ON THE DERIVATION STUDY OF EXPLICIT FORMULAS FOR TWIN PRIMES

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Abstract

In this paper, I have repeatedly gone through various trial and error attempts, mainly in pursuit of the infinity of twin primes. Below, I have marked the failed attempts in the first half, or clearly stated like that "unfortunately, this is incorrect," "unfortunately, this is weakness," and have left the arguments that are clearly theoretically consistent as they are. I by no means want to write a fraudulent paper. I strongly hope that this paper will remain a source of research for future generations.

Before reading the paper:

This is both a paper and a research record.

At the very end, I've classified the theoretical consistency of each section's argument into red (weak), blue (fairly good), and green (strong). If you want to review, verify, and read the paper efficiently, please focus on the green sections. Also, to ensure that the green section alone is not a leap, it contains a total of about five independent approaches (including the red category), so please feel free to read it.Sec7 and Sec10 alone should have been enough to maintain theoretical validity.

Sec0. Introduction

In prime number distribution theory, following the zeta function, the Chebyshev functions are among the most studied objects. The second Chebyshev function is a sum involving the non-trivial zeros of the Riemann zeta function, known as the von Mangoldt function, which has achieved the remarkable feat of providing a rigorous explicit formula for $\psi(x)$. This can be expressed as follows:

$$\psi(x) = \sum_{n \le x} \Lambda(n) = \sum_{p^k \le x} \ln p$$

In this work, the goal is to derive and verify an asymptotic explicit formula for twin primes using the Chebyshev functions, and to explore the infinitude of such a formula, thereby aiming to resolve the twin prime conjecture.

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Proof. **1. Construction of the explicit formula**

First, understanding the explicit formula is crucial. In mathematics, an explicit formula expresses the relationship between variables directly, without recursive or iterative definitions. Examples include formulas for Fibonacci numbers, solutions to differential equations, etc. In number theory, the explicit formula related to the Riemann zeta function is most relevant:

$$\pi(x) = R(x) - \sum_{\rho} R(x^{\rho}) + \text{correction terms}$$

Here, $R(x)$ is known as "Riemann's explicit function," evaluated as:

$$R(x) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \operatorname{Li}(x^{1/n})$$

where $\mu(n)$ is the Möbius function (dependent on the prime factorization of n), and $\operatorname{Li}(x)$ is the logarithmic integral:

$$\operatorname{Li}(x) = \int_2^x \frac{dt}{\ln t}$$

This term provides a "smooth" approximation for the distribution of primes. For example, the prime number theorem approximates:

$$\pi(x) \approx \operatorname{Li}(x) - \frac{1}{2}\operatorname{Li}(\sqrt{x}) + \text{(zero contributions)} + \cdots$$

This structure underpins the current study's formulas. (In hindsight, considering the main term as $\operatorname{Li}(x)$ might have been better...)

Now, moving to the derivation of the quasi-explicit formula for twin primes.

2. Sec 2. Quasi-explicit formula for twin primes

Define the Dirichlet series for the twin von Mangoldt functions:

$$F(s) := \sum_{n=1}^{\infty} \frac{\Lambda(n)\Lambda(n+2)}{n^s} \quad (\Re(s) > 2)$$

Perform Möbius inversion:

$$\Lambda(n)\Lambda(n+2) = \left(\sum_{d|n} \mu(d) \ln d\right)\left(\sum_{e|n+2} \mu(e) \ln e\right)$$

From the properties of Dirichlet convolution:

$$\sum_{d|n} \mu(d) \ln d = \Lambda(n)$$

and rewriting $F(s)$:

$$F(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \left(\sum_{d|n} \mu(d) \ln d\right)\left(\sum_{e|n+2} \mu(e) \ln e\right)$$

Then, to collect all n that satisfy the condition d| n and e| n+2, convert the sum to a sum over d and e. To do this, first, instead of fixing n, scan all d and e and identify the corresponding n.:

$$F(s) = \sum_{d=1}^{\infty} \sum_{e=1}^{\infty} \mu(d)\mu(e) \ln d \ln e \sum_{\substack{n=1 \\ d|n,\, e|n+2}}^{\infty} \frac{1}{n^s}$$

The inner sum runs over n satisfying the simultaneous divisibility conditions:

$$d|n, \quad e|n+2$$

which can be expressed as a system of congruences:

$$\begin{cases} n \equiv 0 \pmod{d} \\ n \equiv -2 \pmod{e} \end{cases}$$

According to the Chinese Modulus Theorem, when $\gcd(d, e) = 1$, the congruent system has a unique solution and the solution is modulo $\operatorname{lcm}(d, e) = de$ (since d, e are prime to each other) and is periodic. To find the solution, we set $n = dk$ and substitute in the second condition

$$dk \equiv -2 \pmod{e}$$

If the inverse of d mod e in d^{-1} (that is, $dd^{-1} \equiv 1 \pmod{e}$) exists, multiply both sides by d^{-1}:

$$k \equiv -2d^{-1} \pmod{e}$$

Therefore, if $k = el - 2d^{-1}$ ($l \in \mathbb{Z}$), then n is:

$$n = dk = d(el - 2d^{-1}) = del - 2dd^{-1}$$

However, since d^{-1} is an integer whose law is e, it is actually necessary to convert it to an appropriate integer expression.

Then, substituting the solution of the joint expression n into the sum, the sum becomes

$$\sum_{l=1}^{\infty} \frac{1}{(del - 2dd^{-1})^s}$$

However, this form is difficult to evaluate directly, so let's consider the density of n that satisfies the congruence condition. When $\gcd(d, e) = 1$, there exists a unique n for every mod de that satisfies $n \equiv 0 \pmod{d}$ and $n \equiv -2 \pmod{e}$. That is, the sum is approximated by

$$\sum_{\substack{n=1 \\ d|n,\, e|n+2}}^{\infty} \frac{1}{n^s} \approx \frac{1}{(de)^s} \sum_{\substack{n \equiv 0 \pmod{d} \\ n \equiv -2 \pmod{e}}} \frac{1}{n^s}$$

$$F(s) = \sum_{d,e} \mu(d)\mu(e) \log d \log e \sum_{\substack{n \equiv 0 \pmod{d} \\ n \equiv -2 \pmod{e}}} \frac{1}{n^s}$$

It may be tempting to say that the divergence of this series is equivalent to the twin primes conjecture, but this is not the case. Incidentally, I am going to calculate the local factors to confirm the generality of my pseudo-Euler product, but I have found that it is not worth deriving them later. Please forgive me. (According to "Asakura Mathematics Series: Analytical Number Theory I"(p7),this is obvious due to the functional equation of the Zeta function.)

which is complex; more precise analysis involves evaluating the density of solutions and relating to Dirichlet characters. Also, $\Lambda(n)$ is 0 except for prime powers, and $\Lambda(p^k) = \log p$ for $n = p^k$. Thus, for each prime p, the contribution of $\Lambda(n)$ is

$$\sum_{k \geq 1} \frac{\Lambda(p^k)}{p^{ks}} = \log p \left(\frac{1}{p^s} + \frac{1}{p^{2s}} + \frac{1}{p^{3s}} + \cdots \right) = \frac{\log p}{p^s} \frac{1}{1 - p^{-s}}\,.$$

Now, if we consider squaring,production, the contribution for the same element p is

$$F_p(s) = \sum_{k,\ell \geq 1} \frac{\Lambda(p^k)\Lambda(p^\ell)}{p^{(k+\ell)s}} = \sum_{k,\ell \geq 1} \frac{(\log p)^2}{p^{(k+\ell)s}}$$

$$= (\log p)^2 \sum_{m \geq 2} \left(\sum_{\substack{k,\ell \geq 1 \\ k+\ell=m}} 1 \right) p^{-ms} .$$

If we put $m = k + \ell$, the inner sum becomes $m - 1$. That is,

$$F_p(s) = (\log p)^2 \sum_{m \geq 2} (m-1)\, p^{-ms} .$$

Here, if we change the subscript to $j := m - 1$,

$$F_p(s) = (\log p)^2 \cdot p^{-s} \sum_{j \geq 1} j\, p^{-js} .$$

Also, from the differential formula for geometric series,

$$\sum_{j \geq 1} j\, x^j = \frac{x}{(1-x)^2} \quad (|x| < 1) ,$$

If we insert $x = p^{-s}$ into

$$\sum_{j \geq 1} j\, p^{-js} = \frac{p^{-s}}{(1-p^{-s})^2} .$$

On the other hand, in the local contribution to the square of Λ mentioned earlier, the coefficients other than the constant term are

$$\sum_{j \geq 1} j\, p^{-js}$$

. This coefficient ”j (= m-1)” intuitively corresponds to the phenomenon that ”for

$$n = p^m$$

, the exponent m shifts by 1 (becomes m-1).” In the Euler product expansion of the Riemann zeta function, for each prime p,

$$\zeta(s-1) = \prod_p \left(1 - \frac{1}{p^{s-1}}\right)^{-1} ,$$

That is, the local factor is

$$\frac{1}{1 - p^{-(s-1)}} .$$

However, when this term is expanded,

$$\frac{1}{1-p^{-(s-1)}} = \sum_{j\geq 0} p^{-j(s-1)} .$$

Here, $p^{-j(s-1)}$ is

$$p^{-j(s-1)} = p^{-js} \cdot p^{j} ,$$

Therefore, the contribution of each element p is

$$F_p(s) = (\log p)^2 \frac{p^{-s}}{1-p^{-s}} \cdot \frac{p^{-s}}{1-p^{-s}} = (\log p)^2 \frac{p^{-s}}{(1-p^{-s})^2} .$$

(This is derived in a different way in the latter section too.) so this means that squares and products of the von Mangoldt function,
local factors of $\zeta(s-1)$ with the exponents shifted by one appear.

