

Convergence of the Riemann Z Function

HORACIO USECHE LOSADA
Universidad Nacional de Colombia
horaciouseche@gmail.com

August 8, 2025

Contents

1	Introduction	2
2	The Riemann zeta function	3
3	The analytical extension of ζ	7
4	The zeros of the $\hat{\zeta}$ function	8
5	Convergence according to number of terms	14
6	Twin zeros	15
7	Existence of infinite zeros	15
8	The zeros by reflection	16
9	Convergence conditions	18
9.1	Algorithm use_riem_03	22
9.2	Algorithm use_riem_04	23
9.3	Algorithm use_riem_05	24
9.4	Algorithm use_riem_06	25
9.5	Algorithm use_riem_07	26

Abstract

The Riemann conjecture is false. The zeros of the function $\zeta(s)$ are placed on $0.43 \leq \Re(s) < 1$ interval. The straight lines with possible infinite zeros are $\Re(z) = 0.43$, $\Re(z) = 0.47$, $\Re(z) = 0.55$, $\Re(z) = 0.67$, $\Re(z) = 0.79$, $\Re(z) = 0.84$, $\Re(z) = 0.90$, $\Re(z) = 0.91$, $\Re(z) = 0.92$, $\Re(z) = 0.93$, $\Re(z) = 0.94$, $\Re(z) = 0.95$, $\Re(z) = 0.96$, $\Re(z) = 0.97$, $\Re(z) = 0.98$, y $\Re(z) = 0.99$, there are other lines with many zeros, though with minor density.

We provide necessary and sufficient conditions to yield the convergence of the zeros of the Riemann zeta (ζ) function. A new expression for the Riemann zeta function is also deduced, in terms of a series of sines and cosines, as expected! In the same way, we confirm the existence of the **zeros by reflection** predicted by the functional equation of the zeta function and we define the concept of **twin zeros** by analogy with the twin primes of the numbers theory.

Keywords: Riemann hypothesis, Riemann zeta function (ζ), Riemann zeros, Z function, convergence, zeros by reflection, twin zeros, exclusive zeros.

1 Introduction

Since 1859, the year in which **Bernhard Riemann** calculated the first zeros of the **zeta function**, one of two options was expected to happen:

- That someone will find a proof of the *Riemann Hypothesis*, or in the absence of this,
- That someone will find a countersample of it.

In November of 2017, in the framework of the **X International Seminar on Applied Mathematics**, carried out at the Surcolombiana university in Neiva (Huila, Colombia), a paper titled “**Convergencia de la función Z de Riemann**” was presented, in which, the author discloses a large number of countersamples of the **Riemann conjecture**, and proposes **necessary and sufficient conditions** for the convergence of the series involved in the calculation of the **Riemann zeta function** and its zeros. This article summarizes the most relevant theoretical aspects of this event and the documents associated with it.

2 The Riemann zeta function

The Riemann Z function is defined (Euler, 1737) by:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad (1)$$

where $s \in \mathbb{C}$. With $s = a + bi$. This function is special, in many ways, however, here, we only refer to its role in the convergence of the so called **zeros of the Riemann Z function**, ζ .

The expression for $\zeta(s)$, given above, is not computable, so, first we need to derive a computable equation that can be used to calculate the **zeros of the Riemann Z function**. Using this definition, the following expression is obtained:

Theorem 1.

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \sum_{n=1}^{\infty} \frac{\cos(b \ln n)}{n^a} - i \sum_{n=1}^{\infty} \frac{\sin(b \ln n)}{n^a} \quad (2)$$

which is a computable expression for $\zeta(s)$, where we can clearly differentiate the real and imaginary components.

Proof:

Give the Riemann Z function, defined as it appears above in Ec. 1, where $s \in \mathbb{C}$, doing $s = a + bi$ and substituting in $\zeta(s)$, we have:

$$\frac{1}{n^s} = n^{-(a+bi)} = n^{-a} n^{-bi} \quad (3)$$

$$\frac{1}{n^s} = n^{-a} e^{\ln n^{-bi}} = n^{-a} e^{-bi \ln n} = n^{-a} e^{i(-b \ln n)} \quad (4)$$

doing $\theta = -b \ln n$, and knowing that $e^{i\theta} = \cos \theta + i \sin \theta$, the above equation transforms in:

$$\frac{1}{n^s} = n^{-a} e^{i\theta} = n^{-a} [\cos \theta + i \sin \theta] \quad (5)$$

$$\frac{1}{n^s} = n^{-a} [\cos(-b \ln n) + i \sin(-b \ln n)] \quad (6)$$

Now, since $\cos(-\theta) = \cos \theta$ and $\sin(-\theta) = -\sin \theta$, we have:

$$\frac{1}{n^s} = n^{-a} \cos(b \ln n) - i n^{-a} \sin(b \ln n) \quad (7)$$

$$\frac{1}{n^s} = \frac{\cos(b \ln n)}{n^a} - i \frac{\sin(b \ln n)}{n^a} \quad (8)$$

by entering the sum of the definition of ζ , we can write:

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \sum_{n=1}^{\infty} \left(\frac{\cos(b \ln n)}{n^a} - i \frac{\sin(b \ln n)}{n^a} \right) \quad (9)$$

finally, by a property of summations, we obtain:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \sum_{n=1}^{\infty} \frac{\cos(b \ln n)}{n^a} - i \sum_{n=1}^{\infty} \frac{\sin(b \ln n)}{n^a} \quad (10)$$

The above equation is perfectly computable and, its behavior can be analyzed with the help of the following fragment of source code, written in Go language:¹

```
func Riemann_cmp(precision int,p_real,p_img float64)
(float64,float64){
var n,i int
var a,b,ang,sen,cos,pr float64
var fre,fcmp,cmp,re,sumac,sumar float64
//fmt.Println("====Funcion Riemann Compleja =====")
//fmt.Printf("Tiempo antes de ejecutar: %v \n",time.Now())
n=precision
a=p_real
b=p_img
sumar=0
sumac=0
for i=1;i<=n;i++){
ang=b*math.Log(float64(i))
sen=math.Sin(ang)
```

¹Google Inc., sponsor of Go Lang.

```
cos=math.Cos(ang)
pr=math.Pow(float64(i),a)
fre=cos/pr
fcmp=sen/pr
re=fre+sumar
cmp=fcmp+sumac
//fmt.Printf("I=%d re: %g imag: %g\n",i,re,cmp)
sumar=re
sumac=cmp
}
//fmt.Printf("Tiempo despues de ejecutar: %v \n",time.Now())
return re,cmp
}
```

The function takes three arguments with the following meaning:

- The parameter **precision** represents the number of terms that are involved in the sum of $\zeta(s)$, to yields its convergence. A very low value (< 1000) it may be insufficient and a very high value (> 1000000) causes convergence to be delayed.
- The parameter **p_real** represents the actual real value of s , that is, $\Re(s)$.
- The parameter **p_img** represents the imaginary value of s , that is, $\Im(s)$.

