Sprugnoli's formula

Edgar Valdebenito
August 7 , 2025

Abstract

In this note, we study an identity obtained by R. Sprugnoli in 2006

1 Introduction

In Reference [1], the following identity is presented

$$\sum_{n=0}^{\infty}(-1)^n\binom{4n}{2n}^{-1}=\frac{16}{17}+\frac{4\sqrt{34}\left(\sqrt{17}-2\right)}{289\sqrt{\sqrt{17}-1}}\arctan\left(\frac{\sqrt{2}}{\sqrt{\sqrt{17}-1}}\right)+\frac{2\sqrt{34}\left(\sqrt{17}+2\right)}{289\sqrt{\sqrt{17}+1}}\ln\left(\frac{\sqrt{\sqrt{17}+1}-\sqrt{2}}{\sqrt{\sqrt{17}+1}+\sqrt{2}}\right) \tag{1}$$

Sprugnoli used a bisection formula and a computer algebra system to derive the formula (1).

An even more general identity due to Adegoke et al. is given by

$$\sum_{n=1}^{\infty}(-1)^{n-1}\binom{4n}{2n}^{-1}x^{2n}=\frac{4y^2}{(1+y^2)^2}+\left(\frac{(y-1)^3}{(y^2+1)^2}-\frac{8(y-1)y^2}{(y^2+1)^3}\right)\sqrt{y}\arctan(\sqrt{y})+\left(\frac{(y+1)^3}{(y^2+1)^2}-\frac{8(y+1)y^2}{(y^2+1)^3}\right)\sqrt{y}\operatorname{arctanh}(\sqrt{y}) \tag{2}$$

where

$$y=\frac{\sqrt{16+x^2}-4}{x}\quad,\ |x|\le 4 \tag{3}$$

Sprugnoli's result can be obtained by setting $x=1$ in (2),for details see [2] .

In this note, we give alternative expressions for (1).

Recall that:

$$\pi=4\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1} \tag{4}$$

2 Main formulas

Entry 1. for $u=\frac{\sqrt{\sqrt{17}-1}-\sqrt{2}}{\sqrt{\sqrt{17}-1}+\sqrt{2}}$ we have

$$\sum_{n=0}^{\infty}(-1)^n\binom{4n}{2n}^{-1}=$$
$$\frac{16}{17}+\frac{\pi}{289}\left(4\left(\frac{1+u}{1-u}\right)^3-\left(\frac{1-u}{1+u}\right)\right)-\frac{4}{289}\left(4\left(\frac{1+u}{1-u}\right)^3-\left(\frac{1-u}{1+u}\right)\right)\arctan(u)+\frac{1}{289}\left(8\left(\frac{1-u}{1+u}\right)+17\left(\frac{1+u}{1-u}\right)+2\left(\frac{1+u}{1-u}\right)^3\right)\ln\left(\frac{1-3u}{3-u}\right) \tag{5}$$

Entry 2. for $v=\frac{2\sqrt{2}-\sqrt{\sqrt{17}-1}}{2\sqrt{2}+\sqrt{\sqrt{17}-1}}$ we have

$$\sum_{n=0}^{\infty}(-1)^n\binom{4n}{2n}^{-1}=\frac{16}{17}+\frac{\pi}{289}\left(32\left(\frac{1-v}{1+v}\right)^3-\frac{1}{2}\left(\frac{1+v}{1-v}\right)\right)-$$
$$\frac{4}{289}\left(32\left(\frac{1-v}{1+v}\right)^3-\frac{1}{2}\left(\frac{1+v}{1-v}\right)\right)\arctan\left(\frac{1-3v}{3-v}\right)+\frac{1}{289}\left(4\left(\frac{1+v}{1-v}\right)+34\left(\frac{1-v}{1+v}\right)+16\left(\frac{1-v}{1+v}\right)^3\right)\ln(v) \tag{6}$$

Entry 3.

$$\sum_{n=0}^{\infty}(-1)^n \binom{4n}{2n}^{-1} = \frac{16}{17} - \frac{2}{289}\left(98\left(\sqrt{17}-4\right) + \sum_{n=1}^{\infty}\left(\sqrt{17}-4\right)^{n+1}\left(\frac{7(-1)^n+23}{2n-1} - \frac{79(-1)^n-177}{2n+1}\right)\right) \tag{7}$$

$$\sum_{n=0}^{\infty}(-1)^n \binom{4n}{2n}^{-1} = \frac{495}{578} - \frac{5567}{30\cdot 289}\left(\sqrt{17}-4\right)^2 + \frac{1}{2\cdot 289}\sum_{n=2}^{\infty}\left(\sqrt{17}-4\right)^{2n}\left(\frac{15}{4n-5} - \frac{15}{4n-3} - \frac{1039}{4n-1} - \frac{49}{4n+1}\right) \tag{8}$$

