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# Higher Genus curves in Generalized Riemann hypothesis, and Generalized Birch and Swinnerton-Dyer conjecture

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## Abstract

Whereby all the infinitely-many prime numbers are classified as [well-defined] Incompletely Predictable entities, so must all the infinitely-many nontrivial zeros be classified as such. We outline interesting observations and conjectures about distribution of nontrivial zeros in L-functions; and [optional] use of Sign normalization when computing Hardy Z-function, including their relationship to Analytic rank and Symmetry type of L-functions. When Sign normalization is applied to eligible L-functions, we posit its dependency on even-versus-odd Analytic ranks, degree of L-function, and particular gamma factor present in the functional equations for Genus 1 elliptic curves and higher Genus curves. By carefully applying inclusion-exclusion principle, our mathematical arguments are postulated to satisfy Generalized Riemann hypothesis, and Generalized Birch and Swinnerton-Dyer conjecture. We explicitly mention underlying proven / unproven hypotheses or conjectures. We provide Algebraic-Transcendental proof (Proof by induction) as supplementary material for open problem in Number theory of Riemann hypothesis whereby it is proposed all nontrivial zeros of Riemann zeta function are located on its Critical line.

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Keywords: Birch and Swinnerton-Dyer conjecture, Dirichlet density, Natural density, Polignac's and Twin prime conjectures, Riemann hypothesis, Thick set, Thin set

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## Contents



## 1. **Introduction**

As with many famous open problems in Number theory such as Riemann hypothesis, Polignac's and Twin prime conjectures, and Birch and Swinnerton-Dyer conjecture; the *basic questions are easy to state but difficult to resolve or reconcile.* Sir Isaac Newton in 1675 wrote the expression: "If I have seen further [than others], it is by *standing on the shoulders of Giants* (Latin: *nani gigantum humeris insidentes*)". This famous metaphor meant *discovering truth by building on previous discoveries.* Above colloquial phrases are prophetically true in Mathematics for Incompletely Predictable Problems (MIPP) for these intractable open problems, whereby we explicitly mention relevant underlying proven / unproven hypotheses or conjectures in this paper.

Equations and inequalities are mathematical sentences formed by relating two expressions to each other. In an equation, two expressions are deemed equal as indicated by symbol "=" [viz, an equation contains equality relationship]. MIPP is valid for various chosen functions, equations or algorithms that contain equality relationship[9]. Eligible functions or algorithms are literally quasi-equations containing "analogical" equality relationship. For instance: Origin point intercepts ≡ Gram[x=0, y=0] points in Analytically continued *proxy* Dirichlet eta function = {Set of All Nontrivial zeros of Riemann zeta function}. Algorithm *Sieve-of-Eratosthenes* $\equiv$ All Integers greater than 1 with exactly two factors, 1 and the number itself = {Set of All Prime numbers}.

In an inequality, two expressions are not necessarily equal as indicated by symbols "$>$", "$<$", "$\leq$" or "$\geq$". Various selected number sequences from On-Line Encyclopedia of Integer Sequences (OEIS) are precisely defined by inequalities. In particular, when easily applied to OEIS number sequences such as A100967[6] and A228186[8], one can deductively show using proven mathematical arguments that MIPP is also valid for an inequality.

An L-function is a meromorphic function on complex plane, associated to one out of several categories of mathematical objects. For the Generic L-functions [*aka* General L-functions] that include dual L-functions and self-dual L-functions theoretically arising from Maass forms, Genus 0, 1, 2, 3, 4, 5... curves, etc; we compare and contrast these two types of L-functions, and show the different forms of symmetry being manifested by Z(t) plots of nontrivial zeros (spectrum).

Genus of a connected, orientable surface is an integer representing the maximum number of cuttings along non-intersecting closed simple curves without rendering the resultant manifold disconnected. Topologically, it is equal to number of "holes" or "handles" on it. Alternatively, it is defined in terms of Euler characteristic $\chi$ via relationship $\chi = 2 - 2g$ for closed surfaces where $g$ is Genus. For surfaces with $b$ boundary components, the equation reads $\chi = 2 - 2g - b$. Genus 0, 1, 2, 3, 4, 5,... curves have 0, 1, 2, 3, 4, 5,... holes.

In classical algebraic geometry, the genus-degree formula relates degree $d$ of an irreducible plane curve $C$ with its arithmetic genus $g$ via formula: $g = \frac{1}{2}(d-1)(d-2)$. Here the "plane curve" means that $C$ is a closed curve in the projective plane $\mathbb{P}^2$. If the curve is non-singular, geometric genus and arithmetic genus are equal, but if the curve is singular with only ordinary singularities, geometric genus is smaller. More precisely, an ordinary singularity of multiplicity $r$ decreases the genus by $\frac{1}{2}r(r-1)$.

## 2. **Various Number systems and Inclusion-Exclusion principle**

Hyperreal numbers extend the real numbers to include certain classes of infinite numbers and infinitesimal numbers. Surreal numbers belong to a totally ordered proper class containing real numbers, infinite numbers and infinitesimal numbers that are larger or smaller in absolute value than any positive real number. Quaternion number system extends complex numbers. Quaternions have expression of the form $a + b\,\mathbf{i} + c\,\mathbf{j} + d\,\mathbf{k}$ , where $a$, $b$, $c$, $d$ are real numbers; $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1, \mathbf{i}\,\mathbf{j} = -\mathbf{j}\,\mathbf{i} = \mathbf{k}, \mathbf{j}\,\mathbf{k} = -\mathbf{k}\,\mathbf{j} = \mathbf{i}, \mathbf{k}\,\mathbf{i} = -\mathbf{i}\,\mathbf{k} = \mathbf{j}$.

Integer numbers $\mathbb{Z} \subset$ Rational numbers $\mathbb{Q} \subset$ Real numbers $\mathbb{R} \subset$ Complex numbers $\mathbb{C}$. Natural numbers $\mathbb{N}$ {1, 2, 3, 4, 5...} $\subset$ Whole numbers $\mathbb{W}$ {0, 1, 2, 3, 4, 5...} $\subset$ Integer numbers $\mathbb{Z}$ {...−3, −2, −1, 0, 1, 2, 3...}. The pairing of Even numbers $\mathbb{E}$ {0, 2, 4, 6, 8, 10...} and Odd numbers $\mathbb{O}$ {1, 3, 5, 7, 9, 11...}, and the pairing of Prime numbers $\mathbb{P}$ {2, 3, 5, 7, 11, 13, 17, 19, 23...} and Composite numbers **C** {4, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20...} can be separately combined to form $\mathbb{W}$ whereby {0, 1} are neither prime nor composite. Complex number $z = a + bi$ where imaginary unit $i = \sqrt{-1}$; $a, b \in \mathbb{R}$; and when $b = 0$, $z$ becomes a real number. $\mathbb{Q} = \dfrac{p}{q}$ where $p, q \in \mathbb{Z}$; $\mathbb{Q}$ are $\mathbb{Z}$ when $p = 1$; and when $q = 0$, $\mathbb{Q}$ is undefined.

Irrational numbers $\mathbb{R}\backslash\mathbb{Q} \subset$ Real numbers $\mathbb{R}$ or Complex numbers $\mathbb{C}$. $\mathbb{R}\backslash\mathbb{Q}$ = [I] Algebraic (irrational) numbers [viz, $\mathbb{R}$ or $\mathbb{C}$ that are root of a non-zero polynomial of finite degree in one variable with integer or, equivalently, rational coefficients e.g. golden ratio $(1+\sqrt{5})/2$, $\sqrt{2}$, $\sqrt[3]{2}$, etc] + [II] Transcendental (irrational) numbers [viz, $\mathbb{R}$ or $\mathbb{C}$ that are not the root of a non-zero polynomial of finite degree in one variable with integer or, equivalently, rational coefficients e.g. $\pi$, $e$, $\ln 2$]. The solitary even Prime number {2} forms a Countably Finite Set (CFS). $\mathbb{E}$, $\mathbb{O}$, $\mathbb{P}$, **C**, $\mathbb{N}$, $\mathbb{W}$, $\mathbb{Z}$, $\mathbb{Q}$ and Algebraic numbers form Countably Infinite Sets (CIS). Transcendental numbers, $\mathbb{R}\backslash\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ form Uncountably Infinite Sets (UIS).

In combinatorics [that deals with counting and arrangements], the inclusion-exclusion principle is a counting technique which generalizes the familiar method of obtaining number of elements in union of two or more sets when these sets may have overlaps. In essence, this principle removes all contributions from over-counted elements in sets and subsets. Instead of using raw *cardinality* from Pure Set theory when / if relevant, we should selectively use Measure theory such as length, area, probability (or proportion), Natural density (a.k.a. Asymptotic density, used when [for example] there is no uniform probability distribution over Natural numbers), and Dirichlet density (useful analytic tool for thin sets like set of Prime numbers that do not have well-defined Natural density; and with deep connections to Riemann zeta function, prime distribution and analytic number theory). Only under strict convergence conditions that any resultant infinite alternating series converge absolutely, this principle is valid for CFS, CIS or UIS [irrespective of whether there are finite or infinite number of these CFS, CIS or UIS]. We shall succinctly adapt or adopt this principle into various relevant mathematical arguments, lemmas, propositions, corollaries, axioms or theorems in this paper.

An example based on Measure theory: (Step 1) Define and measure two lengths as two UIS of Set $A$ and Set $B$ using two intervals of Real numbers on number line; viz, two individual "continuous lengths" are both [quantitatively] infinite $\mu(A) = 2 - 0 = 2$ and $\mu(B) = 4 - 1 = 3$. (Step 2) Measure the intersection as $A \cap B = [1, 2] \implies \mu(A \cap B) = 2 - 1 = 1$. (Step 3) Apply inclusion-exclusion principle $\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B) = 2 +$

$3 - 1 = 4 \implies$ total "discrete length" covered by both intervals is 4; viz, total "size" is [qualitatively] finite. Mainly based on Number theory, Mobius function $\mu(n)$ that connects deeply with Euler's totient function, zeta functions, and multiplicative number theory also gives a powerful compact formula for inclusion-exclusion principle over divisibility conditions.

Let $A$, $B$, $C$,... be finitely large sets or infinitely large sets, and $|S|$ indicates the cardinality of a set S ($\equiv$ 'number of elements' for set S). For CFS e.g. *Set* of even Prime number $= \{2\}$ with cardinality $= 1$, *Set* of odd Prime number with last-digit ending in 5 $= \{5\}$ with cardinality $= 1$; CIS e.g. *Set* of odd Prime numbers $= \{3, 5, 7, 11, 13, 17, 19...\}$ with cardinality $= \aleph_0$; and UIS e.g. *Set* of Real numbers with cardinality $= \mathfrak{c}$ (*cardinality of the continuum*). The inclusion-exclusion principle for three sets is given by $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$. This formula expresses the fact that sum of sizes for these three sets may be too large since some elements may be counted twice (two times) or thrice (three times). General formula for a finite number of sets [with alternating signs $+$, $-$, $+$, $-$,... that depends on number of sets in the intersection] is $\left|\bigcup_{i=1}^{n} A_i\right| =$

$$\sum_{i=1}^{n} |A_i| - \sum_{1 \leq i < j \leq n} |A_i \cap A_j| + \sum_{1 \leq i < j < k \leq n} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n+1} |A_1 \cap \cdots \cap A_n|.$$

In Probability theory, this formula for a finite number of sets is $\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) =$

$$\sum_{i=1}^{n} \mathbb{P}(A_i) - \sum_{i<j} \mathbb{P}(A_i \cap A_j) + \sum_{i<j<k} \mathbb{P}(A_i \cap A_j \cap A_k) + \cdots + (-1)^{n-1} \mathbb{P}\left(\bigcap_{i=1}^{n} A_i\right).$$

In closed form, this formula is $\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{k=1}^{n} \left( (-1)^{k-1} \sum_{\substack{I \subseteq \{1,\ldots,n\} \\ |I|=k}} \mathbb{P}(A_I) \right)$, where the last sum runs over all subsets $I$ of indices 1, ..., $n$ which contain exactly $k$ elements, and $A_I := \bigcap_{i \in I} A_i$ denotes intersection of all those $A_i$ with index in $I$. This formula for an infinite number of sets [strict convergence for infinite alternating series] is $\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{k=1}^{\infty} \left( (-1)^{k-1} \sum_{\substack{I \subseteq \{1,\ldots,\infty\} \\ |I|=k}} \mathbb{P}(A_I) \right)$.

For a general measure space $(S, \Sigma, \mu)$ and measurable subsets $A_1, A_2, ..., A_n$ [or $A_1, A_2, ..., A_n, A_{n+1}, A_{n+2}, ..., A_\infty$] of finite [or infinite measure], the above identity also hold when probability measure $\mathbb{P}$ is replaced by the measure $\mu$.

## 3. **Completely Predictable, Incompletely Predictable and Completely Unpredictable entities**

*Definition* 3.1. Where all infinitely-many prime [and composite] numbers are classified as Pseudo-random entities, so must all nontrivial zeros be classified as such. Pseudo-random entities are Incompletely Predictable entities. Largely based on p. 18 of [9], we provide formal definitions for three types of [infinitely-many] entities as Countably Infinite Sets in a succinct manner. With "Entity X" forming a Countably Infinite Set and irrespective of whether "Entity X" are Completely or Incompletely Predictable entities, we consistently define "$n^{th}$ Gap of Entity X" = "$(n+1)^{th}$ Entity X" − "$(n)^{th}$ Entity X".

*Completely Unpredictable* (non-deterministic) entities are [the statistically] defined as entities that are actually random and DO behave like one e.g. [true] random number generator that supply sequences of entities (as non-distinct Sets of numbers) that are not reproducible; viz, these entities do not contain any repeatable spatial or temporal patterns. We work in base-10 system (a.k.a. decimal system) that represent numbers using ten unique digits {0, 1, 2, 3, 4, 5, 6, 7, 8, 9}. For $n$ = {1, 2, 3, 4, 5,...}, and with our [true] random number generator also utilizing these ten unique digits to supply $n^{th}$ Entity as $n \to \infty$; then Probability (P) of independently obtaining each digit is P(0) = P(1) = P(2) = P(3) = P(4) = P(5) = P(6) = P(7) = P(8) = P(9) $\simeq \frac{1}{10} \simeq 0.1 \simeq 10\%$.

*Completely Predictable* (deterministic) entities are defined as entities that are actually not random and DO NOT behave like one e.g. non-overlapping distinct Set of Even numbers {0, 2, 4, 6, 8, 10,...} and Set of Odd numbers {1, 3, 5, 7, 9, 11,...}; viz, these entities are reproducible. Chosen "Even [or Odd] Gap", as [non-varying] integer number value 2 between any two adjacent Even [or Odd] numbers, always consist of a fixed value. The distinct Sets of trivial zeros from various L-functions [as infinitely-many negative integers] are other examples of Completely Predictable entities. Both Riemann zeta function and its *proxy* Dirichlet eta function have simple zeros at each even negative integer $s = -2n$ where $n$ = 1, 2, 3, 4, 5,...; viz, $s = -2, -4, -6, -8, -10, ....$ In addition, the factor $1 - 2^{1-s}$ in Dirichlet eta function adds an infinite number of [Completely Predictable] complex simple zeros, located at equidistant points on the line $\Re(s) = 1$, at $s_n = 1 + \frac{2n\pi i}{\ln(2)}$ whereby $n = ..., -3, -2, -1, 1, 2, 3, ...$ is any nonzero integer and $i$ is the imaginary unit satisfying equation $i^2 = -1$.

*Incompletely Predictable* [or *Pseudo-random*] (deterministic) entities are defined as entities that are actually not random but DO behave like one e.g. non-overlapping distinct Set of Prime numbers {2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31...} and Set of Composite numbers {4, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20, 21, 22, 24, 25, 26, 27, 28...}; viz, these entities are reproducible. Chosen

"Prime [or Composite] Gaps", as [varying] integer number values between any two adjacent Prime [or Composite] numbers, will never consist of a fixed value. Examples: Set of Prime Gaps = {1, 2, 2, 4, 2, 4, 2, 4, 6, 2,...} and Set of Composite Gaps = {2, 2, 1, 1, 2, 2, 1, 1, 2, 2, 1, 1, 2, 1, 1, 1, 1, 2,...}. The (odd) Prime Gap 1 indicates the only (solitary) even Prime number 2. All infinitely many (odd) Prime numbers have (even) Prime Gaps 2, 4, 6, 8, 10,... [to infinitely large size or, more precisely, to an arbitrarily large number] at sufficiently large integer range. One can conveniently and arbitrarily classify small Prime Gaps to be 2 and 4, and large Prime Gaps to be $\geq 6$. Only two finite integer number values {1, 2} represent Composite Gaps. Occurrences of (even) Composite Gap 2 in the specific Composite even number that always follow an (odd) Prime number are thus associated with appearances of all (odd) Prime numbers. The cardinality of consecutive (odd) Composite Gap 1 $\propto$ size of (even) Prime Gaps; viz, cardinality of consecutive Gap 1-Composite numbers = even Prime Gap − 2 with cardinality of Gap 1-Composite even numbers $= \frac{\text{even Prime Gap} - 2}{2}$ and cardinality of Gap 1-Composite odd numbers = $\frac{\text{even Prime Gap} - 2}{2}$. Note that Gap 2-Prime numbers (twin primes) do not have Gap 1-Composite numbers. The inclusion-exclusion principle for two sets $|A \cup B| = |A| + |B| - |A \cap B|$. |All Even numbers $\cap$ All Prime numbers| = 1, which represent the only even Prime number 2. All Prime numbers [with exception of even Prime number 2] are (almost totally) constituted by Odd numbers. All odd Prime numbers are (totally) constituted by Odd numbers [although the majority of Odd numbers are not odd Prime numbers].