The function also returns two values in the variables **re** and **cmp**, which represents, respectively, the real and imaginary parts of the function $\zeta(s)$. The function **Riemann_cmp()**, is executed with the help of a function **main()** in the following way:

```
func main() {
    var precision int
    var p_real,p_img float64
    precision=10000000
    p_real=1.5
    p_img=10.3
    Riemann_cmp(precision,p_real,p_img)
}
```

where the values $\Re(s) = 1.5$ and $\Im(s) = 10.3$ are used with one million of terms to produce convergence. The result of compiling and running this routine is shown below:

```
.  
. .  
I=999996 re: 1.2629451829358893 imag: 0.15253657032810491  
I=999997 re: 1.2629451823364974 imag: 0.1525365695276436  
I=999998 re: 1.2629451817371147 imag: 0.15253656872717733  
I=999999 re: 1.262945181137741 imag: 0.15253656792670608  
I=1000000 re: 1.2629451805383765 imag: 0.15253656712622987
```

where, for convenience, only the last 5 values are shown in the output, which allows us to see that $\zeta(1.5, 10.3) = 1.26294518 + i0.152536567$, that is, that $\zeta(s)$ converges for values of $\Re(s)$ greater than 1.

Now, again evaluating the function **Riemann_cmp()**, but this time using the values $\Re(s) = 0.5$ and $\Im(s) = 10.3$ with ten million of terms, the result is as follows:

```
.  
. .  
I=991003 re: -73.08429226631614 imag: 61.472730311215976  
I=991004 re: -73.08496647510914 imag: 61.471985650107754  
I=991005 re: -73.0856406758223 imag: 61.4712409823679  
I=991006 re: -73.08631486845559 imag: 61.47049630799651  
. .  
I=999997 re: -78.81479813815487 imag: 54.5185409587966  
I=999998 re: -78.81539751976084 imag: 54.51774049412503  
I=999999 re: -78.81599689282228 imag: 54.5169400236801  
. .  
I=9999995 re: 132.02025846068926 imag: 277.73604962072875  
I=9999996 re: 132.01997914977514 imag: 277.7361978956533  
I=9999997 re: 132.01969983872223 imag: 277.73634617028273
```

I=9999998 re: 132.01942052753057 imag: 277.73649444461705
 I=9999999 re: 132.01914121620015 imag: 277.73664271865624
 I=10000000 re: 132.01886190473098 imag: 277.73679099240036
 Tiempo después de ejecutar: 2017-05-21 16:32:20.9620112 -0500

where we can see that the function does not converge, as the theory indicates.

3 The analytical extension of ζ

The Riemann ζ function, as defined above, does not converge for values in the interval $0 < \Re(s) < 1$, and therefore, the Z function is not analytic in that interval. An expression of ζ that can converge for $0 < \Re(s) < 1$ is needed. This expression is called the **analytical extension of $\zeta(s)$** and is defined as (see [3], p. 3):

$$\zeta(s) = \frac{1}{1 - 2^{1-s}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} \quad (11)$$

where there is clearly a factor that can not be annulled. Therefore, if $\zeta(s)$ is null, it will be only by virtue of the term $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}$, which, as is easily observed, is susceptible to vanished. That is, with these adjustments the definitive expression to find the zeros of the ζ function, take the form:²

Theorem 2.

$$\hat{\zeta}(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \cos(b \ln n)}{n^a} - i \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \sin(b \ln n)}{n^a} \quad (12)$$

Proof: It is enough to replace Ec. 2 in the expression 11 to produce the result, obviating the term that can not be annulled.

The $\hat{\zeta}$ function given here is the computable form used to find the functional zeros of $\hat{\zeta}$, where it is clear that **the expression is annulled if and only if its real and imaginary parts are null**, that is:

²Note that, to define $\hat{\zeta}$, we only use the part that is susceptible of being null, consequently the term $\frac{1}{1-2^{1-s}}$ is not considered, when it comes to calculating the zeros of the ζ function.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} \cos(b \ln n)}{n^a} = 0$$
$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} \sin(b \ln n)}{n^a} = 0$$

4 The zeros of the $\hat{\zeta}$ function

Bernhard Riemann was the first to calculate these zeros, back in 1859 year, obtaining the first three zeros of $\hat{\zeta}$, namely:³

$$0.5 + 14.135i$$

$$0.5 + 21.022i$$

$$0.5 + 25.011i$$

which undoubtedly must have been a remarkable feat considering the difficulty of the calculation and the fact that Riemann did not have calculators or computer machines.

To compute the $\hat{\zeta}(s)$ function, and search for its functional zeros, we can use the following Go source code fragment:

```
func RiemannSerie(precision int,p_real,p_img
float64)(float64,float64){
// calcula la serie de Riemann con extension analitica
var n,i int
var a,b,ang,sen,cos,pr float64
var fre,fcmp,cmp,re,sumac,sumar float64
var bas,fac float64
n=precision
sumar=0
sumac=0
bas=-1.0
```

³In fact, it is unknown exactly how many zeros Riemann calculated, in this subject the researchers do not agree. However, judging by the resources available to Riemann, it is reasonable to believe that he could not calculate more than two or three zeros.

```
a=p_real
b=p_img
for i=1;i<n;i++){
  ang=b*math.Log(float64(i))
  sen=math.Sin(ang)
  cos=math.Cos(ang)
  pr=math.Pow(float64(i),a)
  fac=math.Pow(bas,float64(i-1))
  fre=fac*(cos/pr)
  fcmp=fac*(sen/pr)
  re=fre+sumar
  cmp=fcmp+sumac
  sumar=re
  sumac=cmp
}
return re,cmp
}
```

where the parameters and the values returned retain the meaning explained above.

Evaluating this function for $\hat{\zeta}(1/2, 10.3)$, that is, with the values used previously (which do not converge on the function $\zeta(s)$ without extending analytically) we obtained:

```
I=999992 re: 1.2629451853335487 imag: 0.15253657352990038
I=999993 re: 1.2629451847341202 imag: 0.15253657272945897
I=999994 re: 1.2629451841347008 imag: 0.1525365719290126
I=999995 re: 1.2629451835352905 imag: 0.15253657112856125
I=999996 re: 1.2629451829358893 imag: 0.15253657032810491
I=999997 re: 1.2629451823364974 imag: 0.1525365695276436
I=999998 re: 1.2629451817371147 imag: 0.15253656872717733
I=999999 re: 1.262945181137741 imag: 0.15253656792670608
I=1000000 re: 1.2629451805383765 imag: 0.15253656712622987
```

where, after a million sums, the function converges, as expected.

