Residue classes and stopping time of the 3n+1 problem

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Abstract

This paper presents an analysis of the stopping time of the 3n+1 problem based on the residue class of n.

3n + 1 problem (or conjecture)

In the $3\cdot n+1$ problem[1] it is possible to define the function $s: N \rightarrow N$:

$$s(n)=\begin{cases} 3\cdot n+1 & if \ n\equiv 1 (mod\, 2) \\ \frac{n}{2} & if \ n\equiv 0 (mod\, 2) \end{cases}$$

the sequence $s^k(n)$ for $k\in N$ obtained using the function $s(n)$ is as follows:

$$s^k(n)=\begin{cases} n & for \ k=0 \\ s(s^{k-1}(n)) & for \ k>0 \end{cases}$$

Stopping time

The $3\cdot n+1$ conjecture is equivalent to the conjecture that for each $n\in N, n>1$, there exists $k\in N$ such that $s^k(n)<n$. The least $k\in N$ such that $s^k(n)<n$ is called the stopping time of n [1] .

If n is even $n\equiv 0 (mod\, 2)$ then the stopping time $k=1$ and $s^1(n)=\frac{n}{2}$.

Let's analyze the case where n is odd. Let m be the number of odd terms in the first k terms of the $3\cdot n+1$ sequence, and d_i be the number of consecutive even terms immediately following the *i*-th odd term, then the next term $s^k(n)$ in he $3\cdot n+1$ sequence is[1] :

$$s^k(n)=\frac{3^m}{2^{k-m}}\cdot n+\sum_{i=1}^{m}\frac{3^{m-i}}{2^{d_i+\cdots+d_m}}$$

Note that for n odd $k-m=d_1+\cdots+d_m$.

Then

$$s^k(n)=\frac{3^m\cdot n+3^{m-1}+\sum_{i=2}^{m}3^{m-i}\cdot 2^{d_1+\cdots+d_{i-1}}}{2^{k-m}}=\frac{3^m\cdot n+r}{2^{k-m}}$$

where r depends on m and n with $r\geq 3^m-2^m$.

It is possible to observe that if $n \equiv 1 (mod\, 2^2)$ then $m=1$ and the stopping time is $k=3$

$$s^3(n) = \frac{3 \cdot n+1}{2^2}$$

so we have $r=1=3^1-2^1$ and $d_1=2$.

If we now consider the numbers $n \equiv 3 (mod\, 2^2)$ we have:

$$n=3+4 \cdot a$$

$$s^1(n)=3 \cdot n+1=10+12 \cdot a$$

$$s^2(n)=\frac{3 \cdot n+1}{2}=5+6 \cdot a=2+2 \cdot a+n$$ odd number

$$s^4(n)=\frac{3 \cdot \frac{3 \cdot n+1}{2}+1}{2}=2+a+2n$$

if a odd then $d_1=1$, $d_2=1$ and $m>2$ since $s^2(n), s^4(n)$ odd and $s^4(n)>n$

if a even then $a=2 \cdot b$

$$s^5(n)=\frac{3 \cdot \frac{3 \cdot n+1}{2}+1}{4}=1+b+n$$

if b odd then $d_1=1$, $d_2=2$ and $m>2$ since $s^2(n), s^5(n)$ odd and $s^5(n)>n$

if b even $b=2 \cdot c$ then $n \equiv 3 (mod\, 2^4)$

$$s^6(n)=\frac{3 \cdot \frac{3 \cdot n+1}{2}+1}{8}=c+\frac{(n+1)}{2}=\frac{3^2 \cdot n+5}{2^4}$$

then $k=6$, $d_1=1$, $d_2=3$ and $m=2$ with $r=5=3+2^1=3^2-2^2$.