Apart from integers, Incompletely Predictable entities can also be constituted from other number systems e.g. distinct Sets of $t$-valued transcendental numbers that faithfully represent infinitely-many nontrivial zeros (spectrum) of dual or self-dual L-functions. Geometrically, all nontrivial zeros of L-functions are simply the "Origin point intercepts". L-functions [e.g. from the Genus 1 elliptic curves representing self-dual L-functions] can have Analytic rank 0, 1, 2, 3, 4, 5,... [to an arbitrarily large number]; viz, have "solitary" (zero) Analytic rank and "all other" (nonzero) Analytic rank. Thus, it seems that most L-functions should "qualitatively" have MORE (nonzero) Analytic rank = 1, 2, 3, 4, 5,... and LESS (zero) Analytic rank = 0. Only (zero) Analytic rank L-functions, such as from Genus 0 (non-elliptic) Riemann zeta function [and its *proxy* Dirichlet eta function] and selected (Analytic rank 0) Genus 1 elliptic curves, DO NOT HAVE first nontrivial zeros with $t$ value = 0 [viz, an algebraic number]. Then, all (nonzero) Analytic rank L-functions DO HAVE first nontrivial zeros with $t$ value = 0 [viz, an algebraic number].

In Riemann hypothesis or Generalized Riemann hypothesis, all nontrivial zeros (NTZ) are conjecturally *only* located on $\Re(s) = \frac{1}{2}$-Critical line or Analytically normalized $\Re(s) = \frac{1}{2}$-Critical line. Chosen "NTZ Gaps", as [varying] transcendental number values, between any two adjacent NTZ never consist of a fixed value. All infinitely-many NTZ are Incompletely Predictable entities. Note the infinitely-many digits after decimal point of each (algebraic) or (transcendental) irrational number are also Incompletely Predictable entities whereby individual irrational number has greater precision or accuracy when it is computed as having increasing number of digits.

*Remark* 3.1. We use abbreviations: CP = Completely Predictable, IP = Incompletely Predictable, CFS = Countably finite sets, CIS = Countably infinite sets, UIS = Uncountably infinite sets. We compare and contrast Sets, Subsets, Even $k$-tuple and Prime $k$-tuple when derived from CP entities versus IP entities. There is only one mathematical possibility for CIS having CP or IP entities: Cardinality of *different or changing values* denoting the "Gaps" between any two adjacent elements in CIS with CP entities must be CFS. Cardinality of *different or changing values* denoting the "Gaps" between any two adjacent elements in CIS with IP entities must be CIS. Broadly applying inclusion-exclusion principle to two or more [mutually exclusive] cardinalities:

We can never obtain CIS having both CP entities and IP entities. In a similar manner, irrespective of having CP entities or IP entities, a given set must simply be UIS, CIS or CFS [and cannot be a mixture of UIS, CIS and/or CFS]. Subsets of CP entities are "non-unique and overlapping" e.g. Derived from Set of Gap 2-Even numbers (Twin Even numbers) = {0, 2, 4, 6, 8, 10,...}: Subset of Gap 4-Even numbers (Cousin Even numbers) = {0, 4, 8, 12, 16, 20,...}, Subset of Gap 6-Even numbers (Sexy Even numbers) = {0, 6, 12, 18, 24, 30,...}, etc. Subsets of IP entities are "unique and non-overlapping" e.g. Derived from Set of All Prime numbers = {2, 3, 5, 7, 11, 13,...}: Subset of Gap 2-Prime numbers (Twin Primes) = {3, 5, 11, 17, 29, 41,...}, Subset of Gap 4-Prime numbers (Cousin Primes) = {7, 13, 19, 37, 43, 67,...}, Subset of Gap 6-Prime numbers (Sexy Primes) = {23, 31, 47, 53, 61, 73,...}, etc.

For $k$ = 2, 3, 4, 5, 6,... and $n$ = 2, 3, 5, 7, 11, 13, 17, 19, 23, 29,...; the diameter of a Prime $k$-tuple is difference of its largest and smallest elements. Note the special case of $k = 2$ simply corresponds to Gap 2-prime numbers (Twin primes). An admissible Prime $k$-tuple with smallest possible diameter $d$ (among all admissible Prime $k$-tuples) is a Prime constellation $\equiv$ Prime $k$-tuplet. Prime constellations manifest the Incompletely Predictable property whereby certain prime numbers are "non-unique and overlapping" represented e.g. When $k = 3$, $d = 6$: Constellation (0, 2, 6) $\equiv$ [smallest] prime numbers (5, 7, 11) with chosen $n = 5$; Constellation (0, 4, 6) ≡ [smallest] prime numbers (7, 11, 13) with chosen $n = 7$. When $k = 4$, $d = 8$: Constellation (0, 2, 6, 8) ≡ [smallest] prime numbers (5, 7, 11, 13) with chosen $n = 5$. For all $n \geq k$ this will always produce consecutive Primes. Recall from above that all $n$ are integers for which values $(n+a, n+b, n+c, ...)$ are prime numbers. This means that, for large $n$: $p_{n+k-1} - p_n \geq d$ where $p_n$ is the $n^{th}$ prime number.

We intuitively infer from above synopsis in previous two paragraphs that only by analyzing non-overlapping Subsets of even Prime gaps 2, 4, 6, 8, 10,... [instead of analyzing overlapping Prime $k$-tuples or Prime $k$-tuplets] would we obtain the rigorous proofs for Polignac's and Twin prime conjectures.

For $k = 2, 3, 4, 5, 6,...$ and $n = 0, 2, 4, 6, 8, 10, 12, 14, 16,...$; the diameter of an Even $k$-tuple is difference of its largest and smallest elements. An admissible Even $k$-tuple with smallest possible diameter $d$ (among all admissible Even $k$-tuples) is an Even constellation ≡ Even $k$-tuplet. Even constellations manifest the Completely Predictable property whereby even numbers are "unique and non-overlapping" represented e.g. When $k = 4$, $d = 2(k-1) = 6$: Constellation (0, 2, 4, 6) ≡ [smallest] even numbers (0, 2, 4, 6) with chosen $n = 0$ or [using larger] even numbers (102, 104, 106, 108) with arbitrarily chosen $n = 102$. For all $n$ [as fully obtained from $n < k$ and $n \geq k$], this will always produce consecutive even numbers. Recall from above that all $n$ are integers for which values $(n + 2, n + 4, n + 6, ...)$ are even numbers. This means that, for all $n$: $E_{n+k-1} - E_n = d$ where $E_n$ is the $n^{th}$ even number. Observe we could instead use odd numbers that will also produce the same equally valid deductions.

Gram's rule says there is exactly one nontrivial zero (NTZ) ≡ Gram[x=0, y=0] point in Riemann zeta function between any two Gram points ≡ Gram[y=0] points. A Gram block is an interval bounded by two "good" Gram points such that all Gram points between them are "bad". Rosser's rule says Gram blocks often have the expected number of NTZ in them [viz, NTZ is "conserved" and is the same as the number of Gram intervals], even though some individual Gram intervals in the block may not have exactly one NTZ in them [viz, some of the individual Gram intervals in the block violate Gram's rule]. Both Gram's rule and Rosser's rule say in some sense NTZ do not stray too far from their expected positions. Violations of Gram's rule equate to intermittently observable geometric variants of two consecutive (+ve first and then −ve) Gram points [≡ missing NTZ] that is alternatingly followed by two consecutive NTZ [≡ extra NTZ]. The rarer violations of Rosser's Rule equate to intermittently observable geometric variants of reduction in expected number of x-axis intercept points. They both fail infinitely many times in a +ve proportion of cases. We expect in ∼66% one NTZ is enclosed by two successive Gram points, but in ∼17% no NTZ and in ∼17% two NTZ are in such a Gram interval on the long run.

The success and failures of both Gram's rule and Rosser's rule occur in Dirichlet eta function [*proxy* for Riemann zeta function] on $\sigma = \frac{1}{2}$-Critical line. An insightful inference with deep connection to Riemann hypothesis: Only by analyzing non-overlapping Subset of "One NTZ" = ~66%, Subset of "Zero NTZ" = ~17%, and Subset of "Two NTZ" = ~17% as precisely derived from Set of "All NTZ" = "conserved" 100% [instead of analyzing various overlapping Gram blocks and Gram intervals containing "good" or "bad" Gram points, missing NTZ or extra NTZ] can we rigorously prove Riemann hypothesis.

$INPUT \rightarrow \boxed{\textit{White Box}}$ or $\boxed{\textit{Black Box}} \rightarrow OUTPUT$. In general, "$\rightarrow$" must be replaced by "$\rightleftharpoons$" to indicate bidirectional reversibility. White Box (or Black Box) is a system where its [unique] inner components or logic are (or are not) available for inspection. Key ideas for computer & mathematical systems as White Box or Black Box $INPUT \rightarrow$ (unique) CP vs IP $\boxed{\textit{Information processor}}$ & $\boxed{\textit{Mathematical function, equation or algorithm}} \rightarrow$ (reproducible) CP vs IP $OUTPUT$. Examples of Mathematical function, equation or algorithm:

CP CIS $k^{th}$ Even numbers are Integers $\{0, \pm 2, \pm 4, \pm 6, \pm 8, \pm 10, ...\}$ [$\equiv$ $OUTPUT$] faithfully given by equation $n = \pm 2k$ [$\equiv$ *White Box*], where $k$ are Integers $\{0, 1, 2, 3, 4, 5, ...\}$ [$\equiv$ $INPUT$]. Since Even $n$ are integrally divisible by 2, congruence $n = 0 (\text{mod } 2)$ holds for Even $n$. The generating function of Even numbers is $\frac{2x}{(x-1)^2} = 2x^1 + 4x^2 + 6x^3 + 8x^4 + \cdots$.

CP CIS $k^{th}$ Odd numbers are Integers $\{\pm 1, \pm 3, \pm 5, \pm 7, \pm 9, \pm 11, ...\}$ [$\equiv$ $OUTPUT$] faithfully given by equation $n = \pm(2k - 1)$ [$\equiv$ *White Box*], where $k$ are Integers $\{1, 2, 3, 4, 5, ...\}$ [$\equiv$ $INPUT$]. Since Odd $n$ when divided by 2 leave a remainder 1, congruence $n = 1 (\text{mod } 2)$ holds for Odd $n$. The generating function of Odd numbers is $\frac{x(1+x)}{(x-1)^2} = 1x^1 + 3x^2 + 5x^3 + 7x^4 + \cdots$. The oddness of a number is called its parity, so an Odd number has parity 1 (Odd Parity), while an Even number has parity 0 (Even Parity). The product of an Even number and an Odd number is always Even, as can be seen by writing $(2k)(2l + 1) = 2[k(2l + 1)]$, which is divisible by 2 and hence is Even.

IP "decelerating"-CIS $k^{th}$ Prime numbers are Integers $\{\pm 2, \pm 3, \pm 5, \pm 7, \pm 11, \pm 13, \pm 17, \pm 19, \pm 23, \pm 29, \pm 31, \pm 37, ...\}$ [$\equiv$ $OUTPUT$] faithfully given by algorithm $\pm$*"Sieve-of-Eratosthenes"* [$\equiv$ *White Box*], where $k$ are Integers $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, ...\}$ [$\equiv$ $INPUT$]. A Prime number is an Integer greater than 1 with exactly two factors, 1 and the number itself.

IP "accelerating"-CIS $k^{th}$ Composite numbers are Integers $\{\pm 4, \pm 6, \pm 8, \pm 9, \pm 10, \pm 12, \pm 14, \pm 15, \pm 16, \pm 18, \pm 20, \pm 21, ...\}$ [$\equiv$ $OUTPUT$] faithfully given by algorithm $\pm$*"Complement-Sieve-of-Eratosthenes"* [$\equiv$ *White Box*], where $k$ are Integers $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, ...\}$ [$\equiv$ $INPUT$]. A Composite number is an Integer greater than 1 with more than two factors (including 1 and the number itself).

IP CIS $k^{th}$ NTZ are Complex numbers $s = \sigma \pm it = \frac{1}{2} \pm it$ that are traditionally denoted by $t$-valued Transcendental numbers $\{\pm 14.13, \pm 21.02, \pm 25.01, \pm 30.42, \pm 32.93, \pm 37.58,...\}$ [$\equiv$ *OUTPUT*] as faithfully satisfied by equation *"Riemann zeta function $\zeta(s) = 0$" / "Dirichlet eta function $\eta(s) = 0$"* [$\equiv$ *White Box*], where $k$ are Integers $\{1, 2, 3, 4, 5, 6,...\}$ [$\equiv$ *INPUT*]. All NTZ are proposed in 1859 Riemann hypothesis to lie on $\sigma = \frac{1}{2}$-Critical Line.

For i = 1, 2, 3,..., $\infty$; let i$^{th}$ Even number = $E_i$ and i$^{th}$ Odd number = $O_i$. We can precisely, easily and independently calculate e.g. $E_5 = (2 \times 5) = 10$ and e.g. $O_5 = (2 \times 5) - 1 = 9$. A generated CP number is *locationally defined* as a number whose i$^{th}$ position is independently determined by simple calculations without needing to know related positions of all preceding numbers - this is a "reproducible" Universal Property. The congruence n $\equiv$ 0 (mod 2) holds for positive even numbers ($n$). The congruence n $\equiv$ 1 (mod 2) holds for positive odd numbers ($n$). Then the zeroeth Even number $E_0 = (2 \times 0) = 0$ must exist.

For i = 1, 2, 3,..., $\infty$; let i$^{th}$ Prime number = $P_i$ and i$^{th}$ Composite number = $C_i$. We can precisely, tediously and dependently compute e.g. $C_6 = 12$ and $P_6 = 13$: 2 is 1$^{st}$ prime, 3 is 2$^{nd}$ prime, 4 is 1$^{st}$ composite, 5 is 3$^{rd}$ prime, 6 is 2$^{nd}$ composite, 7 is 4$^{th}$ prime, 8 is 3$^{rd}$ composite, 9 is 4$^{th}$ composite, 10 is 5$^{th}$ composite, 11 is 5$^{th}$ prime, 12 is 6$^{th}$ composite, 13 is 6$^{th}$ prime, etc. Our desired integer 12 is the 6$^{th}$ composite and integer 13 is the 6$^{th}$ prime. A generated IP number is *locationally defined* as a number whose i$^{th}$ position is dependently determined by complex calculations with needing to know related positions of all preceding numbers - this is a "reproducible" Universal Property. Observe that integers $\{0, 1\}$ are neither prime nor composite.

*Remark* 3.2. **Natural and Dirichlet density in Thin and Thick set:** Natural density of a Set $A \subseteq \mathbb{N}$ is: $d(A) = \lim_{n\to\infty} \frac{|A \cap \{1, 2, 3, ..., n\}|}{n}$. If this limit exist, it measure how "large" a subset of the set of natural numbers is. It relies chiefly on the probability of encountering members of the desired subset when combing through the interval $[1, n]$ as $n$ grows large. We have $0 \leq d(A) \leq 1$: If $d(A) = 1$, the set is thick or co-dense (almost everything is in $A$). If $d(A) = 0$, the set is thin or sparse.

Let $A \subseteq \mathbb{P}$ be a subset of prime numbers. Dirichlet density of $A$ is defined as: $\delta(A) = \lim_{s\to 1^+} \frac{\sum_{p\in A} \frac{1}{p^s}}{\sum_{p} \frac{1}{p^s}}$ provided this limit exist. Since the prime zeta function [an analogue of Riemann zeta function] $\sum_{p} \frac{1}{p^s} \sim \log_e(\frac{1}{s-1})$ as $s \to 1^+$, we also have $\delta(A) = \lim_{s\to 1^+} \frac{\sum_{p\in A} \frac{1}{p^s}}{\log_e(\frac{1}{s-1})}$. This expression is usually the order of the "pole" of $\prod_{p\in A} \frac{1}{1-\frac{1}{p^s}}$ at $s = 1$, (though in general it is not really a pole as it has non-integral order), at least if this function is a holomorphic function times a (real) power of $s-1$ near $s = 1$. Dirichlet density is useful when Natural density is undefined or hard to compute. It is especially effective for sets of primes in arithmetic progression. It bridges combinatorics with complex analysis through the zeta and $L$-functions. If Natural density exists, Dirichlet density also exists, and they are equal [but not the opposite way]. If $A$ is the set of all primes, it is the Riemann zeta function which has a pole of order 1 at $s = 1$, so the set of all primes has Dirichlet density 1.

Set $A$ is a *thin set* if it has zero Natural density, and is sparse or rare among Natural numbers; viz, it becomes vanishingly small compared to Natural numbers as you go to infinity. Example: The [earliest] ancient Euclid's Proof of the infinitude of $\mathbb{P}$ (c. 300 BC) utilize *reductio ad absurdum*. Set of all (odd) Prime numbers $\mathbb{P}$ is a "decelerating CIS" and a "thin set". Let $\mathbb{P}$-$\pi(n)$ be the Prime counting function defined as number of primes $\leq n$. Prime number theorem tells us $\mathbb{P}\text{-}\pi(n) \sim \frac{n}{\log_e n}$. With Prime Gaps = Set of $\mathbb{E}$ = $\{2, 4, 6, 8, 10...\}$ being Arbitrarily Large in Number as you go to infinity, the Natural density for All odd $\mathbb{P}$ Set [$= \sum_{n\in\mathbb{E}}$ Gap $n$-$\mathbb{P}$] that "decelerates to an infinitesimal small number value just above zero" is given by $\lim_{n\to\infty} \frac{\mathbb{P}\text{-}\pi(n)}{n} = \lim_{n\to\infty} \frac{1}{\log_e n} = 0$. We recognize that Gap 2-$\mathbb{P}$ Subset, Gap 4-$\mathbb{P}$ Subset, Gap 6-$\mathbb{P}$ Subset, Gap 8-$\mathbb{P}$ Subset, Gap 10-$\mathbb{P}$ Subset,... being proposed to all consist of "decelerating CIS" and "thin sets" would imply the 1849-dated Polignac's conjecture [regarding all even Prime Gaps 2, 4, 6, 8, 10...] and the 1846-dated Twin prime conjecture [regarding "subset" even Prime Gaps 2] to both be true. We ultimately observe *Dimensional analysis homogeneity* when "decelerating CIS" and "thin set" properties are uniformly applicable to all quantities from both sides of the equation: All odd $\mathbb{P}$ Set = Gap 2-$\mathbb{P}$ Subset + Gap 4-$\mathbb{P}$ Subset + Gap 6-$\mathbb{P}$ Subset + Gap 8-$\mathbb{P}$ Subset + Gap 10-$\mathbb{P}$ Subset + $\cdots$.