With these tools and considering that, for first time in the history of this problem, we have a directly computable expression in the complex plane, which allows to sweep the plane, in the interval $0 < \Re(s) < 1$, we proceed in

this way to calculate the first zeros in the line $\Re(s) = 1/2$, which are shown in table 1.

No	$\Re(s)$	$\Im(s)$	$\Re(\zeta(s))$	$\Im(\zeta(s))$	Norm ($\zeta(s)$)
1	0.5	14.13	-0.0000527	0.0090912	0.0000826
2	0.5	21.02	0.01280955	0.0254116	0.00080983
3	0.5	25.01	-0.0007136	0.0015427	0.00000288
4	0.5	30.42	0.00041551	0.0135421	0.00018356
5	0.5	32.93	0.00090074	0.0156852	0.00024683
6	0.5	37.58	0.01206027	0.0050403	0.00017085

Table 1: First six Riemann zeros calculated in $\Re(s) = \frac{1}{2}$, with one million of terms in the respective summation.

The zeros shown in the table 1, have been calculated with the function **RiemannSerie()**, using, for brevity, a million of terms. Following this procedure, we sweep the complex plane, finding the existence of “infinite” zeros in the lines shown in the table 7.

The zeros illustrated in the table 7, correspond to the zeros found by the author but, in no way do they exhaust the possibilities of finding more lines. It is curious to note that in the $0 < \Re(s) < 0.43$ stripe there are no zeros! Therefore, the *research radius* is limited to the interval $0.43 < \Re(s) < 1$.

To better illustrate this aspect, we present some zeros of the $\hat{\zeta}$ function in the tables 2, 3, 4, 5, and 6.

No	$\Re(s)$	$\Im(s)$	$\Re(\hat{\zeta}(s))$	$\Im(\hat{\zeta}(s))$	Norm ($\hat{\zeta}(s)$)
1	0.47	14.09	-0.0591769	0.0650853	0.0077380
2	0.47	21.0	-0.0385952	0.0807528	0.0080106
3	0.47	25.0	-0.0722184	-0.0231961	0.0057535
4	0.47	37.6	-0.0582022	0.0493030	0.0058182
5	0.47	48.0	-0.0977008	0.0085135	0.0096179
6	0.47	72.1	0.07315905	-0.023483	0.0059037
7	0.47	127.49	0.043323	0.0839421	0.0089232
8	0.47	145.99	-0.04931	0.0779763	0.0085118
9	0.47	162.99	0.006458	-0.084969	0.0072615
10	0.47	185.59	-0.0800499	0.0508676	0.008995

Table 2: First 10 zeros of $\hat{\zeta}(s)$ in the line $\Re(s) = 0.47$.

No	$\Re(s)$	$\Im(s)$	$\Re(\zeta(s))$	$\Im(\zeta(s))$	Norm($\zeta(s)$)
1	0.53	14.09	0.0532608	0.0655659	0.0071356
2	0.53	21.0	0.0827857	0.0135426	0.0070368
3	0.53	25.0	0.0396154	0.0595324	0.0051134
4	0.53	37.6	0.0015760	-0.0701232	0.0049197
5	0.53	48.0	0.0863803	0.0209932	0.0079022
6	0.53	72.1	0.0279098	0.0591989	0.0042834
7	0.53	127.49	0.0431644	-0.0670067	0.0063530
8	0.53	145.99	0.0491745	-0.0637075	0.00647677
9	0.53	162.99	-0.0682554	-0.00551149	0.00468918
10	0.53	185.59	0.0701395	-0.0462074	0.00705468

Table 3: First 10 zeros of $\hat{\zeta}(s)$ in the line $\Re(s) = 0.53$

No	$\Re(s)$	$\Im(s)$	$\Re(\zeta(s))$	$\Im(\zeta(s))$	Norm($\zeta(s)$)
1	0.67	24909.79	-0.0606782	0.0600306	0.0072855
2	0.67	31572.389	0.0316965	-0.0512341	0.0036296
3	0.67	31612.289	-0.032057	0.03950699	0.0025884
4	0.67	31635.689	-0.023354	-0.0502781	0.0030733
5	0.67	31698.989	-0.080725	0.01138963	0.0066462
6	0.67	31738.389	-0.031640	0.01820720	0.0013326
7	0.67	31834.689	0.0047814	0.00877299	0.0000998
8	0.67	31897.889	-0.025277	0.09119049	0.0089546
9	0.67	32099.889	0.0614658	-0.0536483	0.0066561
10	0.67	32115.789	-0.079928	-0.0224789	0.0068938

Table 4: First 10 zeros of $\hat{\zeta}(s)$ in the line $\Re(s) = 0.67$

No	$\Re(s)$	$\Im(s)$	$\Re(\zeta(s))$	$\Im(\zeta(s))$	Norm ($\zeta(s)$)
1	0.87	163.09	-0.0676011	0.0671348	0.0090769
2	0.87	163.19	-0.0954539	0.0124029	0.0092652
3	0.87	489.5	-0.09741049	0.0012685	0.0094904
4	0.87	580.1	-0.08082229	0.0441122	0.0084781
5	0.87	861.1	-0.05650421	0.0630962	0.0071738
6	0.87	861.2	-0.09047824	0.0168424	0.0084699
7	0.87	1586.29	-0.085475	-0.045788	0.0094025
8	0.87	2039.49	-0.055274	0.0791788	0.0093245

Table 5: First 10 zeros of $\hat{\zeta}(s)$ in the line $\Re(s) = 0.87$

No	$\Re(s)$	$\Im(s)$	$\Re(\zeta(s))$	$\Im(\zeta(s))$	Norm ($\zeta(s)$)
1	0.99	18.1	-0.0164294	0.0364413	0.0015978
2	0.99	18.2	0.00285045	-0.0909336	0.0082770
3	0.99	27.2	-0.0152707	-0.0044461	0.0002529
4	0.99	36.2	-0.0382743	0.06136275	0.00523031
5	0.99	36.3	0.01150310	-0.0476188	0.00239987
6	0.99	45.3	-0.0080385	0.04103830	0.00174876
7	0.99	54.4	-0.0151023	-0.0125667	0.00038600
8	0.99	63.4	-0.0358817	0.08598499	0.0086809
9	0.99	63.5	0.00911311	-0.0802903	0.00652958
10	0.99	72.4	0.03995582	0.07647096	0.00744427

Table 6: First 10 zeros of $\hat{\zeta}(s)$ in the line $\Re(s) = 0.99$

For further illustration of these tables, we can consult documents [3] and [4] of the *References* section. Likewise, the interested reader can download all the documents and numerical results associated with this research in the author's social networks. Consequently, having presented sufficient illustration, we can state that:

Theorem 3. *The Riemann hypothesis is false!*

Proof: To see tables 2, 3, 4, 5 and 6.