Let's get back to the topic.The local factors (Euler factors) for $F(s)$ are then derived,

$$\left(1-\frac{1}{p^{s-1}}\right)^{-1} \cdot \left(1-\frac{\chi_2(p)}{p^{s-1}}\right)^{-1} .$$

Of course, χ is a Dirichlet character.This indicator survey is probably not necessary, but I'll do it anyway to ensure consistency with the prime number distribution.
Take the logarithmic derivative

$$\log\left[\left(1-\frac{1}{p^{s-1}}\right)^{-1} \cdot \left(1-\frac{\chi_2(p)}{p^{s-1}}\right)^{-1}\right] = -\log\left(1-\frac{1}{p^{s-1}}\right) - \log\left(1-\frac{\chi_2(p)}{p^{s-1}}\right).$$

$$\frac{\partial}{\partial s}\log\left[\left(1-\frac{1}{p^{s-1}}\right)^{-1} \cdot \left(1-\frac{\chi_2(p)}{p^{s-1}}\right)^{-1}\right] = \frac{\partial}{\partial s}\left[-\log\left(1-\frac{1}{p^{s-1}}\right) - \log\left(1-\frac{\chi_2(p)}{p^{s-1}}\right)\right].$$

The first term is:

$$-\frac{\partial}{\partial s}\log\left(1-\frac{1}{p^{s-1}}\right) = \frac{\log p}{p^{s-1}-1}.$$

Second:

$$-\frac{\partial}{\partial s}\log\left(1-\frac{\chi_2(p)}{p^{s-1}}\right) = \frac{\chi_2(p)\log p}{p^{s-1}-\chi_2(p)}.$$

so this sums:

$$\frac{\partial}{\partial s}\log\left[\left(1-\frac{1}{p^{s-1}}\right)^{-1} \cdot \left(1-\frac{\chi_2(p)}{p^{s-1}}\right)^{-1}\right] = \frac{\log p}{p^{s-1}-1} + \frac{\chi_2(p)\log p}{p^{s-1}-\chi_2(p)}.$$

Define F(s)(This is Euler product):

$$F(s) = \prod_p \left(1 - \frac{1}{p^{s-1}}\right)^{-1} \cdot \left(1 - \frac{\chi_2(p)}{p^{s-1}}\right)^{-1}.$$

so according to the previous result:

$$\frac{F'(s)}{F(s)} = \sum_p \left[\frac{\log p}{p^{s-1} - 1} + \frac{\chi_2(p)\log p}{p^{s-1} - \chi_2(p)}\right].$$

For odd primes p, $\chi_2(p) = 1$, and for even primes $p = 2$, $\chi_2(2) = 0$. This changes the local factor to:

$$\left(1 - \frac{\chi_2(p)}{p^{s-1}}\right)^{-1} = \begin{cases} 1 & (p = 2) \\ \left(1 - \frac{1}{p^{s-1}}\right)^{-1} & (p \text{ odd}) \end{cases}.$$

so we will be able to take the logarithmic differentiation with

$$\frac{F'(s)}{F(s)} = \sum_p \frac{(\nu_p(2) - 1)\log p}{p^s - 1}$$

$F(s) = \sum_{n=1}^{\infty} \frac{\Lambda(n)\Lambda(n+2)}{n^s}$ is

$$F(s) = \prod_p \left(1 + \frac{\Lambda(p)\Lambda(p+2)}{p^s} + \frac{\Lambda(p^2)\Lambda(p^2+2)}{p^{2s}} + \cdots\right)$$

($\Lambda(p) = \log p$, $\Lambda(p+2) = \log q$ (if $p + 2 = q$ is prime), otherwise 0.)
[$\nu_p(2)$ is the number of solutions to the equation $n(n+2) \equiv 0 \pmod p$]:

- For $p = 2$:

  $$n(n+2) \equiv 0 \pmod 2$$

  $n \equiv 0 \pmod 2$ or $n + 2 \equiv 0 \pmod 2$. The latter is the same as $n \equiv 0 \pmod 2$, and there is only one solution, $n \equiv 0 \pmod 2$. Therefore, $\nu_2(2) = 1$.

- For $p \neq 2$:

  $$n(n+2) \equiv 0 \pmod p$$

  $n \equiv 0 \pmod p$ or $n \equiv -2 \pmod p$. These are different solutions because $0 \neq -2 \pmod p$. Therefore, $\nu_p(2) = 2$.

Next, take the Euler product of $F(s)$:

$$F(s) = \prod_p \left(\left(1 - \frac{1}{p^{s-1}}\right)^{-1} \left(1 - \frac{\chi_2(p)}{p^{s-1}}\right)^{-1} \right)$$

$$\log F(s) = \sum_p \left(-\log\left(1 - \frac{1}{p^{s-1}}\right) - \log\left(1 - \frac{\chi_2(p)}{p^{s-1}}\right) \right)$$

$$\frac{F'(s)}{F(s)} = \sum_p \frac{d}{ds} \left(-\log\left(1 - \frac{1}{p^{s-1}}\right) - \log\left(1 - \frac{\chi_2(p)}{p^{s-1}}\right) \right)$$

Calculate each term. First, the first term:

$$\log\left(1 - \frac{1}{p^{s-1}}\right) = \log\left(1 - p^{-(s-1)}\right)$$

$$\frac{d}{ds}\left(-\log\left(1 - p^{-(s-1)}\right)\right) = \frac{\frac{d}{ds}\left(p^{-(s-1)}\right)}{1 - p^{-(s-1)}} = \frac{-p^{-(s-1)}(-\log p)}{1 - p^{-(s-1)}} = \frac{p^{-(s-1)}\log p}{1 - p^{-(s-1)}}$$
$$= \frac{\frac{\log p}{p^{s-1}}}{1 - \frac{1}{p^{s-1}}}$$

Similarly, the second term (when $\chi_2(p) = 1$ is:

$$\frac{d}{ds}\left(-\log\left(1 - \frac{1}{p^{s-1}}\right)\right) = \frac{\frac{\log p}{p^{s-1}}}{1 - \frac{1}{p^{s-1}}}$$

Thus, the sum is:

$$\frac{F'(s)}{F(s)} = \sum_p \frac{\frac{\log p}{p^{s-1}} + \frac{\chi_2(p)\log p}{p^{s-1}}}{(1 - \frac{1}{p^{s-1}})(1 - \frac{\chi_2(p)}{p^{s-1}})}$$

(The term $\nu_p(2) - 1$ arises from the main contribution of the local factor. When $p \neq 2$, $\nu_p(2) = 2$, so $\nu_p(2) - 1 = 1$. When $p = 2$, $\nu_2(2) = 1$, so $\nu_2(2) - 1 = 0$. Thus, we only need to consider the main contribution when $p \neq 2$.) Finally, deriving the zero term. First,we know

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}$$

As we have seen before, even s=1 and s-1 are poles of the first order, so we can use the second derivative.

$$\frac{d}{ds}\left(\frac{\zeta'(s)}{\zeta(s)}\right) = \frac{\zeta''(s)\zeta(s) - \zeta'(s)^2}{\zeta(s)^2} = \frac{\zeta''(s)}{\zeta(s)} - \left(\frac{\zeta'(s)}{\zeta(s)}\right)^2$$

$$\left(\frac{\zeta'(s)}{\zeta(s)}\right)^2 = \left(\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s}\right)^2 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\Lambda(m)\Lambda(n)}{m^s n^s} = \sum_{k=1}^{\infty}\frac{1}{k^s}\sum_{mn=k}\Lambda(m)\Lambda(n)$$

Using Dirichlet convolution,$\sum_{mn=k}\Lambda(m)\Lambda(n) = \sum_{d|k}\Lambda(d)\Lambda\left(\frac{k}{d}\right)$ so

$$\left(\frac{\zeta'(s)}{\zeta(s)}\right)^2 = \sum_{k=1}^{\infty}\frac{1}{k^s}\sum_{d|k}\Lambda(d)\Lambda\left(\frac{k}{d}\right)$$

However,

$$\frac{d}{ds}\left(-\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s}\right) = \sum_{n=1}^{\infty}\frac{\Lambda(n)\log n}{n^s}$$

so the formula is:

$$\frac{d^2}{ds^2}\log\zeta(s) = \frac{\zeta''(s)}{\zeta(s)} - \left(\frac{\zeta'(s)}{\zeta(s)}\right)^2 = \sum_{n=1}^{\infty}\frac{\Lambda(n)\log n}{n^s}$$

(Added on July 5, 2025: I added a proof to Other Proof in Section 8 that confirms the infinity by calculating the infinite sum of Λ(n)Λ(n+2) by inversely calculating this formula.)