As seen for numbers $n \equiv 7 (mod\, 2^3)$

$$s^4(n)=\frac{3 \cdot \frac{3 \cdot n+1}{2}+1}{2}=\frac{9 \cdot n+5}{4}$$ odd number and $d_1=1$ and $d_2=1$

$$s^6(n)=\frac{3 \cdot \frac{9 \cdot n+5}{4}+1}{2}=\frac{27 \cdot n+19}{8}$$

if $n \equiv 7 (mod\, 2^4)$ then $s^6(n)$ even and

$$s^7(n)=\frac{27 \cdot n+19}{16}$$

if $n \equiv 23 (mod\, 2^5)$ then $s^7(n)$ even and

$$s^8(n)=\frac{27\cdot n+19}{32}<n$$

then $k=8$, $d_1=1$, $d_2=1$, $d_3=3$ and $m=3$ from which

$$r=19=3^3-2^3=3^2+3\cdot 2^1+2^{(1+1)}=3\cdot(3+2^1)+2^{(1+1)}=3\cdot 5+2^2$$

Note that the numbers $n\equiv 15\left(mod\, 2^4\right)$ and $n\equiv 7\left(mod\, 2^5\right)$ remain to be analyzed.

For numbers $n\equiv 11\left(mod\, 2^4\right)$

$$s^5(n)=\frac{3\cdot\frac{3\cdot n+1}{2}+1}{4}=\frac{9\cdot n+5}{8}$$ odd number and $d_1=1$ and $d_2=2$

$$s^7(n)=\frac{3\cdot\frac{9\cdot n+5}{8}+1}{2}=\frac{27\cdot n+23}{16}$$

if $n\equiv 11\left(mod\, 2^5\right)$ then $s^7(n)$ even and

$$s^8(n)=\frac{3\cdot\frac{9\cdot n+5}{8}+1}{2}=\frac{27\cdot n+23}{32}<n$$

then $k=8$, $d_1=1$, $d_2=2$, $d_3=2$ and $m=3$ from which

$$r=23=3^2+3\cdot 2^1+2^{(1+2)}=3\cdot(3+2^1)+2^{(1+2)}=3\cdot 5+2^3$$

For numbers $n\equiv 27\left(mod\, 2^5\right)$ then $s^7(n)$ odd and $d_3=1$

$$s^7(n)=\frac{3\cdot\frac{9\cdot n+5}{8}+1}{2}=\frac{27\cdot n+23}{16}$$

$$s^9(n)=\frac{3\cdot\frac{27\cdot n+23}{16}+1}{2}=\frac{81\cdot n+85}{32}$$

if $n\equiv 59\left(mod\, 2^6\right)$ then $s^9(n)$ even

$$s^{10}(n)=\frac{81\cdot n+85}{64}$$

if $n\equiv 59\left(mod\, 2^7\right)$ then $s^{10}(n)$ even

$$s^{11}(n)=\frac{81\cdot n+85}{128}<n$$

then $k=11$, $d_1=1$, $d_2=2$, $d_3=1$, $d_4=3$ and $m=4$ from which

$$r=85=3^3+3^2\cdot 2^1+3\cdot 2^{(1+2)}+2^{(1+2+1)}=3\cdot\left(3^2+3\cdot 2^1+2^{(1+2)}\right)+2^{(1+2+1)}+2^{(1+2)}=3\cdot 23+2^4$$

Note that the numbers $n \equiv 27 \left(mod\, 2^6\right)$ and $n \equiv 123 \left(mod\, 2^7\right)$ remain to be analyzed.

For numbers $n \equiv 15 \left(mod\, 2^4\right)$

$$s^6(n) = \frac{3 \cdot \frac{9 \cdot n + 5}{4} + 1}{2} = \frac{27 \cdot n + 19}{8}$$ odd and $d_3 = 1$

$$s^8(n) = \frac{3 \cdot \frac{27 \cdot n + 19}{8} + 1}{2} = \frac{81 \cdot n + 65}{16}$$