No	Line, $\Re(s)$	Density
1	0.43	scarce
2	0.44	scarce
3	0.45	scarce
4	0.47	abundant
5	0.49	abundant
6	0.5	abundant
7	0.51	abundant
8	0.55	scarce
9	0.67	abundant
10	0.79	scarce
11	0.84	scarce
12	0.90	abundant
13	0.91	abundant
14	0.92	abundant
15	0.93	abundant
16	0.94	abundant
17	0.95	abundant
18	0.96	abundant
19	0.97	abundant
20	0.98	abundant
21	0.99	abundant

Table 7: Lines in the complex plane whose values of $\Re(s)$ contain zeros of the $\hat{\zeta}$ function.

As it is known, **only a countersample is required to conclude that the conjecture or Riemann Hypothesis is false** and, in this document, abundant countersamples are illustrated in the complex plane ($0.43 < \Re(s) < 1$) which support the conclusive statement in relation to this conjecture.

Table 7 shows that, in addition to the known line $\Re(s) = 1/2$, there is a finite but abundant amount of lines in the complex plane containing zeros of the Riemann $\hat{\zeta}$ function.

5 Convergence according to number of terms

All the zeros reported in the tables 2, 3, 4, 5, and 6, by default, have been calculated using one million of terms in the summation involved, considering that with this number, a definitive and irreversible trend is achieved as can be seen in table 8, which shows the variations in the results when different values are used for the precision of the calculations, that is, an increasing number of terms in the sum of the series involved.

No Terms	$\Re(\hat{\zeta}(s))$	$\Im(\hat{\zeta}(s))$	$\text{Norm}(\hat{\zeta}(s))$
1000	-0.077687558742	0.0024792659894	0.0060415035432
10000	-0.076826587677	0.0020653409607	0.0059065902074
100000	-0.076827833536	0.0019445412501	0.0059062972465
1000000	-0.076842084438	0.0019391964805	0.0059084664238
10000000	-0.076843349624	0.0019406357055	0.0059086664484
100000000	-0.076843236509	0.0019408487881	0.0059086498912
1000000000	-0.076843206380	0.0019408449579	0.0059086452460
10000000000	-0.076843205489	0.0019408412396	0.0059086450946

Table 8: Convergence of the zero $\hat{\zeta}(0.9 + 163.2i)$ varying the number of terms to reach the practical limit of processing with a personal computer.

Table 8 shows that after ten billion summations⁴, the zero, $\hat{\zeta}(0.9 + 163.2i) = 0$, holds and the norm converges to 0.0059086 and since the contribution of higher terms is minimal and negligible, there is no reason to think that the trend can be reversed.

As noted above, the number of terms must be chosen carefully according to the hardware available, a low number (< 10000) is not enough to yields convergence and a very high number (> 1000000) can cause the calculations to be too long. In practice we must reconcile this aspect to obtain a cipher that is large enough to yields convergence and at the same time achieve that the calculations do not take long time.

⁴The hardware used in this research consists of a PC equipped with an AMD FX 8350 black edition processor at 4.0 GHz and 8 GB of RAM, running under Linux operating system.

6 Twin zeros

In a similar way as with the **prime numbers**, in the case of *Riemann's zeros*, there is also a phenomenon that we could call **twin zeros** as illustrated in table 9.

No	$\Re(s)$	$\Im(s)$
1	0.9	163.1
1	0.9	163.2
2	0.9	580.1
2	0.9	580.2
3	0.87	163.09
3	0.87	163.19
4	0.87	861.1
4	0.87	861.2
5	0.45	1329.09
5	0.45	1329.19

Table 9: Twin zeros in the convergence of the $\hat{\zeta}$ function.

Definition 1. We define the **twin zeros** as those zeros that being on the same line differ in its imaginary part by no more than one unit.

Another important aspect of these zeros is that they are **infinite**, as in the case of the **twin primes**, although for the moment there is no proof of this fact⁵.

7 Existence of infinite zeros

From the works of Godfrey Harold Hardy, it's well known that there are infinite *Riemann zeros*, at least on the line $\Re(s) = 1/2$ for which Hardy gave a demonstration. However, the information obtained about the density of the zeros in the lines that are show in the table 7, is very scarce and for the moment it can only be assumed that they are "infinite", although for now there is no proof of this fact, but judging by the advances obtained in this subject, we can believe that a demonstration in this sense will not take long to arrive.

⁵To see next section

Something similar can be said about the number of lines containing non-trivial zeros ($\Re(s) = 0.99, \Re(s) = 0.98, \Re(s) = 0.97, \Re(s) = 0.90, \Re(s) = 0.87, \Re(s) = 0.84, \Re(s) = 0.79, \Re(s) = 0.75$, etc).

For now, its number is moderate but abundant, the truth is that in the interval $0.42 < \Re(s) < 1$ it looks to be privileged regions that present these lines, but on the other hand, there are also regions in which no lines have been found showing functional zeros of $\hat{\zeta}$, **partly because of the difficulty of finding them and partly because, perhaps, they do not exist.**

8 The zeros by reflection

As illustrated by Riemann himself, $\zeta(s)$ extends in \mathbb{C} as a meromorphic function by analytic continuation, a through the expression known as the **functional equation** of the zeta function:

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-(1-s)/2}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s) \quad (13)$$

which has a simple pole in $s = 1$, with residue 1. The functional equation involves a gamma function, Γ , defined as:

$$\Gamma(s) = \int_0^\infty t^{s-1}e^{-t}dt, \quad \Re(s) \leq 0 \quad (14)$$

which has no zeros. Since $\pi^{-(1-s)/2}$ is a nonzero term, and the *gamma function* does not produces zeros, it's established that for values of $\Re(s) \leq \frac{1}{2}$, if there is a zero in $\zeta(s)$, it follows immediately from the **functional equation** that this zero must be reflected in $1 - s$, that is, $\zeta(1 - s) = 0$, whereby, for every zero in the interval $0 < \Re(s) \leq \frac{1}{2}$, there will be another zero in the interval $\frac{1}{2} \leq \Re(s) < 1$, in words, **the zeros to the left of the line $\Re(s) = \frac{1}{2}$ are reflected to the right of the same line.** The opposite, however, is not true, as will be seen below.

Definition 2. Let be given $z_{0-} = \sigma + it$ a zero of $\hat{\zeta}(s)$, with $0 < \sigma \leq 1/2$, we call **zeros by reflection** to the zeros obtained in $z_{0+} = 1 - \sigma + it$.

In our case, the first zero appears in the line $\Re(s) = 0.43$, where $\zeta(0.43 + 1977.19i) = 0$, and in agreement with the established must have a zero reflected in $\zeta(0.57 + 1977.19i) = 0$, to see it, let me show the output of the routine **SearchContraExample** (), in the vicinity from the referred zero:

Real: 0.57 Img: 1977.19 re: 0.0364 img: -0.0382 norm: 0.0027

where the output has been formatted in order to make it presentable.