A thick set is a set of integers that contains arbitrarily long intervals; viz, long blocks of consecutive integers [even if it also skips large chunks elsewhere]. Given a thick set $A$, for every $p \in \mathbb{N}$, there is some $n \in \mathbb{N}$ such that $\{n, n+1, n+2, ..., n+p\} \subset A$. Trivially Natural numbers $\mathbb{N}$, as Completely Predictable entities having "Natural Gap" = non-varying integer number value 1, is a thick set with Natural density being exactly 1. Other well-known sets that are thick include non-primes and non-squares. Thick sets can also be sparse, e.g. $\bigcup_{n\in\mathbb{N}} \{x : x = 10^n + m : 0 \leq m \leq n\}$. Thus a thick set has Natural density which can be 0 or $> 0$; viz, can be sparse or dense overall. It must always have long intervals (large chunks) but its sparsity can be low [or can be high]; viz, having Natural density close to 1 [or close to 0]. Set of Incompletely Predictable Composite numbers $\mathbf{C}$ is both thick and dense. Let $\mathbf{C}$-$\pi(n)$ be the Composite counting function defined as number of composites $\leq n$. Analogical "Composite number theorem" tells us $\mathbf{C}$-$\pi(n) \approx n - \mathbb{P}$-$\pi(n)$. Since $\mathbb{P}$-$\pi(n) \sim \frac{n}{\log_e n}$, we get $|\mathbf{C} \cap [1, n]| \approx n - \frac{n}{\log_e n} \implies \frac{|\mathbf{C} \cap [1, n]|}{n} \to 1$ as $n \to \infty$; viz, Composite numbers as an "accelerating CIS" and "thick set" that "accelerate to an infinitesimal small number value just below one" have Natural density 1. An exception is specific subset of Gap 2-Composite even numbers that follow, and are associated with, every odd Prime numbers: This unique subset is "decelerating CIS", and is a "thin set" with Natural density 0.

Both the Completely Predictable sets of Even numbers $\mathbb{E}$ and Odd numbers $\mathbb{O}$ are neither *thin set* nor *thick set*. There are never any arbitrarily long blocks of consecutive $\mathbb{E}$ or $\mathbb{O}$ $\implies$ both sets are not thick set. Natural density of both $\mathbb{E}$ or $\mathbb{O}$ is $\frac{1}{2}$ [viz, $\neq 0$] $\implies$ both sets are not thin set.

Numbers with their last digit ending in (i) 1, 3, 7 or 9 [which can be either primes or composites] constitute ∼40% of all integers; and (ii) 0, 2, 4, 5, 6 or 8 [which must be composites] constitute ∼60% of all integers. We validly ignore the only single-digit even prime number 2 and odd prime number 5. We note ≥2-digit Odd Primes can only have their last digit ending in 1, 3, 7 or 9 but not in 0, 2, 4, 5, 6 or 8. **List of eligible Last digit of Odd Primes**:

- The last digit of Odd Primes having their Prime gaps with last digit ending in 2 [viz, Gap 2, Gap 12, Gap 22, Gap 32...] can only be 1, 7 or 9 [but not (5) or 3] as three choices.
- The last digit of Odd Primes having their Prime gaps with last digit ending in 4 [viz, Gap 4, Gap 14, Gap 24, Gap 34...] can only be 3, 7 or 9 [but not (5) or 1] as three choices.

- The last digit of Odd Primes having their Prime gaps with last digit ending in 6 [viz, Gap 6, Gap 16, Gap 26, Gap 36...] can only be 1, 3 or 7 [but not (5) or 9] as three choices.
- The last digit of Odd Primes having their Prime gaps with last digit ending in 8 [viz, Gap 8, Gap 18, Gap 28, Gap 38...] can only be 1, 3 or 9 [but not (5) or 7] as three choices.
- The last digit of Odd Primes having their Prime gaps with last digit ending in 0 [viz, Gap 10, Gap 20, Gap 30, Gap 40...] can only be 1, 3, 7 or 9 [but not (5)] as four choices.

*Remark* 3.3. Yitang Zhang proved a landmark result [announced on April 17, 2013]: There are infinitely many pairs of (odd) Prime numbers that differ by unknown even number $N \leq 70$ million[10]; viz, there is a "decelerating CIS" and "thin set" of Gap $N$-Prime numbers with unknown even number $N \leq 70$ million. This solitary $N$ value as an existing "privileged" but unknown even Prime gap must, without exception, comply with the imposed Odd Prime-Prime Gap constraint on "eligible last digit of Odd Primes" (see itemized List above). Aesthetically, this $N$ value by itself is insufficient since its "decelerating CIS" Odd Primes simply cannot exist alone amongst the large range of prime numbers. Always as finite [but NOT infinite] length, we observe as a side note that two or more consecutive Odd Primes can validly and rarely be constituted by [same] even Prime gap of 6 or multiples of 6. Hence there must be at least two, if not three, existing even Prime gaps generating their corresponding "decelerating CIS" Odd Primes. Polignac's and Twin prime conjectures refers to all even Prime gaps 2, 4, 6, 8, 10... generating corresponding "decelerating CIS" Odd Primes [which are, *by default*, all "thin sets"].

Polymath8a "Bounded gaps between primes" (4 June 2013 – 17 November 2014) was a project to improve $N$ by developing the techniques of Zhang [viz, constructing an "admissible $k$-tuple" whose diameter was bounded by 70 million]. This project concluded with obtaining $N = 4{,}680$.

Polymath8b "Bounded intervals with many primes" (19 November 2013 – 19 June 2014) was a project to further improve $N$ by combining Polymath8a results with the techniques of James Maynard [viz, introducing a refinement of GPY sieve method for studying prime $k$-tuples and small gaps between primes which establishes that "a positive proportion of admissible $m$-tuples satisfy the prime $m$-tuples conjecture for every $m$"]. This project concluded with a bound of $N = 246$; and by assuming Elliott-Halberstam conjecture and its generalized form further lower $N$ to 12 and 6, respectively. Regarded as "Zhang's optimized result", these lowering of $N$ involve studying *overlapping* $k$-tuples. But maximally lowering $N$ to 2 will likely require clever breakthrough concepts that involve studying *non-overlapping* even Prime gaps.

*Remark* 3.4. The notion of *thin set* and *thick set* typically apply to subsets of [discrete] $\mathbb{N}$ (Natural numbers), or more generally, [discrete] $\mathbb{Z}$ (integer numbers). Forming a CIS, the Set of Incompletely Predictable [discrete] nontrivial zeros from e.g. Riemann zeta function $\zeta(s)$ are derived from complex solutions to $\zeta(s) = 0$, whereby $s = \sigma \pm it$. Traditionally given as $\pm t$-valued transcendental numbers; nontrivial zeros conceptually form a "discrete" and "sparse" ("small") set in [continuous] 1-dimensional $\sigma = \frac{1}{2}$-Critical Line [viz, constituted by $\mathbb{R}$ of infinite length] or in [continuous] 2-dimensional Complex plane [viz, constituted by $\mathbb{C}$ of infinite area]. CIS nontrivial zeros has zero Natural density in UIS $\mathbb{R}$ or UIS $\mathbb{C}$, and do not form dense clusters or intervals. We intuitively and meaningfully interpret Set of nontrivial zeros as a *thin set*.

## 4. Symmetry properties of Z(t) plots containing nontrivial zeros from various L-functions

Lemma 4.1. *Plots of Z-function from general L-functions [and from L-function for Riemann zeta function] manifest unique distributions of both $Z(t)$ positivity and $Z(t)$ negativity that depend on the choice of sqrt(root number) being correctly and arbitrarily chosen from $+1$ or $-1$ value for even Analytic rank L-functions AND on the choice of sqrt(root number) being correctly and arbitrarily chosen from $+i$ or $-i$ value for odd Analytic rank L-functions.*

**Proof.** Riemann zeta function is Genus 0 curve having Analytic rank 0 [of degree 1]. Elliptic curves are Genus 1 curves having Analytic rank 0, 1, 2, 3, 4, 5,... [of degree 2], and there are other higher Genus 2, 3, 4, 5, 6... curves having Analytic rank 0, 1, 2, 3, 4, 5,... [of higher degree]. They all have associated self-dual L-functions generating unique nontrivial zeros (spectrum) with $t$ values being fully independent of the chosen Z(t) positivity [or Z(t) negativity].

A product $P$, having positive ($+ve$) or negative ($-ve$) value, is the multiplication of two or more factors $A$, $B$, $C$, $D$, ...; viz, $P = A \times B \times C \times D \times \cdots$. Let $P = Z(t)$, $A = \overline{\varepsilon}^{\frac{1}{2}}$ {viz, [optional] "Sign normalization" from L-functions and modular forms database (LMFDB)}, $B = \dfrac{\gamma(\frac{1}{2}+it)}{|\gamma(\frac{1}{2}+it)|}$ and $C = L(\frac{1}{2}+it)$. LMFDB's $Z(t)$ is defined by $P = A \times B \times C$ whereby $A = \pm 1$ for L-functions with even Analytic rank 0, 2, 4, 6, 8... and $A = \pm i$ for L-functions with odd Analytic rank 1, 3, 5, 7, 9....

The epsilon ($\epsilon$) [$= +1$ for even Analytic ranks $/-1$ for odd Analytic ranks] is also known as Sign or Root number in functional equation for an analytic L-function. Then sqrt(root number) [or square root of epsilon] has values of $+1$ or $-1$ for even Analytic ranks and $+i$ or $-i$ for odd Analytic ranks. This value is arbitrarily chosen under LMFDB's stated convention so that $Z(t) > 0$ for sufficiently small $t > 0$; viz, manifesting Z(t) positivity [$\equiv$ Sign normalization].

The corollary convention so that $Z(t) < 0$ for sufficiently small $t > 0$ refers to arbitrarily choosing this value to instead manifest Z(t) negativity. Thus, sqrt(root number) give rise to two opposite choices for two complementary Z(t) plots of nontrivial zeros (spectrum) in both even and odd Analytic rank L-functions. Consequently, both Z(t) positivity and Z(t) negativity are validly and inherently present in Z(t) plots of L-functions when we use $\epsilon^{\frac{1}{2}} = \sqrt{\epsilon}$ in relevant Z-functions [instead of just using $\epsilon$ in relevant Z-functions], and with unique distributions specified by chosen $+1$ or $-1$ choices and $+i$ or $-i$ choices.

As per Axiom 4.3, Z(t) plots from L-functions using $\epsilon$ without adopting, or $\sqrt{\epsilon}$ with adopting, LMFDB's Sign normalization will NOT affect actual $t$ values of the nontrivial zeros (spectrum). Requiring further confirmatory research studies, we can intuitively propose these unique distributions have various [unknown] deterministic Incompletely Predictable properties.

***The proof is now complete for Lemma 4.1***□.

PROPOSITION 4.2. *Self-deal L-functions are special cases of dual L-functions where they both have unique $Z(t)$ plots of nontrivial zeros (spectrum) with LMFDB's enforced [optional] Sign normalization that manifest different symmetry types.*

**Proof.** Sign (root number) or epsilon for dual L-functions is a complex number $a + bi$ being the "Root of Unity". Then for self-dual L-functions [which are usefully considered as special cases of dual L-functions], $\epsilon = +1 + 0i = +1$ with $\overline{\varepsilon}^{\frac{1}{2}} = \pm 1$ for even Analytic rank AND $\epsilon = -1 + 0i = -1$ with $\overline{\varepsilon}^{\frac{1}{2}} = \pm i$ for odd Analytic rank. Both dual and self-dual concepts deal with relationships between L-functions and their duals. Self-dual L-functions represent a stronger condition of symmetry:

[1] Symmetry. Dual L-functions may have a more general relation to their duals, while self-dual L-functions exhibit exact symmetry.

[2] Applications. Self-dual L-functions are often directly tied to important conjectures and results in Number theory.

[3] Examples. Many well-known L-functions associated with modular forms or Dirichlet characters are self-dual, whereas others might not exhibit this property.

In Remark 4.1 below, we further discuss using examples of Z(t) plots of nontrivial zeros (spectrum) that manifest unique types of individual / combined symmetry for both dual and self-dual L-functions.

***The proof is now complete for Proposition 4.2***□.

AXIOM 4.3. *The LMFDB's [optional] Sign normalization does not affect the actual values of nontrivial zeros (spectrum) from $Z(t)$ plots of L-functions.*

**Proof.** Root number or Sign $\epsilon$ directly govern the functional equation; viz, [1] If $\epsilon = 1$, L-function is symmetric (even functional equation); and [2]

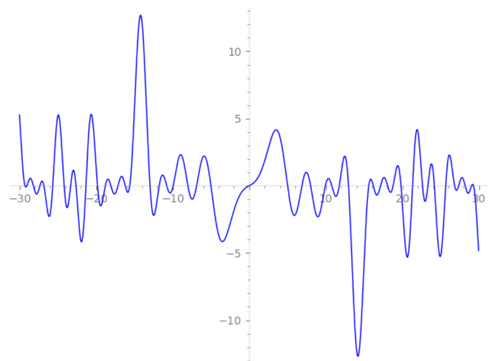


Figure 1. Graph of $Z$-function along [Analytically normalized] $\Re(s) = \frac{1}{2}$-critical line for $-\infty < t < \infty$ nontrivial zeros (spectrum) in Genus 1 odd Analytic rank 1 semistable Elliptic curve 37.a1 of degree 2. Point Symmetry of Origin point, trajectory intersect Origin point, and manifest Z(t) positivity. Integral points (−1, 0), (−1, −1), (0, 0), (0, −1), (1, 0), (1, −1), (2, 2), (2, −3), (6, 14), (6, −15).

If $\epsilon = -1$, L-function is anti-symmetric (odd functional equation). When constructing Z(t), the absolute value $|\Lambda(s)|$ eliminates impact of $\epsilon$ as a Sign, so $\epsilon$ remains unchanged, and we strictly do not need to square root $\epsilon$ for Z(t) to be valid. Why do LMFDB use $\sqrt{\epsilon}$ to obtain two choices so we can arbitrarily choose one of them under Sign normalization to manifest Z(t) positivity?

[1] Ensure symmetry: By incorporating $\sqrt{\epsilon}$ *whereby the required phase adjustment DOES NOT change magnitude of* $\Lambda(E, s)$, the functional equation becomes symmetric $Z(-t) = Z(t)$.

[2] Numerical stability: The square root ensures the phase of $\Lambda(s)$ along Critical line is correctly adjusted for numerical computations.

[3] Nontrivial zeros being unaffected: The $t$ values for infinitely-many nontrivial zeros (spectrum) are independent of using $\epsilon$ versus $\sqrt{\epsilon}$.

[4] Historical context: Similar constructions occur in Analytic Number theory e.g. for Riemann zeta function.

***The proof is now complete for Axiom 4.3*□.**

An old bug in the code for computing some of (Hardy or Riemann-Siegel) Z(t) plots in LMFDB website had previously resulted in failure to follow LMFDB's stated convention that Z(t) > 0 as t → $0^+$. In particular, Z(t) plots affected by this bug are, firstly, from L-functions of all Genus 1 elliptic curves having (odd) Analytic rank 3 [except for the very first listed Analytic rank 3 Elliptic curve 5077.a1 being not affected]; and, secondly, from solitary L-function of (non-elliptic) Genus 0 curve Riemann zeta function having (even) Analytic rank 0.

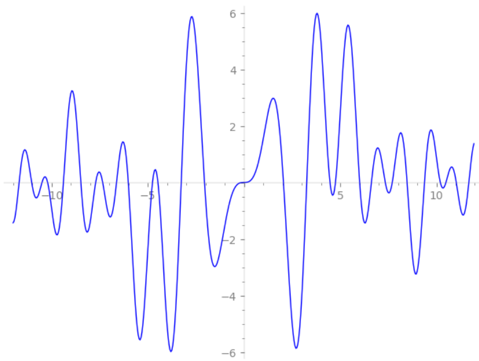


Figure 2. Graph of $Z$-function along [Analytically normalized] $\Re(s) = \frac{1}{2}$-critical line for $-\infty < t < \infty$ nontrivial zeros (spectrum) in Genus 1 odd Analytic rank 3 semistable Elliptic curve 5077.a1 of degree 2. Point Symmetry of Origin point, trajectory intersect Origin point, and manifest Z(t) positivity as Pseudo-transitional curve. Integral points (−3, 0), (−3, −1), (−2, 3), (−2, −4), (−1, 3), (−1, −4), (0, 2), (0, −3), (1, 0), (1, −1), (2, 0), (2, −1), (3, 3), (3, −4), (4, 6), (4, −7), (8, 21), (8, −22), (11, 35), (11, −36), (14, 51), (14, −52), (21, 95), (21, −96), (37, 224), (37, −225), (52, 374), (52, −375), (93, 896), (93, −897), (342, 6324), (342, −6325), (406, 8180), (406, −8181), (816, 23309), (816, −23310).

We assign this stated convention as definition for 'Z(t) positivity' [also called 'Sign normalization'] whereby the complementary 'Z(t) negativity' is defined by (corollary) convention Z(t) $<$ 0 as t $\to 0^+$. We thus acknowledge this Sign normalization [so that Z(t) $>$ 0 for sufficiently small t $>$ 0] used in LMFDB (which is explicitly noted to be arbitrary) should not, in general, be used as a basis for definitive mathematical arguments.

As obvious randomly chosen example correctly manifesting Z(t) positivity, Z(t) plot of nontrivial zeros (spectrum) in Figure 1 for Degree 2 Genus 1 (odd) Analytic rank 1 Elliptic curve 37.a1 [NOT affected by the bug] is uniquely determined by choosing sqrt(root number) $= +i$ choice in self-dual L-function 2-37-1.1-c1-0-1.

Regarding these incorrectly depicted Z(t) plots manifesting Z(t) negativity [instead of Z(t) positivity] in affected Analytic rank 3 elliptic curves, they were first alerted by us in August 2024. The culprit bug in the code causing this problem was subsequently discovered by LMFDB Associate Editor Dr. Edgar Costa in conjunction with LMFDB Managing Editor Prof. Andrew Sutherland, and was largely fixed in October 2024.

