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Abstract

This paper demonstrates that many results about the Riemann Zeta Function in the literature are wrong, claiming an asymptotic estimate without checking the limit, or interchanging integration and summation when the infinite series diverges absolutely. The new Lemma is proved, proving new estimates about Zeta. These contradict known results, conditional on the truth of the Riemann Hypothesis. A stronger result holds too: any open vertical strip that holds the boundary of the Critical Strip also holds a zero of Zeta. The new Lemma can be applied to L-Functions too.

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1 Introduction

Let $s = \sigma + it$ be a complex variable, and let ζ be the Riemann Zeta Function.

The Functional Equation for ζ [1, p.95] is

$$\zeta(s) = \chi(s)\zeta(1-s) \quad (1)$$

$$\chi(s) \sim \left\{ \frac{t}{2\pi} \right\}^{\frac{1}{2}-\sigma} \quad (2)$$

In [1, thm.14.2], for all $\sigma > \frac{1}{2}$ and for all $\epsilon > 0$, if the Riemann Hypothesis is true, the estimate $\log|\zeta(s)| < A \log t$ alone proves

$$\frac{1}{\zeta(s)} = O(t^\epsilon) \quad (3)$$

2 Errors in literature

2.1 The first example

[1, eq.8.4.4], valid for any fixed natural integers N and n , and with $\sigma > 1$, reads

$$\begin{aligned} |\zeta(s)| &\geq \left\{ \cos\left(\frac{2\pi}{n}\right) - 2N^{1-\sigma} \right\} \zeta(\sigma) \\ &= \{1 - \epsilon\} \zeta(\sigma) \end{aligned} \quad (4)$$

The argument is put forward that $\zeta(1+it)$ is unbounded as t increases, because $\zeta(\sigma)$ is unbounded, and so is $\zeta(s)$ in the limit $\sigma \rightarrow 1$.

This argument is false. When $\sigma \rightarrow 1+$, then $N^{1-\sigma}$ tends to 1, and the inequality (4) merely becomes $|\zeta(s)| \geq 0$.

In the limit $\sigma \rightarrow 1+$, the series $\sum_n n^{-s}$ no longer converges absolutely, and the tail $\sum_N^\infty n^{-s}$ is not necessarily arbitrarily small for arbitrarily large N .

2.2 The second example

In [1, p.142], an asymptotic result is obtained by the interchange of the integral and the sum, as follows.

$$\int_0^T \sum_{n < x(t)} n^{-s} dt = \sum_{n < X(T)} \int_0^T n^{-s} dt \quad (5)$$

As in the previous example, the asymptotic result requires the result to stand true in the limit $T \rightarrow \infty$. For instance, $f(x) \sim g(x)$ means $\lim_{x \rightarrow \infty} \left| \frac{f(x)}{g(x)} \right| = 1$. It requires a limit.

In the limit $T \rightarrow \infty$, the sum in Eq. (5) becomes an infinite series. The interchange of integration and summation is no longer allowed, since the infinite series diverges absolutely for $\sigma \leq 1$.

Hence, the final asymptotic result is not necessarily true.

3 Lemma

Let $\text{trig}(x)$ stand for $\cos(x)$ or $\sin(x)$.

Lemma 1. *Let $f(y)$ belong to $L(a, \infty)$. Then, as $|t| \rightarrow \infty$,*

$$\int_a^\infty f(y) \text{trig}(ty) dy = O\left(\frac{1}{t}\right) \quad (6)$$

Proof. One can always define a function $g(y)$, strictly positive, strictly decreasing, so that $g'(y) < 0$ in (a, ∞) , such that for some arbitrary fixed

real $A > 0$,

$$\int_a^\infty |f(y) - g(y)| dy < \frac{A}{|t|} \quad (7)$$

Hence,

$$\left| t \int_a^\infty \{f(y) - g(y)\} \text{trig}(ty) dy \right| < A \quad (8)$$

By partial integration,

$$\begin{aligned} \int_a^\infty g(y) \text{trig}(ty) dy &= \pm \frac{g(y) \text{trig}(ty)}{t} \Big|_{y=a}^{y=\infty} \mp \frac{1}{t} \int_a^\infty g'(y) \text{trig}(ty) dy \\ &= O\left\{\frac{g(a)}{t}\right\} + O\left(\frac{1}{t} \int_a^\infty g'(y) dy\right) \\ &= O\left\{\frac{g(a)}{t}\right\} + O\left\{\frac{g(a)}{t}\right\} = O\left(\frac{1}{t}\right) \end{aligned} \quad (9)$$

Thus, by Eqs. (8, 9),

$$\left| O(1) + t \int_a^\infty f(y) \text{trig}(ty) dy \right| < A \quad (10)$$

Hence the result. \square

4 Propositions

Proposition 1. $\zeta(s) = O(1)$ as $t \rightarrow \infty$, for all fixed $\sigma > 0$.

Proof. By substituting $x = e^y$ into [1, eq.2.1.4], valid for $\sigma > 0$,

$$\zeta(s) = s \int_0^{\infty} \frac{\frac{1}{2} + [e^y] - e^y}{e^{ys}} dy + \frac{1}{s-1} + \frac{1}{2} \quad (11)$$

The result follows by Lemma 1. □

Proposition 2. $\zeta(s) = O\left(t^{\frac{1}{2}-\sigma}\right)$ as $t \rightarrow \infty$, for all fixed $\frac{1}{2} < \sigma < 1$, and for all $t > 0$.

Proof. $\zeta(s)$ being $O(1)$ for $0 < \sigma \leq \frac{1}{2}$, the result follows by the Functional Equation (1) and by the asymptotic equation (2). □

Proposition 3. There is a strictly positive, real A , such that $\frac{1}{|\zeta(s)|} \geq At^{\sigma-\frac{1}{2}}$, for all fixed $\frac{1}{2} < \sigma < 1$, and for all $t > 0$.

Proof. This is a simple consequence of Proposition 2. □

5 Disproof of the Riemann Hypothesis

The estimate in Proposition 1 is in the range of the assumption $\log|\zeta(s)| < A \log t$ of Eq. (3). Proposition 3, a consequence of Proposition 1, contradicts Eq. (3). Since Eq. (3) depends on the Riemann Hypothesis being true, this disproves the Riemann Hypothesis.

6 Zeros close to the boundary

The result (3) remains true for $\sigma > B$, when the Riemann Hypothesis in [1, thm.14.2] is replaced by a weaker hypothesis: that there exists a fixed real $\frac{1}{2} < B < 1$, such that the strip $\sigma > B$ is zero-free. Again, Proposition 3 contradicts Eq. (3). Hence, there is no such B . All halfplanes $1 - \epsilon < \sigma$, with an arbitrary but fixed real $\epsilon > 0$, contain a zero of ζ .

7 L-Functions

This method is suitable for L -Functions too.

References

- [1] E.C. Titchmarsh: *The Theory of the Riemann Zeta-Function*, Oxford University Press (1986), Second Edition