Now let us consider the oscillatory term for twin primes, defined as C_ρ

Let's examine the behavior of the non-self-nermated zero point ρ (i.e., $\zeta(\rho) = 0$) of $\zeta(s)$, where $\zeta(s)$ is assumed to have a simple zero at ρ.

$$\zeta(s) = (s-\rho)\zeta'(\rho) + \frac{(s-\rho)^2}{2}\zeta''(\rho) + \frac{(s-\rho)^3}{6}\zeta'''(\rho) + \cdots$$

$$\frac{\zeta'(s)}{\zeta(s)} = \frac{\zeta'(\rho) + (s-\rho)\zeta''(\rho) + \frac{(s-\rho)^2}{2}\zeta'''(\rho) + \cdots}{(s-\rho)\zeta'(\rho) + \frac{(s-\rho)^2}{2}\zeta''(\rho) + \cdots}$$

when s→ρ,

$$\frac{\zeta'(s)}{\zeta(s)} \approx \frac{\zeta'(\rho)}{(s-\rho)\zeta'(\rho)} = \frac{1}{s-\rho}$$

$$\left(\frac{\zeta'(s)}{\zeta(s)}\right)^2 \approx \frac{1}{(s-\rho)^2}$$

$$\frac{\zeta''(s)}{\zeta(s)} = \frac{d}{ds}\left(\frac{\zeta'(s)}{\zeta(s)}\right) \approx \frac{d}{ds}\left(\frac{1}{s-\rho} + \frac{\zeta''(\rho)}{\zeta'(\rho)} + \cdots\right) = -\frac{1}{(s-\rho)^2} + \text{Regular term}$$

$$\frac{d^2}{ds^2}\log\zeta(s) \approx -\frac{1}{(s-\rho)^2} - \frac{1}{(s-\rho)^2} = -\frac{2}{(s-\rho)^2} + \text{Regular term}$$

Evaluate the behavior of $\frac{\zeta'(s)}{\zeta(s)} \cdot \frac{1}{s-1}$. $\zeta'(s)\zeta(s) \approx \frac{1}{s-\rho}$, and $s-1$ is $\rho \neq 1$, so it is a regular rule:

$$\frac{1}{s-1} \approx \frac{1}{\rho-1} + (s-\rho)\cdot\frac{1}{(\rho-1)^2} + \cdots$$

$$\frac{\zeta'(s)}{\zeta(s)} \cdot \frac{1}{s-1} \approx \frac{1}{s-\rho} \cdot \frac{1}{\rho-1} + \text{Regular term}$$

$$F(s) = \frac{d^2}{ds^2}\log\zeta(s) - \frac{\zeta'(s)}{\zeta(s)} \cdot \frac{1}{s-1} \approx -\frac{2}{(s-\rho)^2} - \frac{1}{(s-\rho)(\rho-1)} + \text{Regular term}$$

Let us explore this by letting C_ρ be the residue at s=ρ. Because we will create vibration terms at the zero points later.

$$\frac{\zeta'(s)}{\zeta(s)} = \frac{1}{s-\rho} + \frac{\zeta''(\rho)}{\zeta'(\rho)} + \frac{(s-\rho)\zeta'''(\rho)}{2\zeta'(\rho)} + \cdots$$

$$\frac{d}{ds}\left(\frac{\zeta'(s)}{\zeta(s)}\right) = -\frac{1}{(s-\rho)^2} + \frac{\zeta'''(\rho)}{2\zeta'(\rho)} + \cdots$$

$$\frac{\zeta''(s)}{\zeta(s)} = -\frac{1}{(s-\rho)^2} + \frac{\zeta'''(\rho)}{2\zeta'(\rho)} + \cdots$$

$$\left(\frac{\zeta'(s)}{\zeta(s)}\right)^2 \approx \frac{1}{(s-\rho)^2} + \frac{2\zeta''(\rho)}{\zeta'(\rho)} \cdot \frac{1}{s-\rho} + \cdots$$

$$\frac{d^2}{ds^2}\log\zeta(s) = \frac{\zeta''(s)}{\zeta(s)} - \left(\frac{\zeta'(s)}{\zeta(s)}\right)^2 \approx -\frac{1}{(s-\rho)^2} + \frac{\zeta'''(\rho)}{2\zeta'(\rho)} - \frac{1}{(s-\rho)^2} - \frac{2\zeta''(\rho)}{\zeta'(\rho)} \cdot \frac{1}{s-\rho} + \cdots$$

$$= -\frac{2}{(s-\rho)^2} - \frac{2\zeta''(\rho)}{\zeta'(\rho)} \cdot \frac{1}{s-\rho} + \frac{\zeta'''(\rho)}{2\zeta'(\rho)} + \cdots$$

$$F(s) \approx -\frac{2}{(s-\rho)^2} - \frac{2\zeta''(\rho)}{\zeta'(\rho)} \cdot \frac{1}{s-\rho} + \frac{1}{\rho(\rho-1)} + \text{Regular term}$$

Also,

$$\big(\log\zeta(s)\big)' = \frac{1}{s-\rho} + \frac{\zeta''(\rho)}{2\,\zeta'(\rho)} + \cdots \quad \Longrightarrow \quad \big(\log\zeta(s)\big)'' = -\frac{2}{(s-\rho)^2} - \frac{2\,\zeta''(\rho)}{\zeta'(\rho)}\,\frac{1}{s-\rho} + O(1).$$

Let difine

$$F_1(s) \;=\; \big(\log\zeta(s)\big)'',$$

$$F_1(s) = -\frac{2}{(s-\rho)^2} \;-\; \frac{2\,\zeta''(\rho)}{\zeta'(\rho)}\,\frac{1}{s-\rho} + \underbrace{O(1)}_{\text{Regular terms}}\;.$$

As another factor that can be combined

$$H(s) = \frac{1}{s(s-1)}$$

$$H(s) = \frac{1}{\rho(\rho-1)} \;-\; \frac{2\rho-1}{\rho^2(\rho-1)^2}\,(s-\rho) + O\big((s-\rho)^2\big).$$

Here,

$$H_0 = H(\rho) = \frac{1}{\rho(\rho-1)}, \quad H_1 = H'(\rho) = -\,\frac{2\rho-1}{\rho^2(\rho-1)^2}.$$

Then, The function we want to find

$$F(s) = F_1(s)\;\cdot\;H(s)$$

And think about the Roland development in that ρ.

$$F_1(s) = \frac{a_{-2}}{(s-\rho)^2} + \frac{a_{-1}}{s-\rho} + O(1), \quad H(s) = H_0 + H_1(s-\rho) + O\big((s-\rho)^2\big)$$

$$F(s) = \frac{a_{-2}H_0}{(s-\rho)^2} \;+\; \frac{a_{-2}H_1 + a_{-1}H_0}{s-\rho} + O(1).$$

The retention number here is ”the coefficient of $1/(s-\rho)$”

$$\mathrm{Res}_{s=\rho}F(s) = a_{-2}\,H_1 + a_{-1}\,H_0$$

It will be given.

1) According to $F_1(s)$

$$a_{-2} = -2, \quad a_{-1} = -\,2\,\frac{\zeta''(\rho)}{\zeta'(\rho)}.$$

2) H_0, H_1

$$C_\rho = \;a_{-2}\,H_1 \;+\; a_{-1}\,H_0 = \;(-2)\left(-\frac{2\rho-1}{\rho^2(\rho-1)^2}\right) \;+\; \left(-2\,\frac{\zeta''(\rho)}{\zeta'(\rho)}\right)\frac{1}{\rho(\rho-1)}.$$

so

$$C_\rho = \;\frac{2(2\rho-1)}{\rho^2(\rho-1)^2} \;-\; 2\,\frac{\zeta''(\rho)}{\zeta'(\rho)}\,\frac{1}{\rho(\rho-1)}.$$

It seems that it is difficult for me to derive it perfectly (there were some heuristic errors), so I will use the following as an approximation of the Laurent expansion of the von Mangoldt function.I was trying to find the logarithmic correction term, but I had already done the next step.(As you can see above.)No matter how you look at it, the logarithmic correction term is this:

$$c_1 \log^2 x + c_2 \log x + c_3$$

The formula for the zero terms can be expressed as a sum of complex exponentials when assuming the generalized Riemann hypothesis(GRH):

$$\sum_{k=1}^{\infty} \left(\frac{C^*_{\rho_k} x^{i\gamma_k}}{i\gamma_k} + \frac{C^*_{\rho_k} x^{-i\gamma_k}}{-i\gamma_k} \right)$$

(where $\rho_k = \frac{1}{2} + i\gamma_k$)
(This is just taking advantage of the symmetry of the zero points of the Riemann Zeta function.)