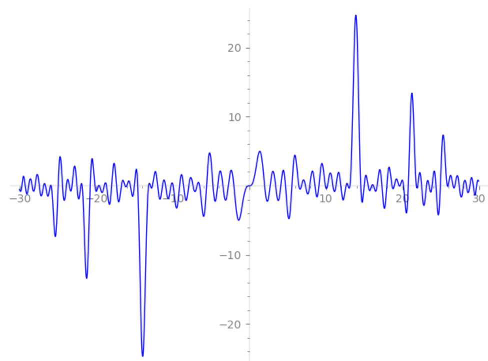


Figure 3. Graph of $Z$-function along [Analytically normalized] $\Re(s) = \dfrac{1}{2}$-critical line for $-\infty < t < \infty$ nontrivial zeros (spectrum) in Genus 1 curve odd Analytic rank 3 semistable Elliptic curve 21858.a1 of degree 2. Point Symmetry of Origin point, trajectory intersect Origin point, and manifest Z(t) positivity [post-bug-fixing]. Integral points are $(-7, 5)$, $(-7, 2)$, $(-6, 12)$, $(-6, -6)$, $(-4, 14)$, $(-4, -10)$, $(-2, 12)$, $(-2, -10)$, $(1, 5)$, $(1, -6)$, $(2, 2)$, $(2, -4)$, $(3, 0)$, $(3, -3)$, $(4, 2)$, $(4, -6)$, $(5, 5)$, $(5, -10)$, $(7, 12)$, $(7, -19)$, $(11, 29)$, $(11, -40)$, $(14, 44)$, $(14, -58)$, $(22, 92)$, $(22, -114)$, $(30, 150)$, $(30, -180)$, $(68, 530)$, $(68, -598)$, $(119, 1244)$, $(119, -1363)$, $(122, 1292)$, $(122, -1414)$, $(137, 1541)$, $(137, -1678)$, $(786, 21660)$, $(786, -22446)$, $(1069, 34437)$, $(1069, -35506)$, $(38746, 7607514)$, $(38746, -7646260)$, $(783868, 693616502)$, $(783868, -694400370)$.

As already mentioned: Apart from smallest conductor Degree 2 Analytic rank 3 Elliptic curve 5077.a1 with its Z(t) plot in Figure 2 of self-dual L-function 2-5077-1.1-c1-0-410 showing Z(t) positivity from arbitrarily choosing sqrt(root number) $= -i$ choice [NOT affected by the bug]; this bug affects Z(t) plots of self-dual L-functions derived from all other Degree 2 (odd) Analytic rank 3 elliptic curves e.g. randomly chosen self-dual L-function 2-21858-1.1-c1-0-3 of Elliptic curve 21858.a1 [depicted as correct Z(t) positivity version post-bug-fixing in Figure 3 & incorrect Z(t) negativity version pre-bug-fixing in Figure 4]. This bug also affected Z(t) plot of self-dual L-function 2-11-1.1-c1-0-0 derived from Degree 1 (even) Analytic rank 0 (non-elliptic) Number field 1.1.1.1: $\mathbb{Q}$ / Riemann zeta function / Dirichlet eta function [depicted as correct Z(t) positivity version post-bug-fixing in Figure 5 & incorrect Z(t) negativity version pre-bug-fixing in Figure 6].

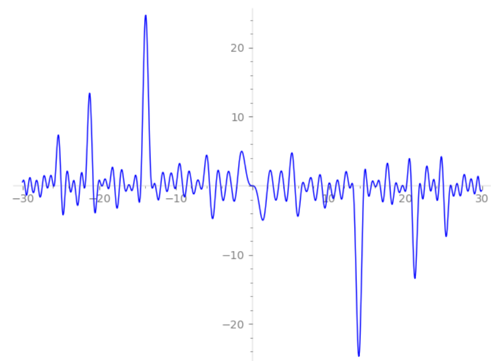


Figure 4. Graph of $Z$-function along [Analytically normalized] $\Re(s) = \frac{1}{2}$-critical line for $-\infty < t < \infty$ nontrivial zeros (spectrum) in Genus 1 curve odd Analytic rank 3 semistable Elliptic curve 21858.a1 of degree 2. Point Symmetry of Origin point; trajectory intersect Origin point; manifest Z(t) negativity [pre-bug-fixing]. Integral points are identical to those in Figure 3.

*Remark* 4.1. In relation to self-dual L-functions, we see horizontal x-axis acting as Line Symmetry for [combined] Figure 3 and Figure 4 with their [paired] sqrt(root number) given by $\pm i$ (for odd Analytic ranks), and [combined] Figure 5 and Figure 6 with their [paired] sqrt(root number) given by $\pm 1$ (for even Analytic ranks).

A character has odd/even parity if it is odd/even as a function. A Dirichlet character $\chi\colon \mathbb{Z} \to \mathbb{C}$ is odd if $\chi(-1) = -1$ and even if $\chi(-1) = 1$. On the chosen examples of Analytic rank 0 dual L-functions from Dirichlet characters having [paired] Sign given by complex number and its conjugate, we depict them via [combined] Figure 7 (L-function 1-5-5.2-r1-0-0) and Figure 8 (L-function 1-5-5.3-r1-0-0) as [respectively] odd Parity $\chi_5(2, \cdot)$ with Sign: $0.850 + 0.525i$ and odd Parity $\chi_5(3, \cdot)$ with Sign: $0.850 - 0.525i$; and via [combined] Figure 9 (L-function 1-7-7.2-r0-0-0) and Figure 10 (L-function 1-7-7.4-r0-0-0) as [respectively] even Parity $\chi_7(2, \cdot)$ with Sign: $0.895 - 0.444i$ and even Parity $\chi_7(4, \cdot)$ with Sign: $0.895 + 0.444i$. In contrast to self-dual L-functions, we instead see vertical y-axis acting as Line Symmetry for these complementary-paired [with "conjugate Signs"] dual L-functions having either even or odd parity, and [combined] "reverse" patterns of nontrivial zeros (spectrum).

Using the very definition of Z(t) for an L-function whereby we [optionally] adopt LMFDB's sqrt(root number) that always provide two choices, we can unambiguously obtain valid mathematical statements or deductions in Lemma

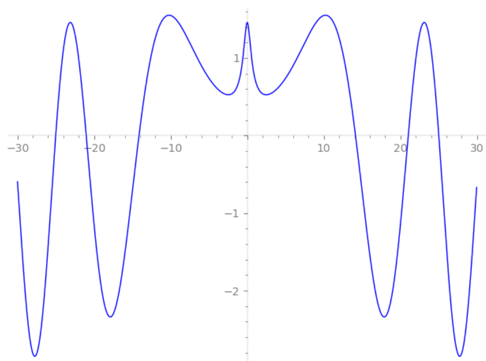


Figure 5. Graph of $Z$-function along $\Re(s) = \frac{1}{2}$-critical line for $-\infty < t < \infty$ nontrivial zeros (spectrum) in even Analytic rank 0 Genus 0 Dirichlet eta function $\eta(s)$ of degree 1 over $K = \mathbb{Q}$ as *Analytic continuation* of Riemann zeta function $\zeta(s)$. Line Symmetry vertical $y$-axis, trajectory DO NOT intersect Origin point, and manifest Z(t) positivity as part of Sign normalization by LMFDB. Integral basis 1. An integral basis of a number field $K$ is a $\mathbb{Z}$-basis for ring of integers of $K$. It is a $\mathbb{Q}$-basis for $K$. Initial $+ve$ nontrivial zeros: 14.13, 21.02, 25.01, 30.42, 32.93, 37.58,.... "Nontrivial Zero Gaps" [as transcendental numbers] between any two adjacent nontrivial zeros never consist of a fixed value $\Longrightarrow$ all infinitely-many nontrivial zeros must be Incompletely Predictable entities.

4.1, Proposition 4.2 and Axiom 4.3. These statements or deductions are thus rigorously proven to be true using simple mathematical arguments.

## 5. Riemann hypothesis with Birch and Swinnerton-Dyer conjecture

Riemann hypothesis (RH) refers to the 1859-dated conjecture on all nontrivial zeros [as a "thin set"] in (self-dual) L-function from Genus 0 curve known as Riemann zeta function [and, via Analytic continuation, its *proxy* Dirichlet eta function] are located on its $\sigma = \frac{1}{2}$-Critical line. Then our posited Generalized RH simply refers to this same conjecture on higher Genus 1, 2, 3, 4, 5... curves [and also, with "overlap", on lower Genus 0 curves]. Proposed during the early 1960's by two British mathematicians Bryan John Birch and Peter Swinnerton-Dyer, Birch and Swinnerton-Dyer (BSD) conjecture refers to a famous conjecture on Analytic ranks of (self-dual) L-functions from Genus 1 curves known as elliptic curves. Then our posited Generalized BSD simply

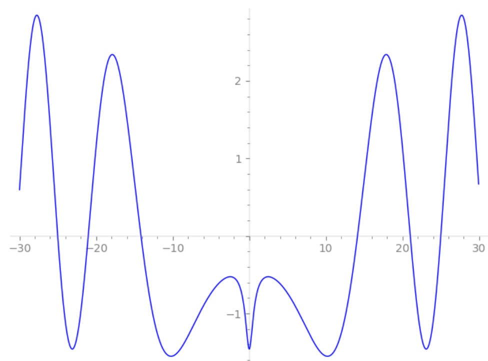


Figure 6. Graph of $Z$-function along $\Re(s) = \frac{1}{2}$-critical line for $-\infty < t < \infty$ nontrivial zeros (spectrum) in even Analytic rank 0 Genus 0 Dirichlet eta function $\eta(s)$ of degree 1 over $K = \mathbb{Q}$ as *Analytic continuation* of Riemann zeta function $\zeta(s)$. Line Symmetry of vertical $y$-axis, trajectory DO NOT intersect Origin point, and manifest Z(t) negativity [pre-bug-fixing] as *Pseudo*-transitional curve. This is the complementary $Z(t)$ plot to that depicted by Figure 5.

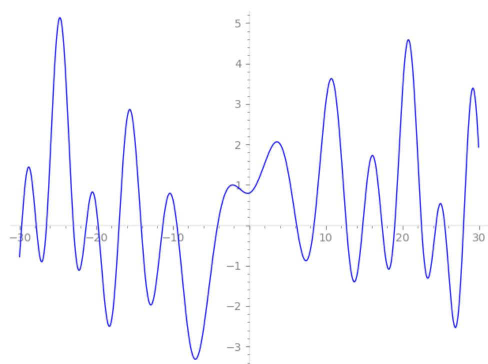


Figure 7. Graph of $Z$-function along $\Re(s) = \frac{1}{2}$-critical line for $-\infty < t < \infty$ nontrivial zeros (spectrum) in even Analytic rank 0 Genus 0 Dirichlet character $\chi_5(2, \cdot)$ with odd Parity. There is neither Line symmetry nor Point symmetry being manifested. Trajectory DO NOT intersect Origin point. This is the complementary $Z(t)$ plot to that depicted by Figure 8.

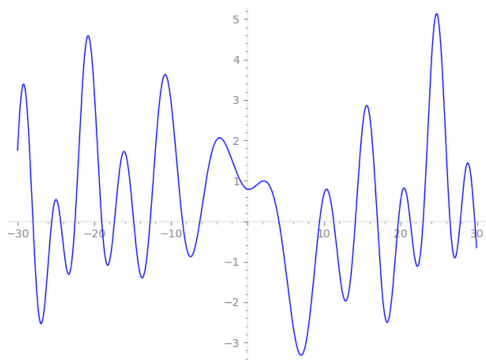


Figure 8. Graph of $Z$-function along $\Re(s) = \frac{1}{2}$-critical line for $-\infty < t < \infty$ nontrivial zeros (spectrum) in even Analytic rank 0 Genus 0 Dirichlet character $\chi_5(3, \cdot)$ with odd Parity. There is neither Line symmetry nor Point symmetry being manifested. Trajectory DO NOT intersect Origin point. This is the complementary $Z(t)$ plot to that depicted by Figure 7.

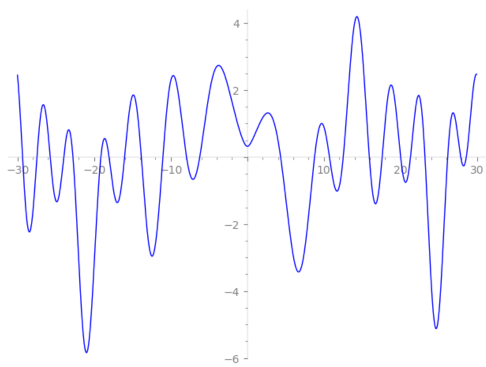


Figure 9. Graph of $Z$-function along $\Re(s) = \frac{1}{2}$-critical line for $-\infty < t < \infty$ nontrivial zeros (spectrum) in even Analytic rank 0 Genus 0 Dirichlet character $\chi_7(2, \cdot)$ with even Parity. There is neither Line symmetry nor Point symmetry being manifested. Trajectory DO NOT intersect Origin point. This is the complementary $Z(t)$ plot to that depicted by Figure 10.

refers to this same conjecture on higher Genus 2, 3, 4, 5, 6... curves [and also on lower Genus 0 curves and, with "overlap", on Genus 1 curves].

Widely studied diverse L-functions [e.g. having to be entire with poles on the edge of $0 < \Re(s) < 1$-Critical strip or in other locations] are those arising

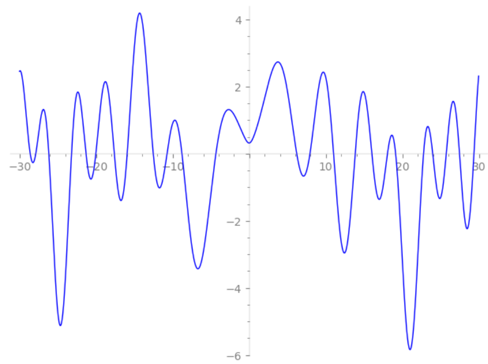


Figure 10. Graph of $Z$-function along $\Re(s) = \frac{1}{2}$-critical line for $-\infty < t < \infty$ nontrivial zeros (spectrum) in even Analytic rank 0 Genus 0 Dirichlet character $\chi_7(4, \cdot)$ with even Parity. There is neither Line symmetry nor Point symmetry being manifested. Trajectory DO NOT intersect Origin point. This is the complementary $Z(t)$ plot to that depicted by Figure 9.

from arithmetic objects such as elliptic and higher-genus curves, holomorphic cusp or modular forms, Maass forms, number fields with their Hecke characters, Artin representations, Galois representations, and motives. The two characterizations of such L-functions are in terms of Dirichlet coefficients and spectral parameters. That every Galois representation arises from an automorphic representation is known as the Modularity Conjecture. Sometimes an L-function may arise from more than one source e.g. L-functions associated with elliptic curves are also associated with weight 2 cusp forms. A big goal of Langlands program ostensibly is to prove any degree $d$ L-function is associated with an automorphic form on $\mathrm{GL}(d)$. Because of this representation theoretic genesis, one can associate an L-function not only to an automorphic representation but also to the symmetric powers, or exterior powers of that representation, or to the tensor product of two representations (the Rankin-Selberg product of two L-functions).

Relevant to (Analytically normalized) $\sigma = \frac{1}{2}$-Critical Line when referenced to positive $0 < t < +\infty$ range in complex variable $s = \sigma \pm it$, the LMFDB's Sign normalization is applicable to eligible L-functions having Z(t) plots of '*OUTPUTS*' as infinitely-many *Incompletely Predictable* nontrivial zeros (spectrum). Using vast [albeit limited] catalogues in LMFDB website[5] for observational study, we propose LMFDB's Sign normalization is ubiquitously satisfied by Genus 0, 1, 2, 3, 4, 5... curves e.g. Genus 1 elliptic curves over Number field $K$ = Rational number $\mathbb{Q}$, real and imaginary quadratic fields; Genus 2 curves over $K = \mathbb{Q}$; etc. LMFDB's Sign normalization [≡ Z(t) positivity] will always, *by default*, be "standardized" on an individual case-by-case basis via arbitrarily applying this normalization in a correct manner [so that Z(t) > 0 for sufficiently small $t > 0$]. As further outlined in section 6, Analytic rank 0 Genus 0 curves of degree 1 and Analytic rank 3 Genus 1 curves of degree 2 have (respective) Pseudo-transitional curves [pre-bug-fixing]: non-elliptic curve Riemann zeta function/Dirichlet eta function and Elliptic curve 5077.a1.

One could adopt Selberg class $\mathcal{S}$ as the set of all Dirichlet series (Generic L-functions) $F(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ that satisfy the four Selberg class axioms [whereby it is often practical to regard Axioms I, II and III to be "essential", and Axiom IV to be "optional"]. As opposed to the very particular cuspidal automorphic representations of GL($n$) by Langlands, this set $\mathcal{S}$ contains the very general analytic axioms defined by Atle Selberg who conjectured its elements all satisfy the (Generalized) Riemann hypothesis.

· Axiom I. Analyticity: $(s-1)^m F(s)$ is an entire function of finite order for some non-negative integer $m$.

· Axiom II. Functional equation: there is a function $\gamma_F(s)$ of form $\gamma_F(s) = \epsilon Q^s \prod_{i=1}^{k} \Gamma(\lambda_i s + \mu_i)$ where $|\epsilon| = 1$, $Q > 0$, $\lambda_i > 0$, and $\mathrm{Re}(\mu_i) \geq 0$ such that $\Lambda(s) = \gamma_F(s)F(s)$ satisfies $\Lambda(s) = \overline{\Lambda}(1-s)$ where $\overline{\Lambda}(s) = \overline{\Lambda(\overline{s})}$.

· Axiom III. Euler product: $a_1 = 1$, and $\log F(s) = \sum_{n=1}^{\infty} \frac{b_n}{n^s}$ where $b_n = 0$ unless $n$ is a positive power of a prime and $b_n \ll n^{\theta}$ for some $\theta < \frac{1}{2}$.

· Axiom IV. Ramanujan hypothesis: $a_n = O_{\epsilon}(n^{\epsilon})$ for any fixed $\epsilon > 0$.

