallel_{\mu\nu}) \quad +$$

$$\left.\frac{i}{s}\left[\left(\cos z - \frac{z}{\sin z}\right)g^\parallel_{\mu\nu} + [(1 - 2\cot z)\cos zv - zv\sin zv]\,g^\perp_{\mu\nu}\right]\right\} \tag{62}$$

Using the v-integration by parts, we write

$$\int_{-1}^1 \frac{dv}{2}e^{-i\varphi_0}\frac{i}{s}\left[(1 - 2\cot z)\cos zv - 2v\sin zv\right] \quad =$$

$$BT + \int_{-1}^1 \frac{dv}{2}e^{-i\varphi_0}\left[\frac{1}{2}(v\cos zv - \cot z\sin zv\right]\left(vk_\parallel^2 + \frac{\sin zv}{\sin z}k_\perp^2\right) \tag{63}$$

Now, using the results of the s-integration and v-integration we get $\Pi_{\mu\nu}$ after some rearrangement

$$\Pi_{\mu\nu}(k) = \frac{\alpha}{2\pi}\int_0^\infty \frac{ds}{s}\int_{-1}^1 \frac{dv}{2}\left\{e^{-i\varphi_0}\tilde{I}_{\mu\nu} + C.T.\right\} \tag{64}$$

where

$$\tilde{I}_{\mu\nu} = \left(g_{\mu\nu}k^2 - k_\mu k_\nu\right)N_0 - \left(g^{\parallel}_{\mu\nu}k_{\parallel}^2 - k_{\parallel\mu}k_{\parallel\nu}\right)N_1 \quad +$$

$$\left(g^{\perp}_{\mu\nu}k_{\perp}^2 - k_{\perp\mu}k_{\perp\nu}\right)N_2 \tag{65}$$

and

$$N_0 = \frac{z}{\sin z}(\cos zv - v\cot z\sin zv) \tag{66}$$

$$N_1 = -z\cot z\left(1 - v^2 + \frac{v\sin zv}{\sin z}\right) + z\,\frac{\cos zv}{\sin z} \tag{67}$$

$$N_2 = -z\,\frac{\cos zv}{\sin z} + zv\,\frac{\cot z}{\sin z}\sin zv + 2z\,\frac{\cos zv - \cos z}{\sin^3 z} \tag{68}$$

and the contact tems in this case involve the boundary conditions. The contact terms involve also the normalization condition

$$\Pi_{\mu\nu}(k)|_{k^2\to 0;B\to 0} = 0 \tag{69}$$

It requires to investigate φ_0:

$$\varphi_0 = m^2 + \frac{1-v^2}{4}k_{\parallel}^2 + \left(\frac{\cos zv - \cos z}{2z\sin z}\right)k_{\perp}^2 \tag{70}$$

Since

$$\left(\frac{\cos zv - \cos z}{2z\sin z}\right)_{z\to 0} = \frac{1}{4}(1-v^2) \tag{71}$$

then, $k_{\perp}^2$ can be joined with $k_{\parallel}^2$ to make k^2 and we have

$$\varphi_0|_{z\to 0} = m^2 + \frac{1}{4}(1-v^2)k^2 \tag{72}$$

and

$$\varphi_0|_{k^2\to 0,B\to 0} = m^2 \tag{73}$$

Then, we have

$$N_0 = 1 - v^2; \quad N_1 = 0; \quad N_2 = 0; \quad \text{for } z\to 0 \tag{74}$$

and the resulting contact terms are

$$C.T. = e^{-ism^2}(1-v^2)(k^2 g_{\mu\nu} - k_\mu k_\nu) \tag{75}$$

and for $\Pi_{\mu\nu}(k)$ we get

$$\Pi_{\mu\nu}(k) = \frac{\alpha}{2\pi}\int_0^\infty \frac{ds}{s}\int_{-1}^1 \frac{dv}{2}\left\{e^{-i\varphi_0}\left[(g_{\mu\nu}k^2 - k_\mu k_\nu)N_0 \quad - \right.\right.$$

$$\left(g^{\|}_{\mu\nu}k^2_{\|} - k_{\|\mu}k_{\|\nu}\right)N_1 + \left(g^{\perp}_{\mu\nu}k^2_{\perp} - k_{\perp\mu}k_{\perp\nu}\right)N_2\Big] \quad -$$

$$\left. e^{-ism^2}(1-v^2)(g_{\mu\nu}k^2 - k_\mu k_\nu)\right\} \tag{76}$$

3 Discussion

The purpose of this paper was to present a complete and explicit result by using a different approach that is an extension of the simple and transparent method proposed by Tsai to calculate the photon mass operator in an external homogeneous magnetic field. The main idea was to calculate the usual vacuum polarization "bubble" of quantum electrodynamics (photon mass operator $\equiv$ $\{\gamma \to (e^- + e^+) \to \gamma\}$) using an explicit momentum representation for the exact electron propagator in the external fields.

References

Akhiezer, A. I. and Berestetskii, V. B. *Quantum Electrodynamics* (Wiley, New York, 1965)

Berestetskii, V. B., Lifschitz, E. M. and Pitaevskii, L. P. *Quantum Electrodynamics*,(2nd ed. Oxford, England: Pergamon Press, p. 596, 1982).

Dittrich, W., (1978). Source methods in quantum field theory, Fortschritte der Physik **26**, 289.

Dittrich, W. and Reuter, M. (1978). *Effective Lagrangians in Quantum electrodynamics*, (Lecture Notes in Physics, Springer Verlag, Berlin, Heidelberg, New York, Tokyo),

Schwinger, J. *Particles and Sources*,
(Gordon and Breach, Science Publishers, New york, London, Paris, 1969).

Schwinger, J. *it Particles, Sources and Fields I.*,
(Addison-Wesley Publishing Company, Reading, Mass. 1970).

Schwinger, J. *Particles, Sources and Fields II.*, (Addison-Wesley Publishing Company, Reading, Mass. 1973).

Schwinger, J. *Particles, Sources, and Fields III.*, (AddisonWesley, Reading, Mass. 1989).

Tsai, Wu-Y. (1974). Modified electron propagation function in strong magnetic fields, Phys. Rev. **10**, No. 4, pp. 1342-1345.

Urrutia, L. F. (1978). Vacuum polarization in parallel homogeneous
electric and magnetic fields, Phys. Rev. D bf 17, No 8.