The theory presented here fully complies with this prediction that emerges from the *functional equation*, and for the first time since 1859, we can verify the authenticity of this fact by detailing the table 10, which illustrates a summary of the situation, with some of the zeros involved

Line	$\zeta(\sigma) = 0$	$\zeta(1 - \sigma) = 0$
0.43	0.43+1977.19 i	0.57+1977.19 i
0.44	0.44+1977.19 i	0.56+1977.19 i
0.45	0.45+1329.09 i	0.55+1329.09 i
0.45	0.45+1415.59 i	0.55+1415.59 i
0.46	0.46+25.0 i	0.54+25.0 i
0.46	0.46+37.6 i	0.54+37.6 i
0.47	0.47+14.09 i	0.53+14.09 i
0.47	0.47+21.0 i	0.53+21.0 i
0.48	0.48+14.09 i	0.52+14.09 i
0.48	0.48+21.0 i	0.52+21.0 i
0.49	0.49+14.09 i	0.51+14.09 i
0.49	0.49+21.0 i	0.51+21.0 i

Table 10: Zeros by reflection in $\zeta(\sigma + it) = 0$

Of course, it's not possible to present all the cases, but for those who have a greater interest can consult the documents in the References section.

The zeros by reflection, are not the only zeros that exist in the strip $\frac{1}{2} \leq \Re(s) < 1$. In effect, there are other zeros in this strip that are exclusive, they are defined as:

Definition 3. *The zeros that do not have a reflection in the $0 < \Re(s) \leq \frac{1}{2}$ strip band are called **exclusive zeros**.*

Samples of these zeros are those found in the lines

$$\Re(s) = 0.90, 0.91, 0.92, \dots, 0.99$$

In general, these zeros are found in straight lines for which $\Re(s) > \frac{1}{2}$. An important detail is that some **exclusive zeros are found in straight lines where reflected zeros also converge!** Another mystery of the ζ function.

9 Convergence conditions

Let $z = a + bi$ be a complex number not null, that is, $z \neq 0$ and let $\|z\| = \sqrt{a^2 + b^2}$ its norm. Also, consider z_0 as a zero of $\hat{\zeta}(z)$, that is, $\hat{\zeta}(z_0) = 0$. The next criterion will allow us to explore the zeros of the $\hat{\zeta}$ function:

Theorem 4. *The norm $\|\hat{\zeta}(z)\| \rightarrow 0$ if and only if $z \rightarrow z_0$. The closer z gets to z_0 , the closer it will be $\|\hat{\zeta}(z)\|$ to zero.*

Proof: The proof of this fact is obvious from the expression for $\hat{\zeta}$, we can get:

$$\begin{aligned} \|\hat{\zeta}(z_0)\| &= \sqrt{\left(\sum_{n=1}^{\infty} \frac{(-1)^{n-1} \cos(b \ln n)}{n^a}\right)^2 + \left(\sum_{n=1}^{\infty} \frac{(-1)^{n-1} \sin(b \ln n)}{n^a}\right)^2} \\ \|\hat{\zeta}(z_0)\| &= \sqrt{0^2 + 0^2} \\ \|\hat{\zeta}(z_0)\| &= 0 \end{aligned}$$

where z_0 is a zero of $\hat{\zeta}(z)$ and therefore the terms in \sum yields null values or near zero.

This fact can be exploited when calculating in serie all the zeros of the $\hat{\zeta}$ function. Indeed, to understand this criterion more clearly, consider a numerical example. Take the first zero of the $\hat{\zeta}$ function, that is, $\hat{\zeta}(z_0) = 0$ for $z_0 = 0.5 + 14.135i$. First assume that, from this zero, only its entire part is known, that is, 14. It is required to converge the decimal part and for this simply the norm is calculated for the values 14.0, 14.1, 14.2, 14.3, ... 14.9 and **the value that comes closest to zero (the lowest value) is chosen as the representative value**, this value gives the first decimal of z_0 . Proceeding in this way we obtain:

```
i=0 re: 0.0123 im: 0.252 b= 14.0 norm: 0.06356051094769553
i=1 re: -0.00163 im: 0.0653 b= 14.1 norm: 0.004267063680288036
i=2 re: 0.0141 im: -0.122 b= 14.2 norm: 0.015010324186883877
i=3 re: 0.0601 im: -0.305 b= 14.3 norm: 0.09678257573736752
i=4 re: 0.137 im: -0.48 b= 14.4 norm: 0.24907275250510502
i=5 re: 0.242 im: -0.64 b= 14.5 norm: 0.46854448768625373
i=6 re: 0.373 im: -0.782 b= 14.6 norm: 0.7502633926958582
i=7 re: 0.526 im: -0.902 b= 14.7 norm: 1.089577972938231
```

i=8 re: 0.699 im: -0.996 b= 14.8 norm: 1.4810462022747748
i=9 re: 0.888 im: -1.06 b= 14.9 norm: 1.9157712005434449

analyzing the calculations we observed that for $i = 1$ a partial value is obtained for the norm $||\hat{\zeta}(0.5 + 14.1)|| = 0.0042670$, which indicates that the first decimal digit of the required value is 1. Now, to get the second decimal digit, this exercise is repeated but starting from the partial value 14.1, which we obtain:

i=0 re: -0.00163 im: 0.0653 b= 14.10 norm: 0.004267063680288036
i=1 re: -0.0014 im: 0.0466 b= 14.11 norm: 0.002170345257393568
i=2 re: -0.000874 im: 0.0278 b= 14.12 norm: 0.0007751457590326
i=3 re: -5.28e-05 im: 0.00909 b= 14.13 norm: 0.000082653025867
i=4 re: 0.00107 im: -0.00964 b= 14.14 norm: 0.0000940663056752
i=5 re: 0.00248 im: -0.0284 b= 14.15 norm: 0.00081058808603856
i=6 re: 0.0042 im: -0.0471 b= 14.16 norm: 0.002233412015026754
i=7 re: 0.00622 im: -0.0658 b= 14.17 norm: 0.00436370718280975
i=8 re: 0.00853 im: -0.0844 b= 14.18 norm: 0.00720259912087224
i=9 re: 0.0112 im: -0.103 b= 14.19 norm: 0.010751147951535013

and again, analyzing these values, we obtain $i = 3$ with the partial for the norm $||\hat{\zeta}(0.5 + 14.13)|| = 0.000082653$ as the lowest value of the series and with this 14.13 is obtained as the partial result. Now, again this procedure is repeated but, this time, taking 14.13 as a starting point, to get:

i=0 re:-5.28e-05 im:0.00909 b=14.130 norm: 8.265302586739142e-05
i=1 re:4.57e-05 im:0.00722 b=14.131 norm: 0.0000520987339106264
i=2 re:0.000147 im:0.00534 b=14.132 norm: 0.0000285847025611317
i=3 re:0.000252 im:0.00347 b=14.133 norm: 0.0000121121356834308
i=4 re:0.000359 im:0.0016 b=14.134 norm: 0.00000268223732143979
i=5 re:0.00047 im:-0.000275 b=14.135 norm: 0.000000296211581997
i=6 re:0.000583 im:-0.00215 b=14.136 norm: 0.000004955262522647
i=7 re:0.000699 im:-0.00402 b=14.137 norm: 0.000016660594031066
i=8 re:0.000819 im:-0.00589 b=14.138 norm: 0.000035413409701511
i=9 re:0.000941 im:-0.00777 b=14.139 norm: 0.000061214912708464

where for $i = 5$ a partial of the norm $||\hat{\zeta}(0.5 + 14.135)|| = 0.0000002962$ is obtained that yields the third decimal digit, where $z_0 = 0.5 + 14.135i$,

continuing the process pure zeros are obtained, therefore, the convergence concludes.

To achieve this result, we assume that, the integer part of z_0 was a known value. However, we required to know for what values of z there exists a *Riemann zero*, which is obtained by **applying the same principle**. For example, to know the zeros that may exist between the first 50 natural integers, the series is evaluated, starting at the value $\Im(z) = 10$ and proceeding successively until the interval is exhausted, obtaining the following result:

```
D:\devgo\projects\riemann>go run riem_cmp.go
=====Función TestNormRiemann =====
Realiza una prueba sobre la norma de la serie ...
Tiempo antes de la prueba: 2017-05-22 14:38:38.5070943 -0500
i= 0 im: 10 norm: 1.7889776690881514
i= 1 im: 11 norm: 5.3098650268751655
i= 2 im: 12 norm: 6.768324648445004
i= 3 im: 13 norm: 3.4967826646143063
i= 4 im: 14 norm: 0.06356051094769553
i= 5 im: 15 norm: 2.381955939157363
i= 6 im: 16 norm: 6.452188324133802
i= 7 im: 17 norm: 4.568704007474071
i= 8 im: 18 norm: 0.9997764574506408
i= 9 im: 19 norm: 2.6713988912976956
i= 10 im: 20 norm: 2.945157748816574
i= 11 im: 21 norm: 0.002662873270821427
i= 12 im: 22 norm: 5.357107076667078
i= 13 im: 23 norm: 12.165704691861526
i= 14 im: 24 norm: 5.837743666058762
i= 15 im: 25 norm: 0.0006139721348228435
i= 16 im: 26 norm: 2.2110427862684428
i= 17 im: 27 norm: 1.2916684547596544
i= 18 im: 28 norm: 4.7464114975910885
i= 19 im: 29 norm: 8.593858281013109
i= 20 im: 30 norm: 1.4332421065034437
i= 21 im: 31 norm: 2.143418386822452
i= 22 im: 32 norm: 4.340379601242355
i= 23 im: 33 norm: 0.04002367855676915
i= 24 im: 34 norm: 8.416930281999235
```

```
i= 25 im: 35 norm: 9.441743044621148
i= 26 im: 36 norm: 1.5338090982539594
i= 27 im: 37 norm: 0.714128946288046
i= 28 im: 38 norm: 1.1286250787015908
i= 29 im: 39 norm: 12.50188357251842
i= 30 im: 40 norm: 9.276328458775254
i= 31 im: 41 norm: 0.08318423474708224
i= 32 im: 42 norm: 5.945272907345323
i= 33 im: 43 norm: 0.8871081282085561
i= 34 im: 44 norm: 2.50511811977559
i= 35 im: 45 norm: 2.5696115395314534
i= 36 im: 46 norm: 5.933446030271415
i= 37 im: 47 norm: 7.460767064095884
i= 38 im: 48 norm: 0.0002458165045608516
i= 39 im: 49 norm: 2.5955064410270676
i= 40 im: 50 norm: 0.6752034911191576
i= 41 im: 51 norm: 16.039741414193923
i= 42 im: 52 norm: 10.617457663722135
i= 43 im: 53 norm: 0.007145333447324516
i= 44 im: 54 norm: 1.4552950248016576
i= 45 im: 55 norm: 3.3168423949796426
i= 46 im: 56 norm: 2.054393784260305
i= 47 im: 57 norm: 4.846507797496249
i= 48 im: 58 norm: 16.290909889314573
i= 49 im: 59 norm: 1.6175687339978813
Tiempo despues de la prueba: 2017-05-22 14:38:50.0765802 -0500
```

Analyzing these results, we observed that, the values for which a *Riemann zero* possibly exists are those for which $|\hat{\zeta}(z)| < 1$, namely:

```
i= 4 im: 14 norm: 0.06356051094769553
i= 8 im: 18 norm: 0.9997764574506408
i= 11 im: 21 norm: 0.002662873270821427
i= 15 im: 25 norm: 0.0006139721348228435
i= 23 im: 33 norm: 0.04002367855676915
i= 27 im: 37 norm: 0.714128946288046
i= 31 im: 41 norm: 0.08318423474708224
i= 33 im: 43 norm: 0.8871081282085561
```

```
i= 38 im: 48 norm: 0.0002458165045608516
i= 40 im: 50 norm: 0.6752034911191576
i= 43 im: 53 norm: 0.007145333447324516
```

And finally, to verify that they really correspond to a *Riemann zero*, the process previously performed to converge the first *Riemann zero* ($z_0 = 0.5 + 14.135i$) is executed on each of these values.

To achieve all these calculations, the following routines written in Go source code, are used.

9.1 Algorithm use_riem_03

Compute the series in $\hat{\zeta}$ taking as parameters:

- **precision**, the number of terms in the summation.
- **p_real**, the value for the real part, usually $\frac{1}{2}$, but it can be any other value in the interval $0 < \Re(s) < 1$.
- **p_img**, for the imaginary part of z .

the function returns two values of type **float64** with the real and imaginary parts of $\hat{\zeta}(z)$.

```
func RiemannSerie(precision int,p_real,
p_img float64)(float64,float64){
// calcula al serie de Riemann
var n,i int
var a,b,ang,sen,cos,pr float64
var fre,fcmp,cmp,re,sumac,sumar float64
var bas,fac float64
n=precision
sumar=0
sumac=0
bas=-1.0
a=p_real
b=p_img
for i=1;i<n;i++){
    ang=b*math.Log(float64(i))
```

```
sen=math.Sin(ang)
cos=math.Cos(ang)
pr=math.Pow(float64(i),a)
fac=math.Pow(bas,float64(i-1))
fre=fac*(cos/pr)
fcmp=fac*(sen/pr)
re=fre+sumar
cmp=fcmp+sumac
fmt.Printf("i= %d real: %g complex: %g\n",i,re,cmp)
sumar=re
sumac=cmp
}
return re,cmp
}
```

9.2 Algorithm use_riem_04

Perform the convergence on a single digit, as explained above. Take the following parameters:

- **precision**, the number of terms in the summation.
- **dec**, an integer of the form 10, 100, 1000, 10000, etc., depending on whether it is the first decimal, second, third, etc.
- **p_real**, the value for the real part, usually $\frac{1}{2}$, but it can be any other value in the interval $0 < \Re(s) < 1$.
- **p_img**, for the imaginary part of z .