In [2], we encounter an attractively useful, but *unavoidably* complex and *inherently quasi*-complete classification on universal 'Generic L-functions' provided by Prof. Farmer and colleagues via dividing Analytic L-functions and $\mathbb{Q}$-automorphic L-functions into arithmetic type and algebraic type based on [extra] collection of axioms. Conjecturally, all four resulting sets of L-functions are equal arising from arithmetic objects of pure motives and geometric Galois representations.

Imperfect commonly accepted scheme on modern classification (taxonomy) is never a mutually exclusive classification system for *Living Things = Life → Domain → Kingdom → Phyllum → Class → Order → Family → Genus → Species*. It is strongly influenced by modern technology e.g. bioinformatics, DNA sequencing, databases, imaging, Artificial Intelligence (AI) software, etc.

Likewise for our primitive arbitrary but insightful and practical (lineage) classification of *Scientific Knowledge = Science → Mathematics → Number* *theory*: Algebraic, Analytic or Geometric $\rightarrow$ *Genus curves*: having different polynomial-degree $\rightarrow$ *Generic L-functions*: dual & self-dual L-functions having different degree, Euler product and gamma factors in functional equations $\rightarrow$ *Analytic ranks*: even versus odd with Line symmetry versus Point symmetry in Z(t) plots of nontrivial zeros (spectrum) $\rightarrow$ *Sign normalization*: adopting the arbitrary decision to have Z(t) positivity $\rightarrow$ *Isogeny class over a field $K$*: e.g. elliptic curves over $\mathbb{Q}$ either have or have not rational isogeny, two elliptic curves are twists if an only if they have same $j$-invariant, etc.

Under Generalized Riemann hypothesis, nontrivial zeros (NTZ) [as actual $\mathbb{C}$ $s$-values] are conventionally denoted by $\mathbb{R}$ $t$-values in $0 < t < +\infty$ range, and lie on Critical Line $\Re(s) = \frac{1}{2}$ (in Analytic normalization). Lowest NTZ of an L-function $L(s)$ is least $t > 0$ for which $L(\frac{1}{2} + it) = 0$. Even when $L(\frac{1}{2}) = 0$, lowest NTZ is by "traditional" definition a positive $t$-valued real number. As functions of complex variable $s$, L-functions for elliptic curves are denoted by $L(E, s)$ or $L_E(s)$, with these symbols often used interchangeably. *They have Analytic rank of zero integer value [whereby $L(1) \neq 0$ and $t \neq 0$ for $1^{st}$ NTZ] or non-zero integer values [whereby $L(1) = 0$ and $t = 0$ for $1^{st}$ NTZ]*. Analytic rank $= 0 \implies$ associated L-functions for elliptic [and non-elliptic] curves NEVER have $1^{st}$ NTZ given by the ($\mathbb{R}$-valued) variable $t = 0$. Analytic rank $\geq 1$ [viz, 1, 2, 3, 4, 5... up to an arbitrarily large number value] $\implies$ associated L-functions for elliptic [and non-elliptic] curves ALWAYS have $1^{st}$ NTZ given by the ($\mathbb{R}$-valued) variable $t = 0$.

Generic L-functions & associated modular forms are usefully regarded as various types of *'infinite series'* in (Generalized) BSD conjecture. The 2001 modularity theorem states that elliptic curves, with their $L_E(s)$, over $\mathbb{Q}$ are uniquely related to [*weight* 2 *for* $\Gamma_0(N)$ *classical*] modular form in a particular way. The rank of a number field $K$ is size of any set of fundamental units of $K$. It is equal to $r = r_1 + r_2 - 1$ where $r_1$ is number of real embeddings of $K$ into $\mathbb{C}$ and $2r_2$ is number of complex embeddings of $K$ into $\mathbb{C}$. The analytic rank of an abelian variety is analytic rank of its L-function $L(A, s)$. The analytic rank of a curve is analytic rank of its Jacobian. The weak form of BSD conjecture $\implies$ Analytic rank = Rank of Mordell-Weil group of abelian variety. Analytic ranks are always computed under assumption that $L(A, s)$ satisfies Hasse-Weil conjecture [they are not necessarily well-defined otherwise]. When $A$ is defined over $\mathbb{Q}$, parity of analytic rank is always compatible with sign of functional equation. In general, analytic ranks stored in LMFDB are only upper bounds on true analytic rank [they could be incorrect if $L(A, s)$ has a zero very close to but not on the central point]. For abelian varieties over $\mathbb{Q}$ of analytic rank

$< 2$ this upper bound is necessarily tight, due to parity. The rank of an elliptic curve $E$ defined over a number field $K$ is rank of its Mordell-Weil group $E(K)$.

Mordell-Weil theorem states that the set of rational points on an abelian variety over a number field forms a finitely generated abelian group, hence isomorphic to a group of form $T \oplus \mathbb{Z}^r$ where $T$ is a finite torsion group. The integer $r \geq 0$ is Mordell-Weil rank of abelian variety. Phrased in another way: This theorem says that $E(K)$ is a finitely-generated abelian group, hence $E(K) \cong E(K)_{\text{tor}} \times \mathbb{Z}^r$ where $E(K)_{\text{tor}}$ is finite torsion subgroup of $E(K)$, and $r \geq 0$ is the rank. Rank is an isogeny invariant: all curves in an isogeny class have the same rank.

A $p$-adic field (or local number field) is a finite extension of $\mathbb{Q}_p$, equivalently, a nonarchimedean local field of characteristic zero. A $p$-group is a group whose order is a power of a prime $p$. A result of Higman and Sims shows that the number of groups of order $p^k$ is $p^{(2/27+o(1))k^3}$, and this can be combined with a result of Pyber to show that, asymptotically, 100% of groups are $p$-groups. For $p$-groups, the rank can be computed by taking the $\mathbb{F}_p$-dimension of the quotient by the Frattini subgroup. Let $A/\mathbb{F}_q$ be an abelian variety where $q = p^r$. The $p$-rank of an abelian variety is the dimension of the geometric $p$-torsion as a $\mathbb{F}_p$-vector space: $p\text{-rank}(A) = \dim_{\mathbb{F}_p}(A(\overline{\mathbb{F}}_p)[p])$. The $p$-rank is at most the dimension of $A$, with equality if and only if $A$ is ordinary; the difference between the two is the $p$-rank deficit of $A$.

*Remark* 5.1. **Formal statements on BSD conjecture:** Central value of an L-function is its value at central point of Critical Strip. Central point of an L-function is the point on real axis of Critical Line. Equivalently, it is the fixed point of functional equation. In its Arithmetic normalization, an L-function $L(s)$ of weight $w$ has its central value at $s = \dfrac{w+1}{2}$ and functional equation relates $s$ to $1+w-s$. For L-functions defined by an Euler product $\prod_p L_p(s)^{-1}$ where coefficients of $L_p$ are algebraic integers, this is the usual normalization implied by definition. Analytic normalization of an L-function is defined by $L_{an}(s) := L(s + \dfrac{w}{2})$, where $L(s)$ is L-function in its arithmetic normalization. This moves the central value to $s = \dfrac{1}{2}$, and the functional equation of $L_{an}(s)$ relates $s$ to $1-s$. Rodriguez-Villegas and Zagier[7] have proven a formula, conjectured by Gross and Zagier[3], for central value of $L(s, \chi^{2n-1})$, namely $L(\dfrac{1}{2}, \chi^{2n-1}) = 2\dfrac{(2\pi\sqrt{7})^n \Omega^{2n-1} A(n)}{(n-1)!}$ where $\Omega = \dfrac{\Gamma(\frac{1}{7})\Gamma(\frac{2}{7})\Gamma(\frac{4}{7})}{4\pi^2}$.

By the functional equation $A(n) = 0$ whenever $n$ is even. For odd $n$ Gross and Zagier conjectured that $A(n)$ is a square [and provide tabulated values using their notation]. Rodriguez-Villegas and Zagier then prove that

$A(n) = B(n)^2$ where $B(1) = \dfrac{1}{2}$ and $B(n)$ is an integer for $n > 1$; and that $A(n)$ is given by a remarkable recursion formula [not stated in this paper]. The accompanying incredible [derived] result of "for odd $n$, $B(n) \equiv -n \bmod 4$", in one fell swoop, proves the non-vanishing of $L(\frac{1}{2}, \chi^{2n-1})$ for all odd $n$.

BSD conjecture relates the order of vanishing (or analytic rank) and the leading coefficient of the L-function associated to an elliptic curve $E$ defined over a number field $K$ at central point $s = 1$ to certain arithmetic data, the BSD invariants of $E$. It is usually stated as two forms. (1) The *weak* form of BSD conjecture states just that the analytic rank $r_{an}$ [that is, the order of vanishing of $L(E, s)$ at $s = 1$], is equal to the rank $r$ of $E/K$. (2) The *strong* form of BSD conjecture states also that the leading coefficient of the L-function is given by the formula

$$\frac{1}{r!}L^{(r)}(E,1) = |d_K|^{1/2} \cdot \frac{\#\text{Ш}(E/K) \cdot \Omega(E/K) \cdot \text{Reg}(E/K) \cdot \prod_{\mathfrak{p}} c_{\mathfrak{p}}}{\#E(K)_{\text{tor}}^2}.$$

The quantities appearing in this formula are: $d_K$ is discriminant of $K$; $r$ is rank of $E(K)$; Ш$(E/K)$ is Tate-Shafarevich group of $E/K$; $\text{Reg}(E/K)$ is regulator of $E/K$; $\Omega(E/K)$ is global period of $E/K$; $c_{\mathfrak{p}}$ is Tamagawa number of $E$ at each prime $\mathfrak{p}$ of $K$; $E(K)_{\text{tor}}$ is torsion order of $E(K)$.

For elliptic curves over $\mathbb{Q}$, a natural normalization for its L-function is the one that yields a functional equation $s \leftrightarrow 2 - s$. This is known as arithmetic normalization, because Dirichlet coefficients are rational integers. We emphasize here that arithmetic normalization is being used by writing L-function as $L(E, s)$. In this notation, the central point is at $s = 1$. "Special value" in LMFDB is the first non-zero value among $L(E, 1)$, $L'(E, 1)$, $L''(E, 1)$, $L'''(E, 1)$, $L''''(E, 1)$, $L'''''(E, 1)$,... *as (correspondingly) listed for Analytic rank* 0, 1, 2, 3, 4, 5... elliptic curves.

Let $A/\mathbb{F}_q$ be an abelian variety of dimension $g$ defined over a finite field. Its L-polynomial is the polynomial $P(A/\mathbb{F}_q, t) = \det(1 - tF_q|H^1((A_{\overline{\mathbb{F}}_q})_{et}, \mathbb{Q}_l))$, where $F_q$ is the inverse of Frobenius acting on cohomology. This is a polynomial of degree $2g$ with integer coefficients. By a theorem of Weil, the complex roots of this polynomial all have norm $1/\sqrt{q}$; this means that there are only finitely many L-polynomials for any fixed pair $(q, g)$. The L-polynomial of $A$ is the reverse of Weil polynomial. Let $K = F_q$ be the finite field with $q$ elements and $E$ an elliptic curve defined over $K$. By Hasse's theorem on elliptic curves, the precise number of rational points $\#E(K)$ of $E$; will comply with inequality $|\#E(K) - (q + 1)| \leq 2\sqrt{q}$. Implicit in the strong form of BSD conjecture is that the Tate-Sharafevich group Ш$(E/K)$ is finite. There is a similar conjecture for abelian varieties over number fields.

## 6. Pseudo-transitional curves: Genus 0 Riemann zeta function and Genus 1 Elliptic curve 5077.a1

*Preliminary note*: Affected mathematical arguments in this section are [falsely] true to the extent if there was [incorrectly] "never a bug in the code for computing Z(t) plots in LMFDB website, whereby Z(t) negativity do exist in some of Z(t) plots irrespective of the LMFDB's stated convention to follow Z(t) positivity [≡ Sign normalization]".

In reference to Z(t) plots of nontrivial zeros (spectrum) as '*OUTPUTS*' from L-functions: Analytic rank 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11... have corresponding Sign $+1$, $+1$, $+1$, $-1$, $+1$, $+1$, $+1$, $-1$, $+1$, $+1$, $+1$, $-1$... with [incorrectly] derived **Sign normalization** being (conjecturally) ONLY satisfied by Genus 1 elliptic curves over $\mathbb{Q}$. However by conforming with this [incorrect] *liberalized* Sign normalization, we devise the following pseudo-conditions:

- [#1.] We expect all even Analytic rank 0, 2, 4, 6, 8, 10... Genus 0, 1, 2, 3, 4, 5... curves to always manifest Z(t) positivity; viz, having Sign $+1$, $+1$, $+1$, $+1$, $+1$....
- [#2.] We expect all odd Analytic rank 1, 3, 5, 7, 9, 11... Genus 0, 1, 2, 3, 4, 5... curves to always manifest alternating Z(t) positivity and Z(t) negativity; viz, having Sign $+1$, $-1$, $+1$, $-1$, $+1$....

Denote $r$ = Analytic rank. Then our [incorrect] Sign normalization is [falsely] represented by $(1)^{r-1}$ for even $r$ with $\epsilon = 1$ and resulting in $+1$; and by $(i)^{r-1}$ for odd $r$ with $\epsilon = i$ [that satisfies $(r-1)^{th}$ "Root of Unity"] resulting in $\pm1$. Intuitively, one anticipate Sign changes to occur exactly when $r \equiv 1, 2 \pmod 4$ but this is not true: [I] For even $r = 0, 2, 4, 6, 8, 10...$; $1^{r-1}$ $= (1)^{-1}, (1)^1, (1)^3, (1)^5, (1)^7...$ = same $+1$ sign [of $+1$, $+1$, $+1$, $+1$, $+1$,...]. c.f. [II] For odd $r = 1, 3, 5, 7, 9, 11...$; $i^{r-1} = (i)^0, (i)^2, (i)^4, (i)^6, (i)^8... =$ alternating $\pm1$ sign [of $+1$, $-1$, $+1$, $-1$, $+1$,...]. Combined signs = $+1$, $+1$, $+1$, $-1$, $+1$, $+1$, $+1$, $-1$, $+1$, $+1$, $+1$, $-1$,... for $r = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11....$ Observe in all Z(t) plots from [5]: Number of nontrivial zeros with '0' value of $\{0, 1, 2, 3, 4, 5...\} = r$ of $\{0, 1, 2, 3, 4, 5...\} \propto$ width of $Z(t) = 0$ value [which is of equal length to the $-ve$ left and $+ve$ right of Origin point in self-dual L-functions].

All (even) Analytic rank 0 Genus 1 elliptic curves manifest Z(t) positivity without exception. But (even) Analytic rank 0 Genus 0 non-elliptic curve Riemann zeta function / Dirichlet eta function manifest Z(t) negativity [pre-bug-fixing], and is called a *Pseudo*-transitional curve (see Figure 6).

Definition: An elliptic curve is *semistable* if it has multiplicative reduction at every "bad" prime. All (odd) Analytic rank 3 Genus 1 elliptic curves manifest Z(t) negativity [pre-bug-fixing] but we observe an exception for *Pseudo*-transitional curve (see Figure 2) of semistable elliptic curve 5077.a1 $\{y^2 + y =$

$x^3 - 7x + 6\}$ that [instead] manifests Z(t) positivity, has smallest conductor 5077 amongst elliptic curves over $\mathbb{Q}$ of Analytic rank 3, 36 Integral points, one "bad" prime at $p = 5077 \equiv F_pT = 1 + O(T)$, Mordell-Weil group structure $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$, Infinite order Mordell-Weil generators $P = (1,\ 0),\ (2,\ 0),\ (0,\ 2)$, Endomorphism ring $\mathbb{Z}$ that is NOT larger than $\mathbb{Z} \Longrightarrow$ DO NOT have Complex Multiplication. With associated L-function of degree 2, elliptic curve 5077.a1 has no rational isogenies and its isogeny class 5077.a consists of this elliptic curve only, which is its own minimal quadratic twist.

**History of Gauss elliptic curve 5077.a1**: In 1985, Buhler, Gross and Zagier used the celebrated Gross-Zagier Theorem on heights of Heegner points (see [4]) to prove L-function of this curve has a zero of order 3 at its critical point $s = 1$, thus establishing first part of BSD conjecture for this curve (see [1]). This was first time that BSD had been established for any elliptic curve of rank 3. To this day, it is not possible, even in principle, to establish BSD for any curve of rank $\geq 4$ since there is no known method for rigorously establishing the value of Analytic rank when it is $> 3$. *We anticipate future Z(t) plots of nontrivial zeros (spectrum) for (odd) Analytic rank 5, 7, 9, 11... elliptic curves over $\mathbb{Q}$ should definitively (dis)prove our [incorrect] Sign normalization*. Via Goldfeld's method, which required use of an L-function of Analytic rank at least 3, elliptic curve 5077.a1 also found an application in context of obtaining explicit lower bounds for the class numbers of imaginary quadratic fields. This solved Gauss's Class Number Problem first posed by Gauss in 1801 in his book Disquisitiones Arithmeticae (Section V, Articles 303 and 304).

Elliptic curves over Number field $\mathbb{Q}$ are classical 2-variable mixed-polynomial-degree 3 Genus 1 curves having degree 2 L-functions of Analytic rank 0, 1, 2, 3, 4, 5.... Number field $\mathbb{Q}$ represented by Normalized defining polynomial $\pm x$ [or simply $x$] is the "simplest" 1-variable polynomial-degree 1 Genus 0 (non-elliptic) curve having L-function of Analytic rank 0. This curve is represented by Analytically continued (self-dual) L-function [LMFDB Number field 1.1.1.1: $\mathbb{Q}$] of Dirichlet eta function $\eta(s)$, which is derived from Riemann zeta function $\zeta(s)$; and DO NOT respect Z(t) positivity under our [incorrect] Sign normalization [see Figure 6]. $\zeta(s)$ is prototypical L-function, the only L-function of degree 1 and conductor 1, and (conjecturally) the only primitive L-function with a pole. It is a self-dual L-function that originated from the Dirichlet character $\chi_1(1, \cdot)$ having even parity. Its unique pole is located at $s = 1$.