Entry 4.

$$\sum_{n=0}^{\infty}(-1)^n \binom{4n}{2n}^{-1} = \frac{16}{17} - \frac{\sqrt{\sqrt{17}-4}}{\sqrt{2}\ 289}\sum_{n=0}^{\infty}\frac{2^{-4n}}{2n+1}\binom{2n}{n}\left(\left(85+23\sqrt{17}\right)(-1)^{\lfloor n/2\rfloor} - \left(51+7\sqrt{17}\right)(-1)^{\lceil n/2\rceil}\right) \tag{9}$$

where $\lfloor\cdot\rfloor$ is the floor function and $\lceil\cdot\rceil$ is the ceiling function.

Entry 5.

$$\sum_{n=0}^{\infty}(-1)^n \binom{4n}{2n}^{-1} = \frac{4}{3} + \frac{2\pi\sqrt{3}}{27} - \sum_{n=0}^{\infty}\left(\binom{8n+2}{4n+1}^{-1} + 2\binom{8n+4}{4n+2}^{-1} + \binom{8n+6}{4n+3}^{-1}\right) \tag{10}$$

Entry 6.

$$\sum_{n=0}^{\infty}(-1)^n \binom{4n}{2n}^{-1} = 2\sum_{n=0}^{\infty}\binom{8n}{4n}^{-1} - \frac{16}{15} - \frac{\pi\sqrt{3}}{27} + \frac{2\sqrt{5}}{25}\ln\left(\frac{1+\sqrt{5}}{2}\right) \tag{11}$$

$$\sum_{n=0}^{\infty}(-1)^n \binom{4n}{2n}^{-1} = \frac{16}{15} + \frac{\pi\sqrt{3}}{27} - \frac{2\sqrt{5}}{25}\ln\left(\frac{1+\sqrt{5}}{2}\right) - 2\sum_{n=0}^{\infty}\binom{8n+4}{4n+2}^{-1} \tag{12}$$

Entry 7.

$$\sum_{n=0}^{\infty}(-1)^n \binom{4n}{2n}^{-1} = \frac{16}{17} - \frac{4}{289}\left(17+8\sqrt{17}\right)\sum_{n=0}^{\infty}\frac{\left(\sqrt{17}-4\right)^{2n+1}}{4n+1} - \frac{4}{289}\left(68+15\sqrt{17}\right)\sum_{n=0}^{\infty}\frac{\left(\sqrt{17}-4\right)^{2n+2}}{4n+3} \tag{13}$$

Entry 8.

$$\sum_{n=0}^{\infty}(-1)^n \binom{4n}{2n}^{-1} = \frac{16}{17} - \frac{16}{289}\cdot\sum_{n=0}^{\infty}17^{-n}\left(2\sum_{k=0}^{2n}\frac{(-2)^k}{2k+1}\binom{2n}{2n-k} - \sum_{k=0}^{2n+1}\frac{(-2)^k}{2k+1}\binom{2n+1}{2n+1-k}\right) \tag{14}$$

Remark:

$$2\sum_{k=0}^{2n}\frac{(-2)^k}{2k+1}\binom{2n}{2n-k} - \sum_{k=0}^{2n+1}\frac{(-2)^k}{2k+1}\binom{2n+1}{2n+1-k} = 2\,{}_2F_1\left(\frac{1}{2}, -2n, \frac{3}{2}, 2\right) - {}_2F_1\left(\frac{1}{2}, -1-2n, \frac{3}{2}, 2\right) \tag{15}$$

where ${}_2F_1$ is the Gauss hypergeometric function.

Entry 9.

$$\sum_{n=0}^{\infty}(-1)^n \binom{4n}{2n}^{-1} = \frac{16}{17} - \frac{4}{289}\left(\left(68-15\sqrt{17}\right){}_2F_1\left(1, 1/4, 5/4, \left(\sqrt{17}-4\right)^2\right) + \frac{\left(204-49\sqrt{17}\right)}{3}\,{}_2F_1\left(1, 3/4, 7/4, \left(\sqrt{17}-4\right)^2\right)\right) \tag{16}$$

where ${}_2F_1$ is the Gauss hypergeometric function.