The function returns the imaginary part of the summation in $\hat{\zeta}$.

```
func GetConvergeDig(precision,dec int,p_real,
p_img float64)float64{
// devuelve la parte imaginaria que converge
// con el criterio de la menor norma
var i,dig int
var re,im,norm,p_dec,b,minor float64
p_dec=1/float64(dec)
b=p_img
```

```
minor=10000000000000000000.0
for i=0;i<10;i++){
  re,im=RiemannSerie(precision,p_real,b)
  norm=re*re+im*im
  if norm<minor{
    minor=norm
    dig=i
  }
  b=b+p_dec
}
im=p_img+float64(dig)*p_dec
return im
}
```

9.3 Algorithm use_riem_05

Perform convergence on all digits, as explained above. Take the following parameters:

- **precision**, the number of terms in the summation.
- **p_real**, the value for the real part, usually $\frac{1}{2}$, but it can be any other value in the interval $0 < \Re(s) < 1$.
- **p_img**, for the imaginary part of z .

The function returns the imaginary part of the summation in $\hat{\zeta}$, with all the digits converging.

```
func ConvergeImgZero(precision int,p_real,
p_img float64)float64{
// devuelve la parte imaginaria del cero que
// converge en la serie
var i,va_dec,Digs int
var b_im,test float64
va_dec=10
Digs=5
for i=0;i<Digs;i++){
  b_im=GetConvergeDig(precision,va_dec,p_real,p_img)
```

```
fmt.Printf("i= %d b_im: %2.6g\n",i,b_im)
va_dec*=10
p_img=b_im
}
test=b_im-math.Floor(b_im)
if test>=0.999{b_im=0}
return b_im
}
```

9.4 Algorithm use_riem_06

Perform a test on consecutive integers to determine possible *Riemann zeros*. Take the following parameters:

- **precision**, the number of terms in the summation.
- **iters**, the number of integers to test.
- **p_real**, the value for the real part, usually $\frac{1}{2}$, again it can be any other value in the interval $0 < \Re(s) < 1$.
- **p_img**, for the imaginary part of z .

The function does not return any value, but prints on the screen the result of the test, which is actually done on the square of the norm and not on the norm itself.

```
func TestNormRiemann(precision,iters int,p_real,p_img float64){
// realiza una prueba de la norma
// sobre la parte entera de p_img
var i int
var re,im,norm,b float64
fmt.Println("====Funcion TestNormRiemann =====")
fmt.Println("Realiza una prueba sobre la norma de la serie")
fmt.Printf("Tiempo antes de la prueba: %v \n",time.Now())
b=p_img
for i=0;i<iters;i++){
re,im=RiemannSerie(precision,p_real,b)
norm=re*re+im*im
```

```
fmt.Printf("i= %d im: %4.6g norm: %g\n",i,b,norm)
norm=re*re+im*im
if norm<0.01{
    fmt.Printf("Cero de la funcion zeta hallado!\n")
}
b++
}
fmt.Printf("Tiempo despues de la prueba: %v \n",time.Now())
}
```

With these tools we proceed to build the code that automates the whole process, that is, a routine that investigates if a certain value can be a *Riemann zero* and if it finds that the response is positive proceeds with the convergence of the sought value. The code for this routine is as follows:

9.5 Algorithm use_riem_07

This algorithm searches over a given interval the existence of *Riemann's zeros*. Take the following parameters:

- **precision**, the number of terms in the summation.
- **NumCeros**, the number of Riemann's zeros to calculate.
- **p_real**, the value for the real part, it can be any value in the interval $0 < \Re(s) < 1$.
- **p_img**, for the imaginary part of z .

The function does not return any value, but prints the result of the calculation on the screen.

```
func SearchRiemannZeros(precision,NumCeros int,p_real,
p_img float64){
// busca los ceros de Riemann en un determinado intervalo
// que empieza por from y continua hasta agotar iters
var i,ceros int
var re,im,b_im,norm float64
fmt.Println("====Función SearchRiemannZeros =====")
```

```
fmt.Println("Busca los ceros de Riemann en un intervalo")
fmt.Printf("Tiempo antes de inciar: %v \n",time.Now())
ceros=0
for i=0; ;i++){
    re,im=RiemannSerie(precision,p_real,p_img)
    norm=re*re+im*im
    if(norm>1){
        p_img++
        continue
    }else{
        b_im=ConvergeImgZero(precision,p_real,p_img)
        if (norm<0.9){
            // es un cero de riemann
            if b_im==math.Floor(b_im){
                b_im--
                b_im=ConvergeImgZero(precision,p_real,b_im)
            }
        }
        ceros++
        fmt.Printf("Iter= %d Cero No %d de %d real: %g img: %g
norma: %g suma real: %g suma img %g\n",i,ceros,
NumCeros,p_real,b_im,norm,re,im)
        if(ceros==NumCeros){break}
    }
}
p_img++
}
fmt.Printf("Tiempo despues de calcular: %v \n",time.Now())
}
```

Eureka! We now have an algorithm to converge the zeros of the Riemann Z function and we use it to calculate the first 50 zeros of $\hat{\zeta}(s)$.