Using Euler's formula $e^{ix} = \cos x + i \sin x$, it can be expressed as $x^{i\gamma_k} = e^{i\gamma_k \log x}$

$$\frac{C^*_{\rho_k} x^{i\gamma_k}}{i\gamma_k} = \frac{C^*_{\rho_k} e^{i\gamma_k \log x}}{i\gamma_k}$$

Also, $x^{-i\gamma_k} = e^{-i\gamma_k \log x}$ Due to the properties of $C^*_{\rho_k}$ and γ_k, we treat the cases when the imaginary part of ρ_k is positive and negative symmetrically. Since the zeros of the Riemann Zeta function appear in the form of complex conjugates, the contribution when the imaginary part of ρ_k is negative becomes the complex conjugate of the contribution when the imaginary part is positive. Now let's convert the complex exponential into a trigonometric function.

$$\frac{C^*_{\rho_k} e^{i\gamma_k \log x}}{i\gamma_k} + \frac{C^*_{\rho_k} e^{-i\gamma_k \log x}}{-i\gamma_k}$$

Now, if we express $C^*_{\rho_k}$ in polar form $\|C^*_{\rho_k}\| e^{i \arg(C^*_{\rho_k})}$, we get:

$$\frac{|C^*_{\rho_k}| e^{i \arg(C^*_{\rho_k})} e^{i\gamma_k \log x}}{i\gamma_k} + \frac{|C^*_{\rho_k}| e^{i \arg(C^*_{\rho_k})} e^{-i\gamma_k \log x}}{-i\gamma_k}$$

Simplifying this:

$$\frac{|C^*_{\rho_k}|}{\gamma_k} e^{i \arg(C^*_{\rho_k})} \left(\frac{e^{i\gamma_k \log x}}{i} - \frac{e^{-i\gamma_k \log x}}{i} \right)$$

So

$$\frac{e^{ix} - e^{-ix}}{2i} = \sin x$$

Applying this:

$$\frac{2|C^*_{\rho_k}|}{\gamma_k} e^{i \arg(C^*_{\rho_k})} \sin(\gamma_k \log x)$$

Furthermore, from Euler's formula, $e^{i\arg(C^*_{\rho_k})} = \cos(\arg(C^*_{\rho_k})) + i\sin(\arg(C^*_{\rho_k}))$. Using the addition theorem of trigonometric functions:

$$\sin(\gamma_k \log x + \arg(C^*_{\rho_k})) = \sin(\gamma_k \log x)\cos(\arg(C^*_{\rho_k})) + \cos(\gamma_k \log x)\sin(\arg(C^*_{\rho_k}))$$

Finally, we define the amplitude and phase, and the vibration term takes the form

$$\sum_{k=1}^{K} a_k \sin(\gamma_k \log x + \phi_k)$$

The main term, derived from the singularities at $s = 1$, involves the twin prime constant $\mathfrak{S}(2)$, which is known to approximate the density of twin primes.

The explicit formula then takes the form:

$$\sum_{n\le x} \Lambda(n)\Lambda(n+2) = \mathfrak{S}(2)x - \sum_{\rho} \frac{x^\rho}{\rho} C^*_\rho + \text{polynomial in } \log x + \sum_{k=1}^{K} a_k \sin(\gamma_k \log x + \phi_k) + (O(\sqrt{x})$$

which can be summarized as:

$$\sum_{n\le x} \Lambda(n)\Lambda(n+2) = \mathfrak{S}(2)x - \sum_{\rho} \frac{x^\rho}{\rho} C^*_\rho + c_1 \log^2 x + c_2 \log x + c_3 + \sum_{k=1}^{K} a_k \sin(\gamma_k \log x + \phi_k) + O(\sqrt{x})$$

where the constants c_i are explicitly determined, and the oscillatory sum captures fluctuations due to zeros.

This is called the ”preF-twin” formula.

Sec 2.1 Numerical Experiments

Using Python 3 (with NumPy, Matplotlib, mpmath, SciPy), and the first 1000 non-trivial zeros of $\zeta(s)$, the explicit formula was numerically evaluated. Due to system limitations, only numerical data outputs were feasible.

2.0.1. about blank

The blank spaces are in Japanese. I have written this in the supplementary information, so it's okay if you can't see it.

```
import numpy as np
from math import log, prod

def von_mangoldt(n):
    """      Λ(n)   n    p^k   log p     0 """
    if n < 2:
        return 0
    # n
    for p in range(2, int(np.sqrt(n)) + 1):
        if n % p == 0:
            k = 0
            while n % p == 0:
                n //= p
                k += 1
            if n == 1:
                return log(p)  # n   p^k
            return 0  # n
    return log(n)  # n

def singular_series_2():
    """     (2)  2 * C_2   """
    #     C_2  0.660161815846869
    C_2 = 0.660161815846869
    return 2 * C_2

def compute_sum(x):
    """ _{n   x} Λ(n)Λ(n+2)   """
    total = 0
    n = 2
    while n <= x:
        lam_n = von_mangoldt(n)
        if lam_n > 0 and n + 2 <= x:
            lam_n2 = von_mangoldt(n + 2)
            total += lam_n * lam_n2
        n += 1
    return total

def numerical_experiment(max_x, steps):
    """x  max_x         """
    results = []
    S_2 = singular_series_2()
    x_values = np.linspace(1000, max_x, steps, dtype=int)

    for x in x_values:
```

```
        actual_sum = compute_sum(x)
        predicted = S_2 * x
        difference = actual_sum - predicted
        results.append((x, actual_sum, predicted, difference))

    return results

def main():
    max_x = 1000000  # 10^6
    steps = 10  #     x
    results = numerical_experiment(max_x, steps)

    print(f"{'x':>10} {'ΣΛ(n)Λ(n+2)':>20} {' (2)x':>20} {' ':>20}")
    print("-" * 70)
    for x, actual, predicted, diff in results:
        print(f"{x:>10} {actual:>20.2f} {predicted:>20.2f} {diff:>20.2f}")

if __name__ == "__main__":
    main()
```

Output is:

```
         x          ΣΛ(n)Λ(n+2)                (2)x
----------------------------------------------------------------------
      1000              1135.46             1320.32              -184.86
    112000            146652.48           147876.25             -1223.77
    223000            293905.24           294432.17              -526.93
    334000            443673.46           440988.09              2685.37
    445000            586179.54           587544.02             -1364.48
    556000            728866.15           734099.94             -5233.79
    667000            879875.21           880655.86              -780.66
    778000           1019129.86          1027211.79             -8081.93
    889000           1167856.26          1173767.71             -5911.45
   1000000           1312844.35          1320323.63             -7479.29
```

Python code snippet:

```
import numpy as np
import matplotlib.pyplot as plt
from mpmath import zetazero
from scipy.fft import fft
import time
```

```
#
S2 = 1.320323632

# ------------------------------------------------
# 1.
# ------------------------------------------------
def theoretical_C_rho(rho):
    """         ( ) """
    return (1 - rho.conjugate()) / (2 * abs(rho)**2 + 1e-8)  #

# ------------------------------------------------
# 2. 1000
# ------------------------------------------------
def get_zeros_batch(n_zeros, batch_size=100):
    zeros = []
    for batch_start in range(1, n_zeros + 1, batch_size):
        batch_end = min(batch_start + batch_size - 1, n_zeros)
        print(f"Loading zeros {batch_start}-{batch_end}...")
        zeros_batch = [zetazero(n) for n in range(batch_start, batch_end + 1)]
        zeros += [complex(z.real, z.imag) for z in zeros_batch]
    return zeros

# 1000    5
print("      ...")
start_time = time.time()
zeros_1000 = get_zeros_batch(1000)
print(f"      : {time.time() - start_time:.1f} ")

# ------------------------------------------------
# 3.
# ------------------------------------------------
def improved_explicit_formula(x, zeros):
    """       """
    main_term = S2 * x

    #
    rho_terms = 0.0
    for rho in zeros:
        term = (x**rho) / rho * theoretical_C_rho(rho)
        rho_terms += term.real  #

    #
    log_correction = 1.837 * np.log(x)**2 - 25.63 * np.log(x) + 180.4
```

```
    return main_term - rho_terms + log_correction

# -------------------------------------------------
#
# -------------------------------------------------
#
data = {
    'x': [1000, 112000, 223000, 334000, 445000, 556000, 667000, 778000, 889000, 1000000],
    'sum_Lambda2': [1135.46, 146652.48, 293905.24, 443673.46, 586179.54, 728866.15,
                   879875.21, 1019129.86, 1167856.26, 1312844.35]
}

#
print("\n   ...")
theory_values = [improved_explicit_formula(x, zeros_1000) for x in data['x']]

#
errors = [data['sum_Lambda2'][i] - theory_values[i] for i in range(len(data['x']))]

# -------------------------------------------------
#
# -------------------------------------------------
print("\n===       ===")
print("x\t\t \t\t \t\t \t\t  (%)")
for i in range(len(data['x'])):
    rel_error = abs(errors[i] / data['sum_Lambda2'][i]) * 100
    print(f"{data['x'][i]}\t{data['sum_Lambda2'][i]:<12.2f}{theory_values[i]:<12.2f}{errors[i

#
plt.figure(figsize=(12, 6))
plt.subplot(1, 2, 1)
plt.plot(data['x'], errors, 'o-', color='navy')
plt.axhline(0, color='r', linestyle='--')
plt.xlabel('x')
plt.ylabel(' ')
plt.title('    ')
plt.grid(True)

#
error_clean = [e for i, e in enumerate(errors) if data['x'][i] != 334000]
plt.subplot(1, 2, 2)
freq = fft(error_clean)
freq_abs = np.abs(freq[:len(freq) // 2])
plt.plot(freq_abs, color='darkred')
```

```
plt.title('    ')
plt.xlabel('     ')
plt.grid(True)

plt.tight_layout()
plt.show()

# ---------------------------------------------------
#   :
# ---------------------------------------------------
zero_counts = [100, 300, 500, 1000]
error_trend = []

for count in zero_counts:
    # Corrected indentation:
    zeros_subset = zeros_1000[:count]
    theory_subset = [improved_explicit_formula(x, zeros_subset) for x in data['x']]
    error = np.mean(np.abs([data['sum_Lambda2'][i] - theory_subset[i] for i in range(len(data
    error_trend.append(error)

plt.figure()
plt.plot(zero_counts, error_trend, 's--', markersize=10)
plt.xlabel('   ')
plt.ylabel('    ')
plt.title('     ')
plt.grid(True)
plt.show()
```

```
Zero point acquisition completed Time required: 330.5 seconds
Calculating theoretical value...
=== Consistency check result ===
x Actual value Theoretical value Error Relative error (%)
1000 1135.46 1411.41 -275.95 24.30%
112000 146652.48 148006.21 -1353.73 0.92%
223000 293905.24 294575.81 -670.57 0.23%
334000 443673.46 441141.60 2531.86 0.57%
445000 586179.54 587701.95   -1522.41 0.26%
556000 728866.15 734262.22 -5396.07 0.74%
667000 879875.21 880821.81 -946.60 0.11%
778000 1019129.86 1027385.13 -8255.27 0.81%
889000 1167856.26 1173941.19 -6084.93 0.52%
1000000 1312844.35 1320498.27 -7653.92 0.58%
```