The first nontrivial zero of Analytic rank 0 $\eta(s)$ [*proxy* function for $\zeta(s)$], at *height* $\approx$14.134, *is higher than that of any other algebraic L-function.* Then *any other algebraic L-function* [*with Analytic rank* 0, 1, 2, 3, 4, 5...] *will comparatively have more frequent nontrivial zeros that first occur at a relatively lower height* [*for L-functions with Analytic rank* 0], *up to and including* (*lowest*) *height of* 0 [*for L-functions with Analytic rank* 1 *or higher*]. As an example of Analytic rank 0 Genus 0 curves of degree 1 respecting Z(t) positivity without exception: LMFDB Analytic rank 0 L-function 1-5-5.4-r0-0-0 Genus 0 curve of degree 1 that originated from Dirichlet character $\chi_5(4,\cdot)$ clearly manifests Z(t) positivity. It has the functional equation $\Lambda(s) = 5^{s/2}\Gamma_{\mathbb{R}}(s)L(s) = \Lambda(1-s)$. After Riemann zeta function, analytic conductor in this self-dual L-function (of even parity with Sign: +1) is the smallest among L-functions of degree 1.

## 7. **Functional equations of Generic L-functions and their associated Gamma factors**

An (analytic) L-function is a Dirichlet series that has an Euler product and satisfies a certain type of functional equation, and allows analytic continuation. This L-function is also called Dirichlet L-function, associated with its Dirichlet L-series, which can be meromorphically continued to the complex plane, have an Euler product $L(s,\chi) = \prod_p (1-\chi(p)p^{-s})^{-1}$, and satisfy a functional equation of form $\Lambda(s,\chi) = q^{\frac{s}{2}}\Gamma_{\mathbb{R}}(s)L(s,\chi) = \varepsilon_\chi\overline{\Lambda}(1-s)$, where $q$ is conductor of $\chi$.

The two complex functions $\Gamma_{\mathbb{R}}(s) := \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})$ and $\Gamma_{\mathbb{C}}(s) := 2(2\pi)^{-s}\Gamma(s)$ that appear in the functional equation of an L-function are known as gamma factors. Here $\Gamma(s) := \int_0^\infty e^{-t}t^{s-1}dt$ is Euler's gamma function, with poles located at $s = 0, -1, -2, -3, -4, -5....$ The gamma factors satisfy $\Gamma_{\mathbb{C}}(s) = \Gamma_{\mathbb{R}}(s)\Gamma_{\mathbb{R}}(s+1)$ and is also viewed as "missing" factors of Euler product of an L-function corresponding to (real or complex) archimedean places. Completely Predictable *trivial zeros* are zeros of an L-function $L(s)$ that occur at poles of its gamma factors. An L-function $L(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ is called arithmetic if its Dirichlet coefficients $a_n$ are algebraic numbers. Thus for arithmetic L-functions, poles are at certain negative integers.

All known analytic L-functions have functional equations that can be written in the form [where $\Lambda(s)$ is now called *completed L-function*] $\Lambda(s) := N^{\frac{s}{2}}\prod_{j=1}^{J}\Gamma_{\mathbb{R}}(s+\mu_j)\prod_{k=1}^{K}\Gamma_{\mathbb{C}}(s+\nu_k)\cdot L(s) = \varepsilon\overline{\Lambda}(1-s)$ where $N$ is an integer, $\Gamma_{\mathbb{R}}$ and $\Gamma_{\mathbb{C}}$ are defined in terms of $\Gamma$-function, $\mathrm{Re}(\mu_j) = 0$ or 1 (assuming Selberg's eigenvalue conjecture), and $\mathrm{Re}(\nu_k)$ is a positive integer or half-integer, $\sum\mu_j + 2\sum\nu_k$ is real, and $\varepsilon$ *is the Sign of functional equation.* With these restrictions on spectral parameters [viz, the numbers $\mu_j$ and $\nu_k$ that appear as shifts in gamma factors $\Gamma_{\mathbb{R}}$ and $\Gamma_{\mathbb{C}}$ (respectively)], the data in the functional equation is specified uniquely. *The integer $d = J + 2K$ is degree of L-function. The integer $N$ is conductor* (*or level*) *of L-function. The pair $[J,K]$ is signature* *of L-function*. *The Sign $\varepsilon$, as complex number, appears as fourth component of the Selberg data of $L(s)$; viz, $(d, N, (\mu_1, \ldots, \mu_J : \nu_1, \ldots, \nu_K), \varepsilon)$. If all the coefficients of Dirichlet series defining $L(s)$ are real, then necessarily $\varepsilon = \pm 1$. If the coefficients are real and $\varepsilon = -1$, then $L(\frac{1}{2}) = 0$*.

The axioms of Selberg class are less restrictive than given above. Note that the functional equation above has the central point at $s = \frac{1}{2}$, and relates $s \leftrightarrow 1 - s$. As already stated, for many L-functions there is another normalization which is natural. The corresponding functional equation relates $s \leftrightarrow w + 1 - s$ for some positive integer $w$, called the motivic weight of the L-function. The central point is at $s = \frac{(w+1)}{2}$, and the arithmetically normalized Dirichlet coefficients $a_n n^{w/2}$ are algebraic integers.

The gamma factor $\Gamma_{\mathbb{R}}(s)$ in functional equation for even Analytic rank 0 polynomial-degree 1 Genus 0 curve with L-function of degree 1 $\eta(s)$ / $\zeta(s)$ over Number field $K = \mathbb{Q}$ as given by Normalized defining polynomial $\pm x$ / $x$ [of polynomial-degree 1] is $\Lambda(s) = \Gamma_{\mathbb{R}}(s)L(s) = \Lambda(1-s)$. An L-function $L(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ is called arithmetic if its Dirichlet coefficients $a_n$ are algebraic numbers. A rational L-function $L(s)$ is an arithmetic L-function with coefficient field $\mathbb{Q}$; equivalently, its Euler product in arithmetic normalization is written as product over rational primes $L(s) = \prod_p L_p(p^{-s})^{-1}$ with $L_p \in \mathbb{Z}[T]$.

The gamma factor $\Gamma_{\mathbb{R}}(s)$ present in functional equations for Degree 3 Conductor 1 Sign 1 Genus 0 curve is seen in (even) Analytic rank 0 dual L-functions 3-1-1.1-r0e3-m0.24m25.28p25.52-0 and related "counterpart" object 3/1/1.1/r0e3/p0.24p25.28m25.52/0 whereby both dual L-functions originated from e.g. GL3 Maass form that are NOT self-dual, rational or arithmetic. Their respective functional equations consist of $\Lambda(s) = \Gamma_{\mathbb{R}}(s - 25.2i)\Gamma_{\mathbb{R}}(s - 0.243i)\Gamma_{\mathbb{R}}(s+25.5i)L(s) = \overline{\Lambda}(1-s)$ and $\Lambda(s) = \Gamma_{\mathbb{R}}(s+25.2i)\Gamma_{\mathbb{R}}(s+0.243i)\Gamma_{\mathbb{R}}(s-25.5i)L(s) = \overline{\Lambda}(1-s)$. The infinitely-many $t$-valued nontrivial zeros (spectrum) derived from them[5] as transcendental (irrational) numbers are...$-22.812865$, $-19.882193$, $-17.687387$, $-16.327596$, $-14.304332$, $-12.718105$, $-9.820639$, $-7.744307$, $-6.757323$, $-3.647261$, 2.721292, 5.404222, 8.838084, 10.034902, 11.938378, 13.965832, 16.042992, 18.823934, 19.919083, 22.010794... and reverse pattern...$-22.010794$, $-19.919083$, $-18.823934$, $-16.042992$, $-13.965832$, $-11.938378$, $-10.034902$, $-8.838084$, $-5.404222$, $-2.721292$, 3.647261, 6.757323, 7.744307, 9.820639, 12.718105, 14.304332, 16.327596, 17.687387, 19.882193, 22.812865... resulting in individual Z(t) plots having Z(t) positivity but manifesting neither Line Symmetry nor Point Symmetry. However, they manifest the [combined] Line Symmetry of vertical y-axis.

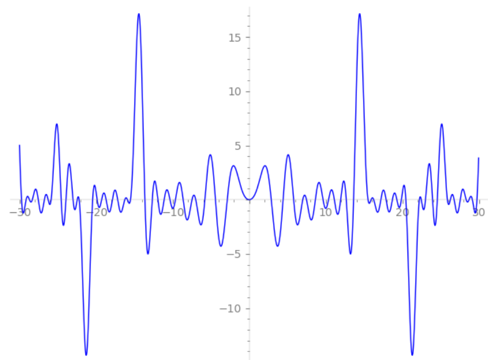


Figure 11. Graph of $Z$-function along [Analytically normalized] $\Re(s) = \dfrac{1}{2}$-critical line for $-\infty < t < \infty$ depicting UNIQUE nontrivial zeros spectrum for Genus 1 Elliptic curve LMFDB label 389.a1 having [non-zero] even Analytic rank 2. Line Symmetry of vertical $y$-axis, trajectory intersect Origin point, and manifest Z(t) positivity [viz, LMFDB's Sign normalization]. Integral points are $(-2, 0)$, $(-2, -1)$, $(-1, 1)$, $(-1, -2)$, $(0, 0)$, $(0, -1)$, $(1, 0)$, $(1, -1)$, $(3, 5)$, $(3, -6)$, $(4, 8)$, $(4, -9)$, $(6, 15)$, $(6, -16)$, $(39, 246)$, $(39, -247)$, $(133, 1539)$, $(133, -1540)$, $(188, 2584)$, $(188, -2585)$.

Examples of gamma factor $\Gamma_{\mathbb{C}}(s)$ in [Analytically normalized] functional equations for polynomial-degree 3 Genus 1 elliptic curves with self-dual L-functions of degree 2 over Number field $K = \mathbb{Q}$:
even Analytic rank 2 $E$ 389.a1 $\{y^2 + y = x^3 + x^2 - 2x\}$ [see Figure 11] is $\Lambda(s) = 389^{s/2}\Gamma_{\mathbb{C}}(s + 1/2)L(s) = \Lambda(2 - s)$
odd Analytic rank 3 $E$ 5077.a1 $\{y^2 + y = x^3 - 7x + 6\}$ [see Figure 2] is $\Lambda(s) = 5077^{s/2}\Gamma_{\mathbb{C}}(s + 1/2)L(s) = -\Lambda(2 - s)$
odd Analytic rank 3 $E$ 21858.a1 $\{y^2 + xy = x^3 + x^2 - 32x + 60\}$ [see Figure 3] is $\Lambda(s) = 21858^{s/2}\Gamma_{\mathbb{C}}(s + 1/2)L(s) = -\Lambda(2 - s)$

The gamma factor $\Gamma_{\mathbb{C}}(s)$ in [Analytically normalized] functional equation for odd Analytic rank 1 polynomial-degree 3 Genus 1 $E$ 14.1-b6 $\{y^2 + xy + y = x^3 - 2731x - 55146\}$ with self-dual L-function of degree 4 over Real quadratic field $K = \mathbb{Q}(\sqrt{7})$ is $\Lambda(s) = 10976^{s/2}\Gamma_{\mathbb{C}}(s + 1/2)^2 L(s) = -\Lambda(2 - s)$

The gamma factor $\Gamma_{\mathbb{C}}(s)$ in [Analytically normalized] functional equation for odd Analytic rank 3 polynomial-degree 3 Genus 1 $E$ 44563.1-a1 $\{y^2 + axy + ay = x^3 - x^2 + (-2a + 1)\,x\}$ with self-dual L-function of degree 4 over Imaginary quadratic field $K = \mathbb{Q}(\sqrt{-3})$ is $\Lambda(s) = 401067^{s/2}\Gamma_{\mathbb{C}}(s + 1/2)^2 L(s) = -\Lambda(2 - s)$

The gamma factor $\Gamma_{\mathbb{C}}(s)$ in [Analytically normalized] functional equation for odd Analytic rank 3 polynomial-degree 4 Genus 2 curve 35131.a.35131.1

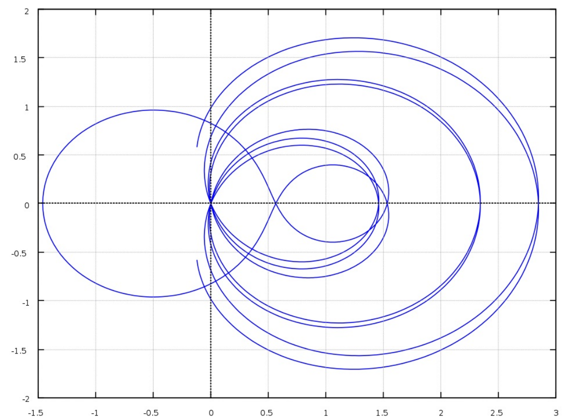


Figure 12. OUTPUT at $\sigma = \frac{1}{2}$-Critical Line. Polar graph of $\zeta(\frac{1}{2} + \imath t)$ / $\eta(\frac{1}{2} + \imath t)$ plotted for real values $t$ between $-30$ and $+30$ from $s = \sigma \pm \imath t$. Horizontal axis: $Re\{\eta(\frac{1}{2} + \imath t)\}$. Vertical axis: $Im\{\eta(\frac{1}{2} + \imath t)\}$. *Indicating Riemann hypothesis*, Origin intercept points $\equiv$ nontrivial zeros are present. Manifesting perfect Mirror (Line) symmetry about horizontal x-axis.

$\{y^2 + x^3 y = x^4 - 3x^3 + 4x^2 - 3x + 1\}$ with self-dual L-function of degree 4 over Number field $K = \mathbb{Q}$ is $\Lambda(s) = 35131^{s/2}\Gamma_{\mathbb{C}}(s + 1/2)^2 L(s) = -\Lambda(2 - s)$

*Remark* 7.1. We document the analyzed Genus 0, 1 and 2 curves with minimal Weierstrass equations. Involving gamma factor $\Gamma_{\mathbb{C}}(s)$, *all Analytic rank* 0, 1, 2, 3, 4, 5... *e.g. polynomial-degree* 3 *Genus* 1 *elliptic curves with L-functions of degree* 2 *over* $\mathbb{Q}$ *satisfy Sign normalization* which depend on even-versus-odd Analytic ranks, (BSD) Invariants, degree of L-function, Special value, etc. We observe the analogical concept akin to satisfying unitary pairing condition at prime $p$ e.g. $\Gamma_{\mathbb{R}}(s-0.2)\Gamma_{\mathbb{R}}(s+0.2)\Gamma_{\mathbb{R}}(s)^3\Gamma_{\mathbb{R}}(s+0.9)\Gamma_{\mathbb{R}}(s+1.1) \times \Gamma_{\mathbb{C}}(s+0.7)\Gamma_{\mathbb{C}}(s+1.3)^2\Gamma_{\mathbb{C}}(s+1.7)\Gamma_{\mathbb{C}}(s+7)$ and $\Gamma_{\mathbb{R}}(s-0.2+3i)\Gamma_{\mathbb{R}}(s+0.2+3i)\Gamma_{\mathbb{R}}(s+1)\Gamma_{\mathbb{R}}(s+1-8i) \times \Gamma_{\mathbb{C}}(s+0.7)\Gamma_{\mathbb{C}}(s+1.3)\Gamma_{\mathbb{C}}(s+1.3-7i)\Gamma_{\mathbb{C}}(s+1.7-7i)$: $\Gamma_{\mathbb{R}}(s) \Leftrightarrow$ "good" primes and $\Gamma_{\mathbb{C}}(s) \Leftrightarrow$ "bad" primes[2].

## 8. **Sign normalization on Graphs of Z-function as Z(t) plots**

We adopt the traditional anti-clockwise notation of Quadrant (Q) I, II, III and IV. We deduce our Q I Z(t) positivity / Q IV Z(t) negativity in Graphs of Z-function can be further shortened, without ambiguity, to Z(t) positivity

/ Z(t) negativity for range $0 < t < +\infty$. *The solutions to* $\sqrt{x}$ *become "larger values" for x sufficiently close to* 0 *e.g.* $\sqrt{9} = 3$, $\sqrt{4} = 2$, $\sqrt{0.002}$ = *"larger value"* 0.0447213..., $\sqrt{0.0002}$ = *"larger value"* 0.014142..., *etc*. Although this statement is true *per se*, it is not the reason for performing LMFDB's Sign normalization on Z(t) plots (see Axiom 4.3).

Analytic rank $r$ of elliptic curves $E$ consist of even $r = 0, 2, 4, 6, 8, 10...$ [with '$+ve$ even' Line symmetry, $\varepsilon = 1$ and $\varepsilon^{\frac{1}{2}} = +1$ or $-1$ that can arbitrarily be chosen to display Z(t) plots in two reciprocal manners "+1 Z(t)" or "$-1$ Z(t)"], and odd $r = 1, 3, 5, 7, 9...$ [with '$-ve$ odd' Point symmetry, $\varepsilon = -1$ and $\varepsilon^{\frac{1}{2}} = +i$ or $-i$ that can arbitrarily be chosen to display Z(t) plots in two reciprocal manners "$+i$ Z(t)" or "$-i$ Z(t)"]. Note: $r = 0$ for (non-elliptic) Riemann zeta function $\zeta(s)$ / Dirichlet eta function $\eta(s)$. Polar graphs e.g. all Analytically normalized $\sigma = \frac{1}{2}$-Critical Line Polar graphs of $E$, Polar graph Figure 12 on $\sigma = \frac{1}{2}$-Critical Line for (non-elliptic) $\zeta(s)$ / $\eta(s)$, etc manifest features of even functions [when having even $r$] and odd functions [when having odd $r$]. Caveat: The horizontal x-axis and vertical y-axis are arbitrarily chosen such that Line Symmetry is [dependently] horizontal x-axis for Polar graphs having even $r$, but Point Symmetry is [independently] Origin point for Polar graphs having odd $r$. Cf: Line Symmetry is [dependently] vertical y-axis for Graphs of Z-functions having even $r$, but Point Symmetry is [independently] Origin point for Graphs of Z-functions having odd $r$.

For $0 < t < +\infty$ range in plotted trajectory of Polar graph or Graph of Z-function, let distance $d$ = difference between $P_1$ (trajectory initially intersecting horizontal x-axis of Polar graph / vertical y-axis in Graph of Z-function) and $P_2$ (trajectory initially intersecting Origin point of Polar graph / Graph of Z-function). Then (i) $d = P_2 - P_1 \neq 0$ for $r = 0$ $\zeta(s)$ / $\eta(s)$ and for $r = 0$ $E$, and (ii) $d = P_2 - P_1 = 0$ for $r = 1, 2, 3, 4, 5...$ $E$ [with these findings also valid for $-\infty < t < 0$ range].