Entry 10.

$$\sum_{n=0}^{\infty}(-1)^n \binom{4n}{2n}^{-1} = \frac{16}{17} + \frac{1}{289}\sqrt{\frac{289\sqrt{17}-799}{2}}\left(\frac{\pi}{2} - 2\arctan\left(\frac{1}{9}\right) - 2\arctan\left(\frac{4\sqrt{\sqrt{17}-1}-5\sqrt{2}}{5\sqrt{\sqrt{17}-1}+4\sqrt{2}}\right)\right) - \frac{1}{289}\sqrt{\frac{289\sqrt{17}+799}{2}}\left(2\,\mathrm{arctanh}\left(\frac{5}{8}\right) - 2\,\mathrm{arctanh}\left(\frac{5\sqrt{\sqrt{17}+1}-8\sqrt{2}}{8\sqrt{\sqrt{17}+1}-5\sqrt{2}}\right)\right) \tag{17}$$

$$\sum_{n=0}^{\infty}(-1)^n\binom{4n}{2n}^{-1}=\frac{16}{17}+\pi\sqrt{\frac{1}{34\sqrt{17}}-\frac{47}{9826}}-$$

$$\frac{2}{17}\sqrt{\frac{94}{17}+2\sqrt{17}}\operatorname{arctanh}\left(\sqrt{\sqrt{17}-4}\right)-\frac{2}{289}(51+7\sqrt{17})\left(\frac{1-\sqrt{\sqrt{17}-4}}{2}+\sum_{n=1}^{\infty}\left(\frac{1-\sqrt{\sqrt{17}-4}}{2}\right)^{n+1}c(n)\right)$$

where

$$c(n)=\sum_{k=0}^{\lfloor n/2\rfloor}\frac{(-1)^k}{2k+1}\binom{n}{n-2k}-2\sum_{k=0}^{\lfloor (n-1)/2\rfloor}\frac{(-1)^k}{2k+1}\binom{n-1}{n-1-2k} \tag{19}$$

$$c(n)=\left(\frac{1+i}{2}\right)\left(\frac{(1-i)^n-i(1+i)^n}{1+n}\right)-\frac{i((1-i)^n-(1+i)^n)}{n}\ ,\ i=\sqrt{-1} \tag{20}$$

Entry 11.

$$\sum_{n=0}^{\infty}(-1)^n\binom{4n}{2n}^{-1}=1-\frac{\pi}{4}+\sum_{n=0}^{\infty}\arctan\left(\frac{\binom{8n+4}{4n+2}-\binom{8n}{4n}}{1+\binom{8n+4}{4n+2}\binom{8n}{4n}}\right)+\sum_{n=1}^{\infty}(-1)^n\sum_{k=1}^{n}\frac{1}{2k+1}\binom{4n-4k+4}{2n-2k+2}^{-2k-1} \tag{21}$$

Entry12.

$$\sum_{n=0}^{\infty}(-1)^n\binom{4n}{2n}^{-1}=\sum_{n=0}^{\infty}2^{-n-1}\,{}_3F_2\left(\left\{\frac{1}{2},1,-n\right\},\left\{\frac{1}{4},\frac{3}{4}\right\},\frac{1}{16}\right) \tag{22}$$

where ${}_3F_2$ is the generalized hypergeometric function.

3 Endnote

Entry 13.

$$\pi=\frac{9\sqrt{3}}{2}\sum_{n=0}^{\infty}\left(\binom{8n+2}{4n+1}^{-1}+2\binom{8n+4}{4n+2}^{-1}+\binom{8n+6}{4n+3}^{-1}\right)-\frac{2481\sqrt{3}}{1156}-$$
$$\frac{16\,701\sqrt{3}}{5780}(\sqrt{17}-4)^2+\frac{9\sqrt{3}}{1156}\sum_{n=2}^{\infty}(\sqrt{17}-4)^{2n}\left(\frac{15}{4n-5}-\frac{15}{4n-3}-\frac{1039}{4n-1}-\frac{49}{4n+1}\right) \tag{23}$$

Entry 14.

$$\sum_{n=0}^{\infty}(-1)^n\binom{4n}{2n}^{-1}=\frac{\pi}{4}+\arctan\left(\frac{1}{16+}\ \frac{1}{3+}\ \frac{1}{6+}\ \frac{1}{4+}\ \frac{1}{54+}\ \frac{1}{2+}\ \frac{1}{1+}\ \frac{1}{9+\ldots}\right) \tag{24}$$

$$\sum_{n=0}^{\infty}(-1)^n\binom{4n}{2n}^{-1}=\frac{7\pi}{26}+\arctan\left(\frac{1}{1256+}\ \frac{1}{4+}\ \frac{1}{9+}\ \frac{1}{1+}\ \frac{1}{1+}\ \frac{1}{3+}\ \frac{1}{1+}\ \frac{1}{1+\ldots}\right) \tag{25}$$

4 References

1. R. Sprugnoli: Sums of reciprocals of the central binomial coefficient, Integers 6, 2006, #A27. http://eudml.org/doc/126928

2. K. Adegoke, R. Frontczak and T. Goy: Evaluation of some alternating series involving the binomial coefficients $C(4n, 2n)$, arXiv:2404.05770v1 [math NT] 7 Apr 2024. https://arxiv.org/abs/2404.05770