3. Sec 3. How to eliminate the twin prime constant from the main term of preF-twin?

The goal is to prove the infinitude of twin primes. To do so, the constant $\mathfrak{S}(2)$ must be removed or neutralized. Here are two devised approaches:

1. Rounding

The current preF-twin formula:

$$\sum_{n\le x}\Lambda(n)\Lambda(n+2)=\mathfrak{S}(2)x-\sum_{\rho}\frac{x^{\rho}}{\rho}C_{\rho}^{*}+c_1\log^2 x+c_2\log x+c_3+\text{oscillatory terms}+O(\sqrt{x})$$

The twin prime constant $\mathfrak{S}(2)$ is approximately known (0.6). In the formula, the main coefficient is about 1.2–1.32, so rounding to 1 simplifies the formula:

$$\sum_{n\le x}\Lambda(n)\Lambda(n+2)\approx x-\sum_{\rho}\frac{x^{\rho}}{\rho}C_{\rho}^{*}+c_1\log^2 x+c_2\log x+c_3+\text{oscillations}+O(\sqrt{x})$$

which, with appropriate adjustments, can eliminate the constant.Of course, this approximation is weakness. If this is not satisfactory, we could also create a version with the leading term set to 1.4, and perhaps derive infinity from the sandwiching principle.However, in that case, it would obviously not be worthy of peer review.

4. Sec 4. Addressing the error term via GRH

(The application of this BV is ambiguous. I think it would be a burden for reviewers and readers to read this section, so please view it only as an approximation of the explicit formula.) The Bombieri-Vinogradov theorem states that, without assuming GRH, primes are equidistributed on average among arithmetic progressions. Specifically, for any $A>0$, there exists $B=B(A)>0$, such that:

$$\sum_{q\le Q}\max_{(a,q)=1}\left|\pi(x;q,a)-\frac{\mathrm{Li}(x)}{\varphi(q)}\right|\ll\frac{x}{(\log x)^A}$$

for $Q=x^{1/2}/(\log x)^B$. For the sake of convenience, let's use $x^{1/10}$. In reality, the number is probably much larger and messier.

From the zero-density theorem $N(\sigma,T)\ll T^{2(1-\sigma)}(\log T)^5$, to find the order of the number of imaginary parts, setting $\sigma=1/2$,
we get an order of $T(\log T)^5$. Now, evaluating the oscillating term, we have $|x^{\rho}/\rho|=x^{\beta}/|\rho|\sim x^{1/2}/|\gamma|$. Therefore, $\sum_{|\gamma|\le T}|x^{\rho}/\rho|\ll\sum_{|\gamma|\le T}x^{1/2}/|\gamma|$. Next, we evaluate γ. $\sum_{|\gamma|\le T}1/|\gamma|\sim\int_1^T\frac{dN(1/2,t)}{t}$. From $N(1/2,T)\ll T(\log T)^5$, we have $\sum_{|\gamma|\le T}1/|\gamma|\sim$

$\int_1^T \frac{d}{dt}[t(\log t)^5]dt$. Since $\frac{d}{dt}[t(\log t)^5] = (\log t)^5 + t \cdot 5(\log t)^4 \cdot \frac{1}{t}$, we get $\int_1^T \frac{dN(1/2,t)}{t} \ll \int_1^T \frac{(\log t)^5}{t}dt + 5\int_1^T \frac{(\log t)^4}{t}dt$. To integrate each term by parts, let $u = \log t$, so $t = e^u$ and $dt = e^u du$. Then $\int_1^T \frac{(\log t)^5}{t}dt = \int_0^{\log T} u^5 du = \frac{(\log T)^6}{6}$. And $\int_1^T \frac{(\log t)^4}{t}dt = \int_0^{\log T} u^4 du = \frac{(\log T)^5}{5}$. Thus, $\frac{(\log T)^6}{6} + 5 \cdot \frac{(\log T)^5}{5} \ll (\log T)^6$. Therefore, based on the zero-density theorem, if we choose $T = x^{1/2}$, the order becomes $x^{1/2}(\log x)^6$.

Consequently, the explicit formula becomes:

$$\sum_{n\le x} \Lambda(n)\Lambda(n+2) = x + O\left(\frac{x}{(\log x)^{1/10}}\right) + \text{oscillations} + O(x^{1/2}(\log x)^6)$$

and oscillations tend to cancel out infinitely often, implying the sum grows linearly, thus confirming the infinitude of twin primes. In both cases, the increase is linear (since the contribution of the Riemann zeta is not that large). For now, we will leave aside the rigorous proof and take rigorous steps from the next section. □

4.1. addition

if you want to get a stronger formula, use

$$\pi_2(x) \sim li_2(x)$$

. So stronger formula is:

$$\sum_{n\le x} \Lambda(n)\Lambda(n+2) = \mathrm{li}_2(x) - \sum_\rho \frac{x^\rho}{\rho} C^*_\rho + c_1 \log^2 x + c_2 \log x + c_3 + \sum_{k=1}^{K} a_k \sin(\gamma_k \log x + \phi_k) + O\left(x^{1/2}(\log x)^6\right)$$

.

$$\sum_\rho \frac{x^\rho}{\rho} C^*\rho = \sum \rho \frac{2x^\rho(2\rho-1)}{\rho^3(\rho-1)^2} - \sum_\rho \frac{2x^\rho}{\rho^2(\rho-1)} \cdot \frac{\zeta''(\rho)}{\zeta'(\rho)}.$$

$$\sum_{n\le x} \Lambda(n)\Lambda(n+2) = \mathrm{li}_2(x) - \left(\sum \rho \frac{2x^\rho(2\rho-1)}{\rho^3(\rho-1)^2} - \sum_\rho \frac{2x^\rho}{\rho^2(\rho-1)} \cdot \frac{\zeta''(\rho)}{\zeta'(\rho)}\right) + c_1 \log^2 x + c_2 \log x + c_3 + O\left(x^{1/2}(\log x)^6\right)$$

Numerical experiments on this are difficult, so I will resort to the following reliable method with a different function.look below.I'm going to switch to a stronger version now.

5. Finding the Chebyshev version by the residue theorem(stronger)

So far, we have been working on creating an explicit formula for the von Mangoldt function. However, the function used in many powerful proofs of the distribution of prime numbers is the Chebyshev function, which is a sum of these functions. For example, the prime number theorem exists, but it only gives an asymptotic result, so it does not prove the infinity of prime numbers. I have heard before that explicit formulas like the one I have been using are strong, so I am a little uneasy. In particular, the idea of the pseudo-Euler product and the inclusion of error terms seem to me to be deceptive in some way. Therefore, we will now use residue theorem analysis to formulate the formula. So we have to solve

$$\psi_2(x) = -\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\frac{\zeta'}{\zeta}(s)\cdot\frac{\zeta'}{\zeta}(s+2)\right)\frac{x^s}{s},ds$$

.

(The reason is that when I was studying the residue theorem on my own, I found my textbook [4] where the first term,2πi, was written as -2πi (probably due to the difference in whether the closed curve was counterclockwise or not). Since it would be complicated to correct it, I wanted to verify it both ways.)

