An Elementary Proof of the Goldbach Conjecture for $n \geq 8$

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Abstract

An elementary proof of the strong form of the Goldbach Conjecture is presented: every even number greater than 2 can be expressed as the sum of two prime numbers. The strategy is based on analyzing the possible pairs of odd numbers that sum up to $2n$ and applying a sieve based on the divisibility of primes less than or equal to $\sqrt{2n}$. It is shown that for $n \geq 8$, at least one of these pairs consists of two prime numbers.

Introduction

The Goldbach Conjecture states that every even number greater than two can be written as the sum of two prime numbers. Despite extensive computational verification up to very large limits, a general accepted proof has yet to be established. In this paper, we present an elementary approach that establishes the conjecture for all $n \in \mathbb{N}$ such that $n \geq 8$.

Construction of the pairs

Let $n \in \mathbb{N}$. We consider the even number $2n$. We generate $\left\lceil \frac{n}{2} \right\rceil$ pairs of odd numbers (r_i, s_i) such that $r_i + s_i = 2n$. For each i from 1 to $\left\lceil \frac{n}{2} \right\rceil$, the pairs are constructed with r_i increasing from 1 (or 3) and $s_i = 2n - r_i$ decreasing accordingly.

Elimination Criterion

We proceed to eliminate all pairs in which at least one of the two elements is divisible by a prime p such that $p \leq \sqrt{2n}$. That is, a pair (r_i, s_i) is eliminated if there exists a $p \leq \sqrt{2n}$ such that $p \mid r_i$ or $p \mid s_i$.

Validity of the sieve

Suppose that a pair (r, s) survives the sieve. Then neither r nor s is divisible by any prime $p \leq \sqrt{2n}$. If either r or s were composite, it would have a prime factor $q > \sqrt{2n}$. But then $r \geq q^2 > 2n$, which contradicts the fact that $r < 2n$. The same holds for s. Therefore, both r and s must be prime.

Number of surviving pairs

Each prime $p_j \leq \sqrt{2n}$ eliminates at most $\frac{2\lceil \frac{n}{2} \rceil}{p_j}$ pairs (one for each multiple in r_i and s_i). Thus, the fraction of pairs that survive the sieve for each p_j is at least $(1 - 2/p_j) = (p_j - 2)/p_j$.
Multiplying all the remaining fractions, we obtain the following.

$$\left\lceil \frac{n}{2} \right\rceil \cdot \prod_{j=1}^{k} \left(\frac{p_j - 2}{p_j} \right) = \left\lceil \frac{n}{2} \right\rceil \cdot \frac{1}{3} \cdot \frac{3}{5} \cdot \frac{5}{7} \cdot \frac{9}{11} \cdots \frac{p_k - 2}{p_k}$$

$$\geq \left\lceil \frac{n}{2} \right\rceil \cdot \frac{1}{3} \cdot \frac{3}{5} \cdot \frac{5}{7} \cdot \frac{7}{11} \cdots \frac{p_{k-1}}{p_k} \geq \frac{n}{2p_k} \geq \frac{n}{2\sqrt{2n}} = \frac{\sqrt{n}}{\sqrt{8}} \geq 1$$

for $n \geq 8$.
Therefore, there is at least one pair that was not eliminated and this pair consists of two prime numbers whose sum is $2n$.

Conclusion

There exists at least one pair (r, s) that was not eliminated, whose elements must be prime. Therefore, $2n = r + s$ is the sum of two primes for all $n \geq 8$.
This proves the Goldbach conjecture for these values.

Q.E.D.

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[1] This article was originally developed in Spanish and later translated for international dissemination.