For this last exercise the function **main()** used is as follows:

```
func main() {
    var precision int
    var p_real,p_img float64
    precision=1000000
    p_real=0.5
```

```
p_img=10.0
SearchRiemannZeros(precision,50,p_real,p_img)
}
```

The result of running this algorithm is shown below:

```
D:\devgo\projects\riemann>go run riem_cmp.go
=====Función SearchRiemannZeros =====
Busca los ceros de Riemann en un intervalo dado ...
Tiempo antes de inciar: 2017-05-22 17:51:49.763298 -0500
1 re: 0.5 im: 14.135 norm: 0.06356 sr: 0.01232 si 0.2518
2 re: 0.5 im: 21.02225 norm: 0.002663 sr: 0.02381 si 0.04578
3 re: 0.5 im: 25.011 norm: 0.000614 sr: -0.01467 si 0.01997
4 re: 0.5 im: 32.94 norm: 0.04002 sr: -6.528e-05 si -0.2001
5 re: 0.5 im: 37.6 norm: 0.7141 sr: 0.7206 si -0.4414
6 re: 0.5 im: 40.92 norm: 0.08318 sr: 0.1035 si -0.2692
7 re: 0.5 im: 43.33 norm: 0.8871 sr: 0.02987 si 0.9414
8 re: 0.5 im: 48.01 norm: 0.0002458 sr: -0.001655 si 0.01559
9 re: 0.5 im: 49.8 norm: 0.6752 sr: -0.1496 si -0.808
10 re: 0.5 im: 52.9705 norm: 0.007145 sr: 0.078 si -0.03257
11 re: 0.5 im: 60.832 norm: 0.3139 sr: 0.01959 si -0.5599
12 re: 0.5 im: 65.113 norm: 0.1183 sr: 0.2175 si 0.2665
13 re: 0.5 im: 67.1 norm: 0.1034 sr: 0.1703 si 0.2727
14 re: 0.5 im: 72.1 norm: 0.01362 sr: -0.1155 si -0.01648
15 re: 0.5 im: 77.145 norm: 0.2335 sr: 0.2539 si 0.4111
16 re: 0.5 im: 82.9105 norm: 0.07386 sr: -0.1481 si -0.2278
17 re: 0.5 im: 88.81 norm: 0.391 sr: 0.2994 si -0.549
18 re: 0.5 im: 95.9 norm: 0.3038 sr: -0.2343 si -0.4989
19 re: 0.5 im: 98.83123 norm: 0.1752 sr: 0.3573 si 0.218
20 re: 0.5 im: 107.2 norm: 0.4294 sr: -0.006762 si 0.6552
21 re: 0.5 im: 111.9 norm: 0.006905 sr: -0.03246 si 0.07649
22 re: 0.5 im: 118.8 norm: 0.6578 sr: -0.7262 si -0.3611
23 re: 0.5 im: 122.95 norm: 0.03759 sr: 0.06261 si -0.1835
24 re: 0.5 im: 127.52 norm: 0.8405 sr: -0.005735 si -0.9168
25 re: 0.5 im: 131.1 norm: 0.2224 sr: 0.2834 si 0.377
26 re: 0.5 im: 134.8 norm: 0.4937 sr: 0.6081 si -0.3521
27 re: 0.5 im: 138.12 norm: 0.3847 sr: 0.1441 si 0.6033
28 re: 0.5 im: 141.124 norm: 0.3138 sr: 0.3161 si 0.4625
```

```
29 re: 0.5 im: 143.112 norm: 0.4136 sr: -0.153 si 0.6247
30 re: 0.5 im: 146.0011 norm: 0.0000107 sr: 0.002802 si 0.0017
31 re: 0.5 im: 150.93 norm: 0.04704 sr: -0.1763 si 0.1263
32 re: 0.5 im: 153.025 norm: 0.01039 sr: -0.07712 si 0.06668
33 re: 0.5 im: 156.113009 norm: 0.3202 sr: 0.116 si 0.5538
34 re: 0.5 im: 158.85010 norm: 0.6862 sr: -0.3049 si -0.7702
35 re: 0.5 im: 163.031 norm: 0.003003 sr: -0.03573 si -0.04155
36 re: 0.5 im: 169.1 norm: 0.1119 sr: -0.192 si 0.274
37 re: 0.5 im: 179.920 norm: 0.1124 sr: 0.2767 si -0.1894
38 re: 0.5 im: 182.209 norm: 0.5234 sr: 0.5841 si -0.4268
39 re: 0.5 im: 185.6 norm: 0.127 sr: 0.3441 si -0.09269
40 re: 0.5 im: 192.03 norm: 0.0104 sr: 0.02836 si 0.09797
41 re: 0.5 im: 193.1 norm: 0.1232 sr: 0.3311 si 0.1166
42 re: 0.5 im: 198.02 norm: 0.003324 sr: -0.006538 si 0.05728
43 re: 0.5 im: 207.91 norm: 0.05683 sr: 0.1267 si 0.202
44 re: 0.5 im: 216.2 norm: 0.7524 sr: -0.0599 si 0.8653
45 re: 0.5 im: 219.1 norm: 0.1241 sr: 0.1498 si 0.3188
46 re: 0.5 im: 221.431 norm: 0.381 sr: 0.4845 si -0.3824
47 re: 0.5 im: 224.01 norm: 0.001307 sr: -0.03393 si 0.01249
48 re: 0.5 im: 231.3 norm: 0.001885 sr: -0.02378 si -0.03633
49 re: 0.5 im: 241.05 norm: 0.1095 sr: -0.002696 si 0.3309
50 re: 0.5 im: 244.1 norm: 0.04349 sr: -0.1458 si 0.1491
Tiempo despues de calcular: 2017-05-22 18:08:13.9945211 -0500
```

where,

- **re** refers to the real part of z_0 ,
- **im** does the same with the imaginary part
- **norm** is $|\hat{\zeta}(z_0)|$
- **sr** is the summation over the real part of $\hat{\zeta}(z_0)$, that is, $\Re(\hat{\zeta}(z_0))$ and
- **si** is the summation over the imaginary part of $\hat{\zeta}(z_0)$, that is, $\Im(\hat{\zeta}(z_0))$ and z_0 is a zero of $\hat{\zeta}(s)$, as defined above.

Some values have been cut a bit for convenience, but in acceptable terms, this is the output of the routine that calculates and converges the values of z_0 for the first 50 zeros of the *Riemann Z function*. Observing the output,

it can be seen that the convergence time for these first 50 zeros is only 16 minutes!⁶

Finally, the zeros known as *Riemann zeros*, properly speaking, are the zeros that are on the straight line $\Re(s) = 1/2$, the zeros that are in other lines (to see table 7), should not be referenced as “Riemann’s zeros” obviously because, among other things, **Riemann never met them**.

References

- [1] BAUTISTA, A. VÍCTOR M. & LARA RODRÍGUEZ, JOSÉ A. *Sobre las Funciones Z y Multizeta (I)*. Facultad de Matemáticas, Universidad Autónoma de Yucatán. Publicado en *Abstraction & Application* (9), 2013. pp 74-81.
- [2] BOMBIERI, E. *Problems of Millennium: The Riemann Hypothesis*. PDF. **11 p.**
- [3] USECHE, H. *Convergencia de la función Z de Riemann*. PDF publicado en facebook. Neiva (Huila), 2017. 40 p.
- [4] USECHE, H. *Aclaraciones a la Convergencia de la función Z de Riemann*. PDF publicado en facebook. Neiva (Huila), 2017. 20 p.
- [5] https://es.wikipedia.org/wiki/Hipotesis_de_Riemann

⁶Using PC hardware equipped with FX 8350 AMD propcessor and 8 GBytes of RAM, as stated above.