**Axes definitions for Polar graph vs Graph of $Z$-function**

The complex variable $s = \sigma \pm it$ refers to its entire range $-\infty < t < +\infty$. For complete validity, we must notationally replace $\zeta(s)$ (with Convergence for $\sigma > 1$) with $\eta(s)$ (with Convergence for $\sigma > 0$) since nontrivial zeros only occur at $\sigma = \frac{1}{2}$-Critical Line [where for elliptic curves, this will require Analytic normalization]. Polar graphs [e.g. represented by Figure 12 *with* 0-*dimensional* $\sigma = \frac{1}{2}$-*Origin point* $\equiv$ 1-*dimensional* $\sigma = \frac{1}{2}$-*Critical Line*, *and thus* *all infinitely-many* $\sigma = \frac{1}{2}$*-Origin intercept points* $\equiv$ *all infinitely-many* $\sigma = \frac{1}{2}$*-nontrivial zeros*]: Horizontal axis is $Re\{\eta(\frac{1}{2} \pm it)\}$. Vertical axis is $Im\{\eta(\frac{1}{2} \pm it)\}$. Graph of $Z$-function: Horizontal axis is variable $t$. Vertical axis is $Z(t)$. We validly use $Z(t) = \overline{\varepsilon}^{\frac{1}{2}} \frac{\gamma(\frac{1}{2}+it)}{|\gamma(\frac{1}{2}+it)|} L(\frac{1}{2}+it)$ [with $\sqrt{\epsilon}$; viz, with LMFDB's Sign normalization]. We can also validly use $Z(t) = \varepsilon \frac{\gamma(\frac{1}{2}+it)}{|\gamma(\frac{1}{2}+it)|} L(\frac{1}{2}+it)$ [without $\sqrt{\epsilon}$; viz, without LMFDB's Sign normalization].

Let $\delta = \frac{1}{\infty}$ [an infinitesimal small number value]. We select the square root that makes Z($\delta$) $+ve$ for very small $+ve$ $\delta$. That is, eventhough it is a completely arbitrary choice, we will always achieve [inevitable] standardization by choosing whichever square root makes Z($\delta$) $> 0 \equiv$ LMFDB's Sign normalization $\equiv$ resultant manifestation of Z(t) positivity.

Let $r$ = Analytic rank. Which square root we take; viz, $\sqrt{-1} = +i$ or $-i$ for odd $r$ and $\sqrt{+1} = +1$ or $-1$ for even $r$ is exactly the one needed to make Z($\delta$) $> 0$. Example 1: Manifesting Z(t) positivity, the $r = 1$ self-dual L-function from semistable elliptic curve 37.a1 (see Figure 1) requires $\sqrt{\epsilon} = +i$. By way of note, this elliptic curve is of minimal conductor with positive rank. It is also a model for quotient of modular curve $X_0(37)$ by its Fricke involution $w_{37}$; this quotient is also denoted $X_0^+(37)$. This is the smallest prime $N \in \mathbb{N}$ such that $X_0(N)/\langle w_N \rangle$ is of positive genus. Example 2: Manifesting Z(t) positivity, the $r = 3$ self-dual L-function from semistable elliptic curve 5077.a1 (see Figure 2 and its famous history in section 6) requires $\sqrt{\epsilon} = -i$.

Recall the following: Sign (root number) of the functional equation of an analytic L-function is complex number $\varepsilon$ that appears in this functional equation $\Lambda(s) = \varepsilon\overline{\Lambda}(1-s)$. An L-function $L(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ is called self-dual if its Dirichlet coefficients $a_n$ are real. Thus self-dual L-functions with odd Analytic rank must have Sign (root number) $-1$, and with even Analytic rank must have Sign (root number) $+1$.

A character has odd/even parity if it is odd/even as a function. The dual of an L-function $L(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ is the complex conjugate $\bar{L}(s) = \sum_{n=1}^{\infty} \frac{\bar{a_n}}{n^s}$. A Dirichlet character $\chi\colon \mathbb{Z} \to \mathbb{C}$ is odd if $\chi(-1) = -1$ and even if $\chi(-1) = 1$. The L-function 1-5-5.2-r1-0-0, as an example of Genus 0 curve dual L-function of Analytic rank 0 degree 1 odd parity, originate from Dirichlet character $\chi_5(2, \cdot)$ [see Figure 7] with having its functional equation as $\Lambda(s) = 5^{s/2}\Gamma_{\mathbb{R}}(s+1)L(s) = (0.850+0.525i)\overline{\Lambda}(1-s)$. Here, the Sign (root number) of $0.850+0.525i$ for [NOT self-dual] L-function 1-5-5.2-r1-0-0 could be anything of modulus 1. In contrast:

Respectively, the Analytic rank 0 degree 1 L-function 1-2e2-4.3-r1-0-0 having odd parity and 1-2e3-8.5-r0-0-0 having even parity as two examples of Genus 0 curve, originating from Dirichlet character $\chi_4(3,\cdot)$ and $\chi_8(5,\cdot)$, have functional equations $\Lambda(s) = 4^{s/2}\Gamma_{\mathbb{R}}(s+1)L(s) = \Lambda(1-s)$ and $\Lambda(s) = 8^{s/2}\Gamma_{\mathbb{R}}(s+1)L(s) = \Lambda(1-s)$. The Sign (root number) $\epsilon$ is $+1$ because both [even Analytic rank 0] L-functions are self-dual.

The nontrivial zeros, as denoted by $+ve$ $\mathbb{R}$ $\gamma$ values, of an L-function $L(s)$ are complex numbers $\rho$ for which $L(\rho) = L(\frac{1}{2} + i\gamma) = 0$. (Hardy or Riemann-Siegel) *Z-function* for Genus 0 curve Riemann zeta-function $\zeta(s)$ / Dirichlet eta function $\eta(s)$ is a real-valued function defined in terms of values of $\zeta(s)$ / $\eta(s)$ on Critical Line via formula $Z(t) := e^{i\theta(t)}\zeta\left(\frac{1}{2} + it\right)$ / $Z(t) := e^{i\theta(t)}\eta\left(\frac{1}{2} + it\right)$, where $\theta(t)$ is *Riemann-Siegel theta function* $\theta(t) := \arg\left(\Gamma\left(\frac{2it+1}{4}\right)\right) - \frac{\log\pi}{2}t$. There is a bijection between zeros $t_0$ of $Z(t)$ and zeros $\frac{1}{2} + it_0$ of $\zeta(s)$ / $\eta(s)$. $\zeta(s) = \frac{\eta(s)}{\gamma} \equiv \eta(s) = \gamma\cdot\zeta(s)$ whereby this particular $\gamma = (1 - 2^{1-s})$ is now representing the proportionality factor [and do not represent nontrivial zeros].

Z-function of a general L-function is a smooth real-valued function of a real variable $t$ such that $|Z(t)| = |L(\frac{1}{2} + it)|$. Specifically, if we write the completed L-function as $\Lambda(s) = \gamma(s)L(s)$ where $\Lambda(s)$ satisfies functional equation $\Lambda(s) = \varepsilon\overline{\Lambda}(1-s)$, then $Z(t)$ is defined by $Z(t) = \overline{\varepsilon}^{\frac{1}{2}}\frac{\gamma(\frac{1}{2}+it)}{|\gamma(\frac{1}{2}+it)|}L(\frac{1}{2} + it)$. In portion $\epsilon^{\frac{1}{2}} = \sqrt{\epsilon}$, the square root is chosen so that $Z(t) > 0$ for sufficiently small $t > 0$ $\equiv$ *Sign normalization*. The multiset of zeros of [*perpetual oscillatory function*] $Z(t)$ matches that of $L(\frac{1}{2}+it)$ and $Z(t)$ changes sign [*for infinitely-many times*] at the zeros of $L(\frac{1}{2} + it)$ of odd multiplicity.

Analogical concepts for LMFDB's Sign normalization: Recall the parity of the (simple) polynomial functions to be EITHER $\pm$ even functions OR $\pm$ odd functions: [I] e.g. $y = \pm x^{0,2,4,6,8,10...}$ being even functions with corresponding entire functions of $-\infty < x < +\infty$ range are located in Quadrant I and II when "$y$ is a $+ve$ function" and in Quadrant III and IV when "$y$ is a $-ve$ function". [II] e.g. $y = \pm x^{1,3,5,7,9,11...}$ being odd functions with corresponding entire functions of $-\infty < x < +\infty$ range are located in Quadrant I and III when "$y$ is a $+ve$ function" and in Quadrant II and IV when "$y$ is a $-ve$ function". Nomenclature: Let elliptic curve be denoted by $E$. Let $y$ and its exponents be denoted by ±Z(t) and $r$. We analyze $0 < t < +\infty$ range utilizing the [so-called] "first sinusoidal wave" of plotted Z-function for $E$ whereby we arbitrarily choose in a consistent *de-facto* manner +Z(t) in even $r$ [viz, Q I

Z(t) positivity], and +Z(t) in odd $r$ [viz, Q I Z(t) positivity]. Our analogical equivalent approach to Sign normalization is valid despite Z(t) plots perpetually oscillating above/below horizontal $t$ axis an infinite number of times after the "first sinusoidal wave".

Additionally via various Incompletely Predictable *complex interactions*, we intuitively expect frequency and complexity of nontrivial zeros (spectrum) and integer $N$ values of conductor (or level) in self-dual L-functions of elliptic curves to be empirically correlated with Analytic rank 0, 1, 2, 3, 4, 5....

## 9. **Conclusions**

With having *Analytic rank* 0 as common overlapping "component" between them, both Riemann hypothesis (RH) and Birch & Swinnerton-Dyer (BSD) conjecture involve proving the unexpected presence of certain [overall] "macro-properties". Statement: *Irrespective of L-function sources and always with one [unique] set of nontrivial zeros as OUTPUTS from each L-function, all the infinitely-many nontrivial zeros as [well-defined] Incompletely Predictable entities are ONLY located on (Analytically normalized)* $\sigma = \frac{1}{2}$*-Critical Line.*

Then with respecting Remark A.2, the above profound statement insightfully describes intractable open problem in Number theory of (Generalized) RH. Graphs of Z-function on Genus 1 elliptic curves with nonzero Analytic rank 1, 2, 3, 4, 5... have trajectories that intersect the Origin point. Graphs of Z-function on Genus 1 elliptic curves with Analytic rank 0 [viz, having zero independent basis point (with infinite order) associated with either finitely many or zero $E(\mathbb{Q})$ solutions] DO NOT have trajectories that intersect the Origin point. *Ditto* for Graph of Z-function on Genus 0 (non-elliptic) Riemann zeta function / Dirichlet eta function with Analytic rank 0 [viz, it DOES NOT have trajectory that intersect the Origin point]. *This implies the "simplest version" of BSD conjecture to be true*; and also simultaneously *implies the "simplest version" of RH to be true* (*with its Geometrical-Mathematical proof in* [9] *and its Algebraic-Transcendental proof in Appendix* A). *Adopting Z(t) positivity* in Graphs of Z-function as part of LMFDB's Sign normalization occurs for both odd and even Analytic rank elliptic curves. Studying nontrivial zeros (spectrum) using *Graphs of Z-function plots versus Polar graphs plots* to detect their underlying altered patterns, symmetry, frequency, etc in a geometrical manner promises to be a useful *"experimental"* tool to characterize L-functions of Analytic rank 0, 1, 2, 3, 4, 5....

L-functions literally encode *arithmetic information* e.g. Riemann zeta function connects through values at $+ve$ even integers (and $-ve$ odd integers) to Bernoulli numbers, with the appropriate generalization of this phenomenon obtained via $p$-adic L-functions, which describe certain Galois modules. The distribution of nontrivial zeros (spectrum), orders, and conductors, often manifesting as self-similarity or large fractal dimension, are theoretically connected to Chaos theory & Fractal geometry, random matrix theory and quantum chaos.

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## References

[1] Buhler J.P., Gross B.H. & Zagier D.B. (1985). On the conjecture of Birch and Swinnerton-Dyer for an elliptic curve of rank 3. *Math. Comp.*, 44 (170), pp. 473 – 481. https://dx.doi.org/10.1090/S0025-5718-1985-0777279-X

[2] Farmer D.W., Pitale A., Ryan N.C. & Schmidt R. (2019). Analytic L-functions: Definitions, theorems, and connections. *Bull. Amer. Math. Soc.*, 56 (2), pp. 261 – 280. https://dx.doi.org/10.1090/bull/1646

[3] Gross B.H. & Zagier D. (1980). On the critical values of Hecke L-series. *Bulletin de la Societe Mathematique de France*, 108(2), pp. 49 – 54.

[4] Gross B.H. & Zagier D.B. (1986). Heegner points and derivatives of L-series. *Invent. Math.*, 84 (2), pp. 225 – 320. https://dx.doi.org/10.1007/BF01388809

[5] LMFDB Collaboration. (2025). *The L-functions and modular forms database*. https://www.lmfdb.org [Online accessed on May 24, 2025]

[6] Noe T. (2004). A100967. The On-line Encyclopedia of Integer Sequences. https://oeis.org/A100967

[7] Rodriguez-Villegas F. & Zagier D. (1993). Square roots of central values of Hecke L-series. In *Advances in number theory* (*Kingston, ON*, 1991), pp. 81 – 99. Oxford Sci. Publ., Oxford Univ. Press, New York.

[8] Ting J. (2013). A228186. The On-Line Encyclopedia of Integer Sequences. https://oeis.org/A228186

[9] Ting J.Y.C. (2024). On the Universal Presence of Mathematics for Incompletely Predictable Problems in Rigorous Proofs for Riemann Hypothesis, Modified Polignac's and Twin Prime Conjectures. *J. Math. Res.*, 16(2) pp. 1 – 61. https://dx.doi.org/10.5539/jmr.v16n2p1

[10] Zhang Y. (2014). Bounded gaps between primes. *Ann. of Math.*, 179, (3), pp. 1121 – 1174. http://dx.doi.org/10.4007/annals.2014.179.3.7

## Appendix A. Algebraic-Transcendental proof for Riemann hypothesis using Algebraic-Transcendental theorem

Riemann zeta function $\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}=\frac{1}{1^s}+\frac{1}{2^s}+\frac{1}{3^s}+\cdots=\prod_{p \text{ prime}}\frac{1}{1-p^{-s}}$ for $\mathrm{Re}(s)>1$. $\zeta(s)$ [via its attached Euler product] is deeply connected to prime numbers [and also, *by default*, "complementary" composite numbers]. Dirichlet eta function $\eta(s)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s}=\frac{1}{1^s}-\frac{1}{2^s}+\frac{1}{3^s}-\frac{1}{4^s}+\cdots$ for $\mathrm{Re}(s)$ $>0$. Where $\Gamma(s)$ is gamma function, $\zeta(s)$ and $\eta(s)$ will satisfy their respective functional equations $\zeta(s)=2^s\pi^{s-1}\,\sin\left(\frac{\pi s}{2}\right)\,\Gamma(1-s)\,\zeta(1-s)$ and $\eta(-s)=2\frac{1-2^{-s-1}}{1-2^{-s}}\pi^{-s-1}s\sin\left(\frac{\pi s}{2}\right)\Gamma(s)\eta(s+1)$. Complex variable $s=\sigma\pm it$ where $\sigma$ and $t$ are real numbers. Critical Line: $\sigma=\frac{1}{2}$. Critical strip: $0<\sigma<1$.

Proposed by German mathematician Bernhard Riemann (17 September 1826 – 20 July 1866) in 1859, Riemann hypothesis states that all infinitely-many nontrivial zeros (NTZ), as a "thin set", of $\zeta(s)$ are located on its $\sigma=\frac{1}{2}$-Critical line. L-function associated to Genus 0 (non-elliptic) curve of $\zeta(s)$ is known to admit an analytic continuation and satisfy a functional equation via its *proxy* $\eta(s)$; viz, we do not need to assume Hasse-Weil conjecture. In [9], we have provided Geometrical-Mathematical proof for Riemann hypothesis.

The success and failures of both Gram's rule and Rosser's rule only occur in Dirichlet eta function [*proxy* for Riemann zeta function] on $\sigma=\frac{1}{2}$-Critical line. To solve Riemann hypothesis, one must analyze non-overlapping Subset of "One NTZ" = ∼66%, Subset of "Zero NTZ" = ∼17%, and Subset of "Two NTZ" = ∼17% as precisely derived from Set of "All NTZ" = ("conserved") 100% [instead of analyzing various overlapping Gram blocks and Gram intervals containing "good" or "bad" Gram points, missing NTZ or extra NTZ].

Transcendental functions $\gg$ Algebraic functions with the Uncountably Infinite Set of Transcendental irrational numbers $\gg$ Countably Infinite Set of Algebraic irrational numbers. From selected mathematical arguments, we formally derive Algebraic-Transcendental theorem which supports the Statement: *Algebraic functions must form a subset of the broader class of Transcendental functions*. We now supply a non-exhaustive list of Algebraic-Transcendental links. This will suffice for our purpose to create Algebraic-Transcendental theorem required to complete the deceptively simple Algebraic-Transcendental proof for Riemann hypothesis.

LEMMA A.1. *We outline relevant Algebraic-Transcendental connections when based on algebraic functions and algebraic numbers, and transcendental functions and transcendental numbers.*

**Proof.** An algebraic function is a function often defined as root of an irreducible polynomial equation. The algebraic functions are usually algebraic expressions using a *finite number of terms*, involving only algebraic operations addition (+), subtraction (−), multiplication (×), division (÷), and raising to a fractional power. Examples of pure algebraic function are: $f(x) = \dfrac{1}{x}$, $f(x) = \sqrt{x}$, $f(x) = \dfrac{\sqrt{1+x^3}}{x^{3/7} - \sqrt{7}x^{1/3}}$, Golden ratio $\phi = \dfrac{1+\sqrt{5}}{2} = 1.6180339887...$ [that is the most irrational number because it's hard to approximate with a rational number], etc. Algebraic functions usually cannot be defined as finite formulas of elementary functions, as shown by the example of Bring radical $f(x)^5 + f(x) + x = 0$ (this is the Abel-Ruffini theorem).