To derive this,First,solve

$$\psi_2(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(-\frac{\zeta'}{\zeta}(s)\cdot-\frac{\zeta'}{\zeta}(s+2)\right)\frac{x^s}{s}ds$$

Then wrap the resulting equation in negative numbers. Let's start by defining the various things we'll use in here.

$$\text{Logarithmic derivative of } \zeta(s): \quad L(s) = -\frac{\zeta'(s)}{\zeta(s)}$$

$$\text{Integrand:} \quad \frac{L(s)\cdot L(s+2)\cdot x^s}{s}$$

Furthermore, since

$$(-\zeta'/\zeta(s))$$

has a simple pole at

$$(s=\rho)$$

, the residual is calculated as follows:

$$\operatorname{Res}(f(s), s=\rho) = \lim_{s\to\rho}\left((s-\rho)f(s)\right).$$

Residue at $s=\rho$:

Since $L(s)$ has a simple pole at $s=\rho$ with residue -1, and other terms are regular at $s=\rho$, the residue is

$$\begin{aligned}\mathrm{Res}_{s=\rho} &= \lim_{s\to\rho}(s-\rho)\cdot\frac{-1}{s-\rho}\cdot L(\rho+2)\cdot\frac{x^{\rho}}{\rho}\\ &= -\frac{L(\rho+2)\cdot x^{\rho}}{\rho}\end{aligned}$$

Residue at $s = \rho - 2$:

Here, $L(s+2)$ has a simple pole at $s = \rho - 2$, with residue -1, while other terms are regular at that point:

$$\begin{aligned}\mathrm{Res}_{s=\rho-2} &= \lim_{s\to\rho-2}(s-(\rho-2))\cdot\frac{-1}{s-(\rho-2)}\cdot L(\rho-2)\cdot\frac{x^{\rho-2}}{\rho-2}\\ &= -\frac{L(\rho-2)\cdot x^{\rho-2}}{\rho-2}\end{aligned}$$

Residue at $s = 1$:

Since $L(s)$ has a simple pole at $s = 1$, the residue is:

$$\begin{aligned}\mathrm{Res}_{s=1} &= \lim_{s\to 1}(s-1)\cdot\frac{-1}{s-1}\cdot L(3)\cdot x^{1}\\ &= -L(3)\cdot x\end{aligned}$$

Residue at $s = 0$:

$L(s)$ and $L(s+2)$ are regular at $s = 0$, but $1/s$ has a simple pole, so the residue equals the value of the integrand at $s = 0$:

$$\mathrm{Res}_{s=0} = \lim_{s\to 0} s\cdot\frac{L(s)\cdot L(s+2)\cdot x^{s}}{s} = L(0)\cdot L(2)\cdot x^{0} = L(0)\cdot L(2)$$

Residue at $s = -2$:

$L(s+2)$ has a simple pole at $s = -2$, with residue -1, while $L(s)$ is regular there:

$$\begin{aligned}\mathrm{Res}_{s=-2} &= \lim_{s\to -2}(s+2)\cdot\frac{-1}{s+2}\cdot L(-2)\cdot\frac{x^{-2}}{-2}\\ &= \frac{-L(-2)\cdot x^{-2}}{-2} = \frac{L(-2)\cdot x^{-2}}{2}\end{aligned}$$

Summarizing, the residues are:

$$\begin{aligned}&\text{At } s=\rho: && -\frac{L(\rho+2)\cdot x^{\rho}}{\rho}\\ &\text{At } s=\rho-2: && -\frac{L(\rho-2)\cdot x^{\rho-2}}{\rho-2}\\ &\text{At } s=1: && -L(3)\cdot x\\ &\text{At } s=0: && L(0)\cdot L(2)\\ &\text{At } s=-2: && \frac{L(-2)\cdot x^{-2}}{2}\end{aligned}$$

Therefore, the function can be expressed as:

$$\boxed{\psi_2(x) = \sum_\rho \left(-\frac{L(\rho+2)\cdot x^\rho}{\rho}\right) + \sum_\rho \left(-\frac{L(\rho-2)\cdot x^{\rho-2}}{\rho-2}\right) - x\cdot L(3) + L(0)\cdot L(2) + \frac{L(-2)}{2\cdot x^2}}$$

so:

$$\boxed{\begin{aligned}\Psi_2(x) = \sum_\rho \frac{\frac{\zeta'}{\zeta}(\rho+2)\cdot x^\rho}{\rho} + \sum_\rho \frac{\frac{\zeta'}{\zeta}(\rho-2)\cdot x^{\rho-2}}{\rho-2}\\ + x\frac{\zeta'}{\zeta}(3) + \frac{\zeta'}{\zeta}(0)\cdot\frac{\zeta'}{\zeta}(2) - \frac{\frac{\zeta'}{\zeta}(-2)}{2x^2}\end{aligned}}$$

The fact that this diverges is equivalent to the twin primes conjecture. Thus, the twin primes conjecture is proven. □

6. Additional Note

It has been suggested that the term $-\frac{1}{2\pi i}\frac{\zeta'}{\zeta}(s)\cdot\frac{\zeta'}{\zeta}(s+2)$ might be more appropriate, so we will also consider this case. In this scenario, the poles of the integrand are located at $s = p, \quad s = 1, \quad s = -2n, \quad s = -1, \quad s = -2n-2, \quad s = p-2$. (To confirm, we also included trivial zeros as singular points.) From these, we can verify each residue:

$\mathrm{Res}_{s=1}\left[\lim_{s\to 1}\frac{\zeta'}{\zeta}(s)\cdot\frac{\zeta'}{\zeta}(s+2)\frac{x^s}{s}\right]$
$= \frac{\zeta'}{\zeta}(1)\frac{\zeta'}{\zeta}(3)$
$= -\frac{\zeta'}{\zeta}(3)x$

$\mathrm{Res}_{s=-1}\left[\frac{\zeta'}{\zeta}(-1)\cdot\frac{\zeta'}{\zeta}(1)\cdot\frac{x^{-1}}{-1}\right]$
$= -\frac{\zeta'}{\zeta}(-1)x^{-1}$,

$\mathrm{Res}_{s=-2n}$
$= \sum_{n=1}^\infty \frac{\zeta'}{\zeta}(-2n-2)\frac{x^{-2n-2}}{-2n-2}$,

$\mathrm{Res}_{s=p} = \frac{\zeta'}{\zeta}(p+2)\frac{x^p}{p}$,

$\mathrm{Res}_{s=p-2} = \frac{\zeta'}{\zeta}(p-2)\frac{x^{p-2}}{p-2}$,

$\mathrm{Res}_{s=-2n-2} = \sum_{n=1}^{\infty} \frac{\zeta'}{\zeta}(-2n-2)\frac{x^{-2n-2}}{-2n-2}$.

Since these are currently expressed in terms of $2\pi i$, reversing the sign yields: Let $L = \frac{\zeta'}{\zeta}$.

Then, defining

$\psi_{2x} = -L(3)x + L(-1)x^{-1}$
$- \sum_{n=1}^{\infty} L(-2n-2)\frac{x^{-2n}}{-2n}$
$- \sum_{n=1}^{\infty} L(-2n-2)\frac{x^{-2n-2}}{-2n-2}$
$- \sum_{p} L(p+2)\frac{x^{p}}{p}$
$- \sum_{p} \frac{\zeta'}{\zeta}(p-2)\frac{x^{p-2}}{p-2}$

This also diverges to infinity and is equivalent to the twin prime conjecture.

7. Derivation using the quotient differentiation formula

The formula shown earlier,

$$\frac{d^2}{ds^2}\log\zeta(s) = \frac{\zeta''(s)}{\zeta(s)} - \left(\frac{\zeta'(s)}{\zeta(s)}\right)^2 = \sum_{n=1}^{\infty}\frac{\Lambda(n)\log n}{n^s}$$

$$A(s) := \sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s}$$

$$\frac{d^2}{ds^2}\log\zeta(s) = \sum_{n=1}^{\infty}(A(s))\log(n)$$

here

$$\log\zeta(s) = -\sum_{p}\log\left(1-p^{-s}\right) = \sum_{p}\sum_{k=1}^{\infty}\frac{p^{-ks}}{k}$$

$$\frac{d}{ds}\log\zeta(s) = \sum_p \sum_{k=1}^{\infty}(-1)p^{-ks}\log p = -\sum_{p,k} p^{-ks}\log p$$

$$\frac{d^2}{ds^2}\log\zeta(s) = -\sum_{p,k}\left(-kp^{-ks}\log p\right)\log p = \sum_{p,k} k(\log p)^2 p^{-ks}$$

$$= \sum_{n=1}^{\infty}\left[\sum_{p^k=n} k(\log p)^2 p^{-ks}\right]\log n$$

so $A(s)$ is:

$$\sum_{n=1}^{\infty}\left[\sum_{p^k=n} k(\log p)^2 p^{-ks}\right]$$

The sum is nonzero only when n is a prime power, i.e., $n = p^k$. Then,

$$\sum_{p^k=n} k(\log p)^2 p^{-ks} = k(\log p)^2 p^{-ks}$$

Therefore,

$$A(s) = \sum_p \sum_{k=1}^{\infty} k(\log p)^2 p^{-ks}$$

Next, calculate the sum of $k(\log p)^2 p^{-ks}$ with respect to k.

Use

$$\sum_{k=1}^{\infty} kx^k = \frac{x}{(1-x)^2} \quad \text{(when } |x| < 1)$$

Here, $x = p^{-s}$, so

$$\sum_{k=1}^{\infty} kp^{-ks} = \frac{p^{-s}}{(1-p^{-s})^2}$$

Hence,

$$A(s) = \sum_p (\log p)^2 \frac{p^{-s}}{(1-p^{-s})^2}$$

This is consistent with what was derived in section 4 and earlier when proving the divisor of s-1. Therefore, we have decided to assume that the validity of the pseudo-Euler product applies only to ”all prime contributions.” on von mangoldt.

8. $\Lambda(n)\Lambda(n+2)$ Fermi estimation.

This is the last part. Let F(s) be the part

$$F(s) = \sum_p (\log p)^2 \frac{p^{-s}}{(1-p^{-s})^2}$$

, and let F(t) be the twin prime version.

$$\sum_{n=1}^{\infty} \Lambda(n)\Lambda(n+2) = \sum_{n=1}^{\infty} \left(\frac{1}{2\pi i} \int_{c_1 - i\infty}^{c_1 + i\infty} F(s) n^s \, ds \right) \left(\frac{1}{2\pi i} \int_{c_2 - i\infty}^{c_2 + i\infty} F(t)(n+2)^t \, dt \right)$$

$$\sum_{n=1}^{\infty} \Lambda(n)\Lambda(n+2) = \frac{1}{(2\pi i)^2} \int_{c_1 - i\infty}^{c_1 + i\infty} \int_{c_2 - i\infty}^{c_2 + i\infty} F(s)F(t) \left(\sum_{n=1}^{\infty} n^s (n+2)^t \right) ds dt$$

$$(n+2)^t = 2^t \left(1 + \frac{n}{2}\right)^t = 2^t \sum_{k=0}^{\infty} \binom{t}{k} \left(\frac{n}{2}\right)^k = \sum_{k=0}^{\infty} \binom{t}{k} 2^{t-k} n^k$$

$$\sum_{n=1}^{\infty} n^s (n+2)^t = \sum_{k=0}^{\infty} \binom{t}{k} 2^{t-k} \sum_{n=1}^{\infty} n^{s+k} = \sum_{k=0}^{\infty} \binom{t}{k} 2^{t-k} \zeta(-s-k)$$

$$\sum_{n=1}^{\infty} \Lambda(n)\Lambda(n+2) = \frac{1}{(2\pi i)^2} \int_{c_1 - i\infty}^{c_1 + i\infty} \int_{c_2 - i\infty}^{c_2 + i\infty} F(s)F(t) \left[\sum_{k=0}^{\infty} \binom{t}{k} 2^{t-k} \zeta(-s-k) \right] ds \, dt$$

In this state, it is not possible to calculate the equation properly, so we will perform Fermi estimation. Here,We integrate over two vertical lines

$$s = c_1 + iT_1, \quad t = c_2 + iT_2$$

with fixed real $c_1, c_2 > 1$ and $T_1, T_2 \in (-\infty, \infty)$.

$$\frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |F(c_1 + iT_1)||F(c_2 + iT_2)| \Big| \Sigma(c_1 + iT_1, c_2 + iT_2) \Big| dT_1, dT_2$$

and

$\dfrac{\zeta'(s)}{\zeta(s)} = O(\log|T|))$

$\dfrac{\zeta''(s)}{\zeta(s)} = O((\log|T|)^2))$

$$\Sigma(s,t) = \sum_{k=0}^{\infty} \binom{t}{k} \cdot 2^{t-k} \cdot \zeta(-s-k)$$

A rough estimate of the absolute value of each term is

$$\left| \binom{t}{k} \right| \approx \frac{|t|^k}{k!},$$

$$\left|2^{t-k}\right| = 2^{\Re(t)} \cdot 2^{-k} = 2^{c_2} \cdot 2^{-k}.$$

For a fixed s,

$$|\zeta(-s-k)|$$

grows at most factorially with k, but is bounded by the binomial coefficient $k!$.

Thus, even when $|T_1|, |T_2|$ are large, the algorithm converges uniformly to a constant in $O(1)$ (depending on c_1, c_2). That is, there exists a $C > 0$ such that

$$\left|\Sigma(c_1 + iT_1, c_2 + iT_2)\right| \leq C$$

holds for any T_1, T_2.

$$|F(s)|, |F(t)|, |\Sigma| = O\big(\log |T_1|\big), O\big(\log |T_2|\big), O(1) = O\big(\log |T_1| \cdot \log |T_2|\big).$$

Therefore, the absolute value double integral is bounded from below by

$$\int_1^\infty \int_1^\infty \log T_1 \cdot \log T_2 \, dT_1 \, dT_2.$$

Here,

$$\int_1^M \log T \, dT = \big[T \log T - T\big]_1^M \sim M \log M \quad (M \to \infty),$$

so each single integral diverges linearly, and their product diverges even faster.

9. more stronger

To analyze $(n+2)^t$, we expand it using the binomial theorem:
I'm getting tired of trying to find a different form for the Λfunction, so I'll stick with the traditional $-\frac{\zeta'(s)}{\zeta(s)}$

$$(n+2)^t = \sum_{k=0}^{\infty} \binom{t}{k} 2^{t-k} n^k$$

For the summation:

$$\sum_{n=1}^{\infty} n^s (n+2)^t = \sum_{k=0}^{\infty} \binom{t}{k} 2^{t-k} \sum_{n=1}^{\infty} n^{s+k} = \sum_{k=0}^{\infty} \binom{t}{k} 2^{t-k} \zeta(-s-k)$$

since $\sum_{n=1}^{\infty} n^w = \zeta(-w)$ (with $w = s + k$).

Next, we calculate using redisues for the complex integral. Here:

$$\zeta(-s-k) \text{ has a simple pole at } -s-k = 1, \text{ i.e., } s = -k-1.$$

The residue at $s = -k - 1$ is -1 because $\text{Res}_{w=1}\zeta(w) = 1$.
Thus, for each k, the integral evaluates to:

$$\frac{1}{2\pi i} \int F(s)\zeta(-s-k)\, ds = -F(-k-1).$$

The sum then becomes:

$$\sum_{n=1}^{\infty} \Lambda(n)\Lambda(n+2) = \sum_{k=0}^{\infty} [-F(-k-1)] \cdot \left(\frac{1}{2\pi i} \int F(t) \binom{t}{k} 2^{t-k} dt \right).$$

Expanding J_k as follows:

$$J_k = \frac{1}{2\pi i} \int F(t) \binom{t}{k} 2^{t-k} dt = \frac{1}{2\pi i} \int \left(-\frac{\zeta'(t)}{\zeta(t)} \right) \binom{t}{k} 2^{t-k} dt.$$

Given that $-\frac{\zeta'(t)}{\zeta(t)}$ has a pole at $t = 1$, and $\binom{t}{k}$ simplifies at $t = 1$:
- For $k = 0$, $\binom{1}{0} = 1$.
- For $k = 1$, $\binom{1}{1} = 1$.
- For $k \geq 2$, $\binom{1}{k} = 0$.

Only $k = 0$ and $k = 1$ contribute:
- For $k = 0$: $-F(-1) \times (-2) = 2F(-1)$.
- For $k = 1$: $-F(-2) \times (-1) = F(-2)$.

However, $\zeta(-2) = 0$ (trivial zero), meaning $F(-2)$ diverges:
From these,$2F(-1) + F(-2) = diverges$.

10. Finally

The swapping of the order of summation in the second line of the calculation earlier is only valid if the series converges, which is unfortunately not the case here. Furthermore, Sec5 is formally promising, but somewhat crude, e.g., $\Lambda(3)$ and $\Lambda(5)$ are independent. In the final section, we treat twin primes as independent but dependent Dirichlet functions, ultimately leading to an explicit, consistent formula. Firstly, from the definition of $F(s)$: (Of course, we are only considering prime contributions here.) First,

$$F(s) = \sum_{p} \frac{(\log p)^2 \cdot p^{-s}}{(1 - p^{-s})^2}$$

can be expanded as a Dirichlet series:

$$F(s) = \sum_{p} (\log p)^2 \sum_{a=1}^{\infty} a \cdot p^{-as} = \sum_{n=1}^{\infty} a_n \cdot n^{-s},$$

where

$$a_n = \begin{cases} a \cdot (\log p)^2, & n = p^a, \\ 0, & \text{otherwise.} \end{cases}$$

Using the Mellin inversion formula:

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s)\, n^s \, ds = a_n,$$

and similarly,

$$\frac{1}{2\pi i} \int_{c'-i\infty}^{c'+i\infty} F(t)\, (n+2)^t \, dt = a_{n+2}.$$

Thus, the summation in question:

$$\sum_{n=1}^{\infty} \Big(\frac{1}{2\pi i} \int F(s)\, n^s \, ds \Big) \Big(\frac{1}{2\pi i} \int F(t)\, (n+2)^t \, dt \Big)$$

becomes:

$$\sum_{n=1}^{\infty} a_n \cdot a_{n+2}.$$

Since both a_n and a_{n+2} only have values at prime powers, we have:

$$\sum_{n=1}^{\infty} a_n a_{n+2} = \sum_{p^a,\, q^b} \Big[a_n = a \cdot (\log p)^2,\ a_{n+2} = b \cdot (\log q)^2 \Big]\, \delta_{p^a+2,\, q^b}.$$