A transcendental function is an analytic function that does not satisfy a polynomial equation whose coefficients are functions of independent variable written using only the basic operations of addition, subtraction, multiplication, and division (without the need of taking limits). Examples of pure transcendental functions are: logarithm function $\ln x$ or $\log_e x$, exponential function $e^x$, trigonometric functions $\sin x$ and $\cos x$, hyperbolic functions $\sinh x$ and $\cosh x$, generalized hypergeometric functions, class of numbers called Liouville numbers [that can be more closely approximated by rational numbers than can any irrational algebraic number], etc. Equations over these expressions are called transcendental equations. A transcendentally transcendental function or hypertranscendental function is transcendental analytic function which is not the solution of an algebraic differential equation with coefficients in integers $\mathbb{Z}$ and with algebraic initial conditions; e.g. zeta functions of algebraic number fields, in particular, Riemann zeta function $\zeta(s)$ and gamma function $\Gamma(s)$ (cf. Holder's theorem).

The indefinite integral of many algebraic functions is transcendental. For example, integral $\displaystyle\int_{t=1}^{x} \frac{1}{t} dt$ turns out to equal logarithm function $log_e(x)$. Similarly, the limit or the infinite sum of many algebraic function sequences is transcendental. Example, $\lim\limits_{n\to\infty}(1 + x/n)^n$ converges to exponential function $e^x$, and infinite sum $\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$ turns out to equal hyperbolic cosine function $\cosh x$. In fact, it is impossible to define any transcendental function in terms of algebraic functions without using some such "limiting procedure" (integrals, sequential limits, and infinite sums are just a few).

A function that is not a transcendental function must logically be an algebraic function. This implies every algebraic function is algebraic solution to a polynomial equation but transcendental functions are not solutions to any such equation. Stated in another way: The output values of an algebraic function (for specific input values of $x$) are algebraic numbers. This is because algebraic function itself is defined as a solution to an algebraic equation, and any solution to such an equation is [and must be] an algebraic number.

While transcendental functions often produce transcendental numbers as outputs, they also have solutions as algebraic numbers. The composition of transcendental functions in $f(x) = \cos\arcsin x = \sqrt{1-x^2}$ will give an algebraic function. Outputs from transcendental functions as algebraic numbers: Equation $e^x = 1$ has solution $x = 0$, an algebraic number (since 0 is algebraic). Equation $\sin(x) = 0$ has solutions $x = n\pi$, where $n = 0, 1, 2, 3, 4, 5...$ are algebraic numbers (since integers are algebraic). Equation $\ln(x) = 0$ has solution $x = 1$, an algebraic number (since 1 is algebraic). Outputs from transcendental function as transcendental numbers: Equation $e^x = 2$ has solution $x = \ln(2)$, which is transcendental, since $\ln(2)$ is a transcendental number.

*Remark* A.1. Two trigonometric functions in equation $\sin(x) = \cos(x) = \frac{\pi}{4}$ have identical solution $x = \frac{1}{\sqrt{2}}$. This "sweet-spot" property is due to sine-cosine complementary angle relationship for isosceles triangle. The $\frac{1}{\sqrt{2}} \approx 0.70710678$ is (algebraic) irrational number and $\frac{\pi}{4}$ is (transcendental) irrational number. Then $\frac{\pi}{4}$ radian $\approx 0.785398$ radian $\equiv 45^o$.

***The proof is now complete for Lemma A.1□.***

Proposition A.2. *Algebraic functions never give rise to transcendental numbers as outputs unless we start involving transcendental functions.*

**Proof.** Algebraic numbers = {Integers + Rational numbers + Roots of Integers (or Algebraic irrational numbers) as Algebraic (non-complex) numbers} + {$z = a + bi$ as Algebraic (complex) numbers where $a$, $b$ must be Integers or Rational numbers}. Thus certain algebraic functions may involve more complex operations such as roots or radicals, giving complicated outputs that are still algebraic. We deduce from mathematical arguments in Lemma A.1: While a given *de novo* function itself is algebraic [viz, a pure algebraic function], it will never give rise to transcendental numbers unless we involve transcendental functions [viz, create a mixed algebraic-transcendental function].

***The proof is now complete for Proposition A.2□.***

Corollary A.3. *Any outputs as transcendental numbers from a given function must involve transcendental functions, which can be given as either pure transcendental functions or mixed algebraic-transcendental functions.*

**Proof.** Pure algebraic functions always give outputs that are algebraic [but never transcendental]. Both pure transcendental functions and mixed algebraic-transcendental functions give outputs as transcendental numbers $\pm$ algebraic numbers [but never as outputs that are all algebraic numbers].

Examples of mixed algebraic-transcendental functions: $f(x) = x^2 + e^x$ that involves both algebraic and transcendental terms; and $f(x) = e^{\sqrt{2}}$ that involves transcendental operation on algebraic number.

***The proof is now complete for Corollary A.3*** $\square$.

Axiom A.4. *Nontrivial zeros (spectrum) computed [e.g. using Hardy Z-function as $Z(t)$ plots] for any L-function involve transcendental functions in one form or another, and are inherently given as $t$-valued transcendental numbers.*

**Proof.** It is precisely the case that since all infinitely-many nontrivial zeros (spectrum) computed [e.g. using Hardy Z-function as Z(t) plots] for any given L-function will involve transcendental functions in one form or another [often as mixed algebraic-transcendental functions]; then it is simply a mathematical impossibility that nontrivial zeros as outputs will not be given as $t$-valued transcendental numbers. This deduction is completely consistent with our Proposition A.2 and Corollary A.3.

A particular solution with '0' (zero) value from a given function may imply that function to be a pure algebraic function, a pure transcendental function or a mixed algebraic-transcendental function. Integer 0 is an algebraic number (since 0 is algebraic). L-functions are usually mixed algebraic-transcendental functions: [1] Analytic rank 0 L-functions will never have their $1^{st}$ nontrivial zero being endowed with algebraic 0 value. [2] Analytic rank 1, 2, 3, 4, 5... (viz, non-zero $\geq 1$) L-functions will always have their $1^{st}$ nontrivial zero being endowed with algebraic 0 value.

***The proof is now complete for Axiom A.4*** $\square$.

Theorem A.5. *We can categorically formulate Algebraic-Transcendental theorem which states that all infinitely-many nontrivial zeros (spectrum) from Riemann zeta function must be located on its $\sigma = \frac{1}{2}$-Critical Line [as was originally proposed by the 1859-dated Riemann hypothesis].*

**Proof.** Being a self-dual L-function, Riemann zeta function as a Genus 0 curve admits an analytic continuation and satisfy a functional equation via *proxy* Dirichlet eta function. By its very definition, geometrical $\sigma = \frac{1}{2}$-Origin point ≡ mathematical $\sigma = \frac{1}{2}$-Critical line. All infinitely-many Origin intercept points ≡ All infinitely-many Nontrivial zeros are proposed to lie on this Critical line $\implies$ Geometrical-Mathematical proof for Riemann hypothesis as outlined in [9]. Consistent with Axiom A.4 is the fact that all infinitely-many nontrivial zeros from Dirichlet eta function [*proxy* function for Riemann zeta function] are always given as $t$-valued transcendental (irrational) numbers. From previous mathematical arguments in section 3 on properties for Incompletely Predictable entities, there are two occurrences of these entities in nontrivial zeros: (i) The integer numbers representing each and every one of infinitely-many nontrivial zeros, and (ii) Infinitely-many digital numbers after decimal point in each and every one of infinitely-many nontrivial zeros.

$\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$ [as example involving infinite sum of infinitely-many algebraic functions] turns out to equal hyperbolic cosine function $\cosh x$ [which is a transcendental function]. We can now conceptually represent a transcendental function as infinite sum of infinitely-many algebraic function sequences [viz, an 'infinite series']. Thus when based on inclusion-exclusion principle, we validly deduce that $\sigma = \frac{1}{2}$-Dirichlet eta function is an unique $\sigma = \frac{1}{2}$-mixed-algebraic-transcendental function [that contains all nontrivial zeros] AND $\sigma \neq \frac{1}{2}$-Dirichlet eta functions are infinitely-many non-unique $\sigma \neq \frac{1}{2}$-mixed-algebraic-transcendental functions [that cannot contain nontrivial zeros]. We now have the mutually exclusive statement based on $\sigma = \frac{1}{2}$-Dirichlet eta function and $\sigma \neq \frac{1}{2}$-Dirichlet eta functions being completely different 'infinite series': {*It is a mathematical impossibility for any nontrivial zeros to be located away from Critical line.*} ≡ {*It is a mathematical certainty for all nontrivial zeros to be located on Critical line.*}

Euler formula can be stated as $e^{\imath n} = \cos n + \imath \cdot \sin n$. Applying this famous formula to $\sigma = \frac{1}{2}$-Dirichlet eta function results in simplified $\sigma = \frac{1}{2}$-Dirichlet eta function that faithfully contains all $t$-valued nontrivial zeros [whereby this simplified function will clearly identify itself as representing a mixed-algebraic-transcendental function involving both algebraic and transcendental functions]: The simplified $\sigma = \frac{1}{2}$-Dirichlet eta function

$$= \sum_{n=1}^{\infty} (2n)^{-\frac{1}{2}} 2^{\frac{1}{2}} \cos(t \ln(2n) + \frac{1}{4}\pi) - \sum_{n=1}^{\infty} (2n-1)^{-\frac{1}{2}} 2^{\frac{1}{2}} \cos(t \ln(2n-1) + \frac{1}{4}\pi)$$

***The proof is now complete for Theorem A.5□.***

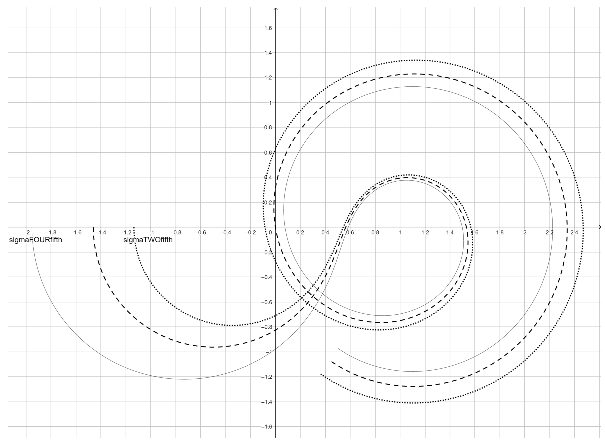


Figure 13. Simulated dynamic trajectories showing Origin intercept points when $\sigma = \frac{1}{2}$ and virtual Origin intercept points when $\sigma = \frac{2}{5}$ and $\sigma = \frac{4}{5}$. Horizontal axis: $Re\{\zeta(\sigma + \imath t)\}$ / $Re\{\eta(\sigma + \imath t)\}$, and vertical axis: $Im\{\zeta(\sigma + \imath t)\}$ / $Im\{\eta(\sigma + \imath t)\}$. Presence of Origin intercept points at [static] Origin point. Presence of virtual Origin intercept points as additional negative virtual Gram[y=0] points on x-axis (e.g. using $\sigma = \frac{2}{5}$ value) at [infinitely-many varying] virtual Origin points; viz, these negative virtual Gram[y=0] points on x-axis cannot exist at Origin point since two trajectories form co-linear lines (or co-lines) [*viz*, *two parallel lines that never cross over near Origin point*].

*Remark* A.2. Hasse-Weil zeta function is a global L-function defined as an Euler product of local zeta functions. Hasse-Weil conjecture states that Hasse-Weil zeta function attached to an algebraic variety $V$ defined over an algebraic number field $K$ should *admit an meromorphic continuation* for all complex $s$ and *satisfy a functional equation*. In, for instance, Genus 2 curves over totally real fields, they have non-regular Hodge numbers and the Taylor-Wiles method that was successful in proving this conjecture for Genus 1 curves (for example) breaks down in several places. Many of the L-functions we consider in this paper (including those associated to curves of Genus $> 1$), are not known to admit an analytic continuation or satisfy a functional equation. To properly discuss nontrivial zeros on the Critical Line and in the Hardy Z-function; we therefore need to, at least, assume this conjecture.

Taking Remark A.2 into perspective consideration; all the correct and complete mathematical arguments in this paper are assumed to comply with two conditions below [that have "Analytic rank 0" component present in both]:

**Condition 1**. Generalized Riemann hypothesis (RH): All the nontrivial zeros (spectrum) of General [or Generic] L-functions from Genus 0, 1, 2, 3, 4, 5... curves with Analytic rank 0, 1, 2, 3, 4, 5... lie on the $\sigma = \frac{1}{2}$-Critical Line or the Analytically normalized $\sigma = \frac{1}{2}$-Critical Line. The 'special case' (*simplest*) RH[9] refers to the [Analytic rank 0] Genus 0 non-elliptic curve called Riemann zeta function / Dirichlet eta function.

**Condition 2**. Generalized Birch and Swinnerton-Dyer (BSD) conjecture: All Generic L-functions from Genus 0, 1, 2, 3, 4, 5... curves satisfy Algebraic (Mordell-Weil) rank = Analytic rank [for even Analytic rank 0, 2, 4, 6, 8, 10... and odd Analytic rank 1, 3, 5, 7, 9, 11...]. The 'special case' (*simplest*) BSD conjecture refers to Genus 1 elliptic curves; expressed as *weak form* and *strong form* of BSD conjecture.

***Analogy for (Generalized) Riemann hypothesis***: Let $\delta = \frac{1}{\infty}$ [which represents an infinitesimal small number value], Geometrical 0-dimensional $\sigma = \frac{1}{2}$-Origin point $\equiv$ Mathematical 1-dimensional $\sigma = \frac{1}{2}$-Critical Line, and Origin intercept points $\equiv$ nontrivial zeros. [Using sine-cosine complementary angle relationship $\sin(\theta) = \cos(\theta - \frac{\pi}{2}) \equiv \cos(\theta) = \sin(\theta - \frac{\pi}{2}) \equiv$ "always having complete set of nontrivial zeros" as alternative analogical explanation: Riemann hypothesis is uniquely denoted by $\theta = \frac{\pi}{4}$ with $\sin(\theta) = \cos(\theta) = \frac{1}{\sqrt{2}}$ whereby all (100%) nontrivial zeros are "conserved" despite success / failure of Gram's rule and Rosser's rule. Then Generalized Riemann hypothesis are non-uniquely denoted by $\theta \neq \frac{\pi}{4}$ with $\sin(\theta) \neq \cos(\theta) \neq \frac{1}{\sqrt{2}}$.]

*Proposition*: Always having Origin point intercept $\Leftrightarrow \sin x = \cos(Ax - \frac{C\pi}{2})$ uniquely when $C = 1$.

*Corollary*: Never having Origin point intercept $\Leftrightarrow \sin x \neq \cos(Ax - \frac{C\pi}{2})$ non-uniquely when $C = 1 \pm \delta$.

Assigned values for $A$ is "inconsequential" in the sense that the solitary $A = 1$ value $\Longrightarrow$ 'special case' Riemann hypothesis [on Genus 0 curve], and the multiple $A \neq 1$ values $\Longrightarrow$ Generalized Riemann hypothesis [on Genus 1, 2, 3, 4, 5... curves].

*Remark* A.3. Geometrical-Mathematical proof[9] for Riemann hypothesis is exemplified by Figure 5, Figure 12 and Figure 13. Let $\delta = \frac{1}{\infty}$ [an infinitesimal small number value] in reference to Figure 13. Then the plotted trajectories arising from inputting $\sigma = \frac{1}{2} + \delta$ and $\sigma = \frac{1}{2} - \delta$ into Riemann zeta function/Dirichlet eta function will always result in two co-linear lines being located (approximately) an infinitesimal small $\delta$ distance, respectively, just to right and left of Origin point [but never touching Origin point $\equiv$ Critical line].

*Remark* A.4. ***Proof by induction for Riemann hypothesis using plotted co-linear lines [that conceptually comply with inclusion-exclusion principle].*** For $n = 0, 1, 2, 3, 4, 5, ..., \infty$ in reference to $-\infty < t < +\infty$ when inputting $\sigma = \frac{1}{2}+n\delta$ [$\equiv$ "To Right of Origin Point"] and $\sigma = \frac{1}{2}-n\delta$ [$\equiv$ "To Left of Origin Point"] into Riemann zeta function/Dirichlet eta function, there are infinitely-many [self-similar] plotted trajectories as co-linear lines using Polar graph in, and to cover, entire $0 < \sigma < 1$-Critical strip.

*Proving the Base case when* $n = 1$: At $n = 0$ [$\equiv$ "On the Origin Point"] in Figure 12 using either $\sigma = \frac{1}{2} + n\delta$ or $\sigma = \frac{1}{2} - n\delta$, this will always represent the Polar graph at $\sigma = \frac{1}{2}$-Critical line with having all (100%) nontrivial zeros, thus implying Riemann hypothesis to be true. At $n = 1$ [$\equiv$ "To Right of Origin Point"] using $\sigma = \frac{1}{2} + n\delta$, this will always represent the Polar graph at $\sigma \neq \frac{1}{2}$-Non-critical line without having any (0%) nontrivial zeros. At $n = 1$ [$\equiv$ "To Left of Origin Point"] using $\sigma = \frac{1}{2} - n\delta$, this will always represent the Polar graph at $\sigma \neq \frac{1}{2}$-Non-critical line without having any (0%) nontrivial zeros.

*Induction step*: Suppose $\sigma = \frac{1}{2} + k\delta \equiv$ "To Right of Origin Point" or $\sigma = \frac{1}{2} - k\delta \equiv$ "To Left of Origin Point" for some $k > 0$ [viz, $k = 1, 2, 3, 4, 5, ..., \infty$]. Based on deviation property "Increasing distance away from Origin Point as $k$ becomes larger", we correctly claim both scenario are valid for next case $k+1$ that always represent Polar graph at $\sigma \neq \frac{1}{2}$-Non-critical line without having any (0%) nontrivial zeros. We now establish the truth of this statement for all natural numbers $k \geq 1$, thus implying Riemann hypothesis to be true.

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