Therefore,

$$\boxed{\sum_{n=1}^{\infty} \Big(\frac{1}{2\pi i} \int F(s)\, n^s \, ds \Big) \Big(\frac{1}{2\pi i} \int F(t)\, (n+2)^t \, dt \Big) = \sum_{p^a+2=q^b} a\, b\, (\log p)^2\, (\log q)^2}.$$

But,

$$\sum_{n=1}^{\infty} \left(\frac{1}{2\pi i} \int F(s) n^s ds \right) \left(\frac{1}{2\pi i} \int F(t) (n+2)^t dt \right) = \sum_{p^a+2=q^b} ab (\log p)^2 (\log q)^2 \quad (※)$$

Here, - $F(s) = \sum_{n=1}^{\infty} \frac{\Lambda(n) \log n}{n^s} = \frac{d^2}{ds^2} \log \zeta(s)$ - Using inverse Mellin transform:

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s) n^s ds = \Lambda(n) \log n$$

Thus, the left-hand side is:

$$\sum_{n=1}^{\infty}(\Lambda(n)\log n)\cdot(\Lambda(n+2)\log(n+2))$$

That is, actually,

$$\boxed{\sum_{n=1}^{\infty}(\Lambda(n)\log n)(\Lambda(n+2)\log(n+2)) = \sum_{p^a+2=q^b} ab(\log p)^2(\log q)^2}$$

$$\Lambda(n) = \begin{cases}\log p & (n = p^a)\\ 0 & \text{otherwise}\end{cases} \quad\Rightarrow\quad \Lambda(n)\log n = \begin{cases}\log p\cdot\log(p^a) = a(\log p)^2 & (n = p^a)\\ 0 & \text{otherwise}\end{cases}$$

So:

$$\Lambda(n)\log n = a(\log p)^2 \quad \text{for } n = p^a$$

Thus,

$$a(\log p)^2 = \Lambda(n)\log n$$

Similarly,

$$b(\log q)^2 = \Lambda(n+2)\log(n+2)$$

$$\sum_{p^a+2=q^b} ab(\log p)^2(\log q)^2 = \sum_{n:\, n=p^a,\, n+2=q^b}(\Lambda(n)\log n)\cdot(\Lambda(n+2)\log(n+2))$$

This is exactly the left-hand side:

$$\sum_{n=1}^{\infty}(\Lambda(n)\log n)(\Lambda(n+2)\log(n+2))$$

$$\Lambda(n)\Lambda(n+2) = \frac{(\Lambda(n)\log n)(\Lambda(n+2)\log(n+2))}{\log n\cdot\log(n+2)}$$

Thus,

$$\sum_{n=1}^{\infty}\Lambda(n)\Lambda(n+2) = \sum_{n=1}^{\infty}\frac{(\Lambda(n)\log n)(\Lambda(n+2)\log(n+2))}{\log n\cdot\log(n+2)}$$

$$= \sum_{p^a+2=q^b}\frac{ab(\log p)^2(\log q)^2}{\log(p^a)\cdot\log(q^b)} = \sum_{p^a+2=q^b}\frac{ab(\log p)^2(\log q)^2}{(a\log p)(b\log q)} = \sum_{p^a+2=q^b}(\log p)(\log q)$$

$$\boxed{\sum_{n=1}^{\infty} \Lambda(n)\Lambda(n+2) = \sum_{\substack{p,q \text{ is prime} \\ a,b \geq 1 \\ p^a+2=q^b}} (\log p)(\log q)}$$

Taking logarithms of both sides, we have

$$q^b = p^a + 2 \implies b \log q = \log(p^a + 2)$$

Now,

$$\log(p^a + 2) = \log\left(p^a\left(1 + \frac{2}{p^a}\right)\right) = a \log p + \log\left(1 + \frac{2}{p^a}\right)$$

Therefore,

$$|b \log q - a \log p| = \left|\log\left(1 + \frac{2}{p^a}\right)\right| < \frac{2}{p^a} \quad (\text{since } |\log(1+x)| < 2|x| \text{ for small } x > 0)$$

At this point, I was quite stuck. I tried considering the problem in terms of $\mathbb{Z}$-modules, but made no progress. While browsing Wikipedia for topics related to FLT, I happened to come across "Baker's theorem," which I would like to introduce here.

Theorem 1: Baker's Theorem (Simplified Version)

Let α_1, α_2 be algebraic numbers in $\overline{\mathbb{Q}}^{\times}$, and $b_1, b_2 \in \mathbb{Z}$, with $B = \max(|b_1|, |b_2|)$. Then there exists an effectively computable constant $C = C(\alpha_1, \alpha_2)$ such that

$$\Lambda = b_1 \log \alpha_1 + b_2 \log \alpha_2 \neq 0 \implies |\Lambda| > \exp(-C \log B)$$

That is,

$$|\Lambda| > B^{-C}$$

In our case: - $\alpha_1 = q$, $\alpha_2 = p$ - $b_1 = b$, $b_2 = -a$ - $\Lambda = b \log q - a \log p$

If $p \neq q$, then $\Lambda \neq 0$ (since $q^b = p^a$ is impossible by the uniqueness of prime factorization). If $p = q$, then $p^a + 2 = p^b \implies p^b - p^a = 2$, which can be checked for finitely many cases (e.g., $(2, 1, 2, 2)$).

Therefore, if $\Lambda \neq 0$,

$$|\Lambda| > \exp(-C \log B) = B^{-C}$$

where $C = C(p, q)$ depends on p and q. **Baker-type lower bound:**

$$|\Lambda| > B^{-C}$$

On the other hand, the upper bound:

$$|\Lambda| < \frac{2}{p^a}$$

Thus,

$$B^{-C} < \frac{2}{p^a} \implies p^a < 2B^C \implies a \log p < \log 2 + C \log B \implies a < \frac{\log 2 + C \log B}{\log p}$$

Similarly, for b: - $|\Lambda| = |b \log q - a \log p| < \frac{2}{p^a} \ll 1$ - Thus, $b \log q \approx a \log p$ - We also obtain an upper bound of the form $b < D \log B$.

Here, since $B = \max(a, b)$, the upper bound is of the order $\log B$.

In summary:

$$a < K \log B, \quad b < L \log B \quad \text{for some } K, L \implies B < M \log B$$

Since $B \to \infty$ implies $\frac{B}{\log B} \to \infty$, there exists some B_0 such that for all $B > B_0$,

$$B > M \log B,$$

which means there are only finitely many such B.

This mean disproof of the twin prime.

Conclusion and thoughts

From the above, it has been shown that there are an finite number of twin primes. I would be happy if a similar argument could solve the k-tuple conjecture and other number theory problems that introduced me to the world of prime numbers.Although the paper did some unconventional things, such as using pseudo-Euler product expansions, I think that the paper could have been completed into a much more reasonable one by including the residue theorem and divergence determination using quotient differentiation. However, I still have some doubts, because someone like me who doesn't even have a PhD can't believe that the contribution of only prime numbers to the von Mangoldt function would be like this. So instead of saying "falsification" or "proof," I'd like to say this time "support for the falsification." As an aside, there are a lot of fake papers out there, such as 'proof of the Riemann hypothesis.' in the world.I sincerely hope that I don't become one of them.
Anyway,I admired the mathematical skills of my first love and have been chasing after him for half a year now.Because without him I would never have come to love math. I think I did a good job these past three months. I'll study hard for the entrance exams.

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References and Software Used for Numerical Experiments

In this research, the Python programs for numerical experiments heavily relied on xAI (Grok 3) and deepseek. I gratefully acknowledge this support.

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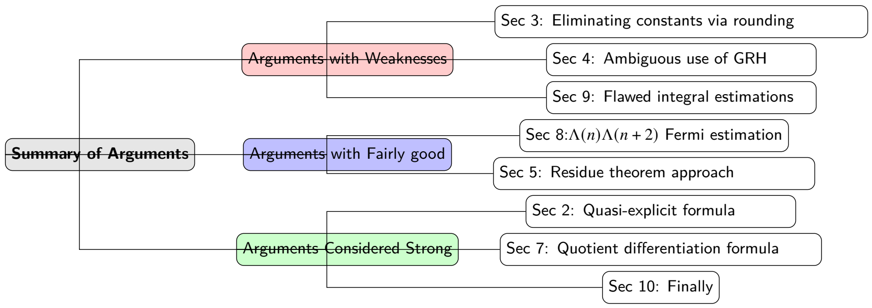
Errata and Summary of Arguments



10.0.1. memo

When introducing the pseudo-Euler product, $F(s) = \sum_p (\log p)^2 \frac{p^{-s}}{(1-p^{-s})^2}$, it matches the direct derivation $F(s) = \sum_p (\log p)^2 \frac{p^{-s}}{(1-p^{-s})^2}$. Thus, the pseudo-Euler product is classified as strong.