if $n \equiv 15 \left(mod\, 2^5\right)$ then $s^8(n)$ even and

$$s^9(n) = \frac{81 \cdot n + 65}{32}$$

if $n \equiv 15 \left(mod\, 2^6\right)$ then $s^9(n)$ even and

$$s^{10}(n) = \frac{81 \cdot n + 65}{64}$$

if $n \equiv 15 \left(mod\, 2^7\right)$ then $s^{10}(n)$ even and

$$s^{11}(n) = \frac{81 \cdot n + 65}{128} < n$$

then $k = 11$, $d_1 = 1$, $d_2 = 1$, $d_3 = 1$, $d_4 = 4$ and $m = 4$ from which

$$r = 65 = 3^4 - 2^4 = 3^3 + 3^2 \cdot 2^1 + 3 \cdot 2^{(1+1)} + 2^{(1+1+1)} = 3 \cdot \left(3^2 + 3 \cdot 2^1 + 2^{(1+1)}\right) + 2^{(1+1+1)} = 3 \cdot 19 + 2^3$$

For numbers $n \equiv 7 \left(mod\, 2^5\right)$

$$s^7(n) = \frac{27 \cdot n + 19}{16}$$ odd and $d_3 = 2$

$$s^9(n) = \frac{3 \cdot \frac{27 \cdot n + 19}{16} + 1}{2} = \frac{81 \cdot n + 73}{32}$$

if $n \equiv 7 \left(mod\, 2^6\right)$ then $s^9(n)$ even and

$$s^{10}(n) = \frac{81 \cdot n + 73}{64} < n$$

if $n \equiv 7 \left(mod\, 2^7\right)$ then $s^{10}(n)$ even and

$$s^{11}(n) = \frac{81 \cdot n + 73}{128} < n$$

then $k = 11$, $d_1 = 1$, $d_2 = 1$, $d_3 = 2$, $d_4 = 3$ and $m = 4$ from which

$$r = 73 = 3^3 + 3^2 \cdot 2^1 + 3 \cdot 2^{(1+1)} + 2^{(1+1+2)} = 3 \cdot \left(3^2 + 3 \cdot 2^1 + 2^{(1+1)}\right) + 2^{(1+1+2)} = 3 \cdot 19 + 2^4$$

Note that the numbers $n \equiv 39 \left(mod\, 2^6\right)$, $n \equiv 71 \left(mod\, 2^7\right)$, $n \equiv 31 \left(mod\, 2^5\right)$, $n \equiv 47 \left(mod\, 2^6\right)$ and $n \equiv 79 \left(mod\, 2^7\right)$ remain to be analyzed.

It can be observed that in all the cases examined $k - m = \lfloor 1 + m \cdot \log_2(3) \rfloor$.

By continuing with this procedure it is easy to verify the following results:

m	k	$n\left(mod\, 2^{k-m}\right)$	d_m	r
1	3	1	2	1
2	6	3	3	$5 = 3 + 2^1$
3	8	11	2	$23 = 3^2 + 3 \cdot 2^1 + 2^{(1+2)}$
		23	3	$19 = 3^2 + 3 \cdot 2^1 + 2^{(1+1)}$
4	11	7	3	$73 = 3^3 + 3^2 \cdot 2^1 + 3 \cdot 2^{(1+1)} + 2^{(1+1+2)}$
		15	4	$65 = 3^3 + 3^2 \cdot 2^1 + 3 \cdot 2^{(1+1)} + 2^{(1+1+1)}$
		59	3	$85 = 3^3 + 3^2 \cdot 2^1 + 3 \cdot 2^{(1+2)} + 2^{(1+2+1)}$
5	13	39	3	$251 = 3^4 + 3^3 \cdot 2^1 + 3^2 \cdot 2^{(1+1)} + 3 \cdot 2^{(1+1+2)} + 2^{(1+1+2+1)}$
		79	2	$259 = 3^4 + 3^3 \cdot 2^1 + 3^2 \cdot 2^{(1+1)} + 3 \cdot 2^{(1+1+1)} + 2^{(1+1+1+3)}$
		95	4	$211 = 3^4 + 3^3 \cdot 2^1 + 3^2 \cdot 2^{(1+1)} + 3 \cdot 2^{(1+1+1)} + 2^{(1+1+1+1)}$
		123	2	$319 = 3^4 + 3^3 \cdot 2^1 + 3^2 \cdot 2^{(1+2)} + 3 \cdot 2^{(1+2+1)} + 2^{(1+2+1+2)}$
		175	3	$227 = 3^4 + 3^3 \cdot 2^1 + 3^2 \cdot 2^{(1+1)} + 3 \cdot 2^{(1+1+1)} + 2^{(1+1+1+2)}$
		199	2	$283 = 3^4 + 3^3 \cdot 2^1 + 3^2 \cdot 2^{(1+1)} + 3 \cdot 2^{(1+1+2)} + 2^{(1+1+2+2)}$
		219	3	$287 = 3^4 + 3^3 \cdot 2^1 + 3^2 \cdot 2^{(1+2)} + 3 \cdot 2^{(1+2+1)} + 2^{(1+2+1+1)}$
6	16	287	4	$697 = 3^5 + 3^4 \cdot 2^1 + 3^3 \cdot 2^{(1+1)} + 3^2 \cdot 2^{(1+1+1)} + 3 \cdot 2^{(1+1+1+1)} + 2^{(1+1+1+1+2)}$
		347	3	$989 = 3^5 + 3^4 \cdot 2^1 + 3^3 \cdot 2^{(1+2)} + 3^2 \cdot 2^{(1+2+1)} + 3 \cdot 2^{(1+2+1+1)} + 2^{(1+2+1+1+2)}$
		367	4	$745 = 3^5 + 3^4 \cdot 2^1 + 3^3 \cdot 2^{(1+1)} + 3^2 \cdot 2^{(1+1+1)} + 3 \cdot 2^{(1+1+1+2)} + 2^{(1+1+1+2+1)}$
		423	3	$881 = 3^5 + 3^4 \cdot 2^1 + 3^3 \cdot 2^{(1+1)} + 3^2 \cdot 2^{(1+1+2)} + 3 \cdot 2^{(1+1+2+1)} + 2^{(1+1+2+1+2)}$
		507	3	$1085 = 3^5 + 3^4 \cdot 2^1 + 3^3 \cdot 2^{(1+2)} + 3^2 \cdot 2^{(1+2+1)} + 3 \cdot 2^{(1+2+1+2)} + 2^{(1+2+1+2+1)}$
		575	5	$665 = 3^5 + 3^4 \cdot 2^1 + 3^3 \cdot 2^{(1+1)} + 3^2 \cdot 2^{(1+1+1)} + 3 \cdot 2^{(1+1+1+1)} + 2^{(1+1+1+1+1)}$
		583	3	$977 = 3^5 + 3^4 \cdot 2^1 + 3^3 \cdot 2^{(1+1)} + 3^2 \cdot 2^{(1+1+2)} + 3 \cdot 2^{(1+1+2+2)} + 2^{(1+1+2+2+1)}$
		735	3	$761 = 3^5 + 3^4 \cdot 2^1 + 3^3 \cdot 2^{(1+1)} + 3^2 \cdot 2^{(1+1+1)} + 3 \cdot 2^{(1+1+1+1)} + 2^{(1+1+1+1+3)}$
		815	3	$809 = 3^5 + 3^4 \cdot 2^1 + 3^3 \cdot 2^{(1+1)} + 3^2 \cdot 2^{(1+1+1)} + 3 \cdot 2^{(1+1+1+2)} + 2^{(1+1+1+2+2)}$
		923	4	$925 = 3^5 + 3^4 \cdot 2^1 + 3^3 \cdot 2^{(1+2)} + 3^2 \cdot 2^{(1+2+1)} + 3 \cdot 2^{(1+2+1+1)} + 2^{(1+2+1+1+1)}$
		975	3	$905 = 3^5 + 3^4 \cdot 2^1 + 3^3 \cdot 2^{(1+1)} + 3^2 \cdot 2^{(1+1+1)} + 3 \cdot 2^{(1+1+1+3)} + 2^{(1+1+1+3+1)}$
		999	4	$817 = 3^5 + 3^4 \cdot 2^1 + 3^3 \cdot 2^{(1+1)} + 3^2 \cdot 2^{(1+1+2)} + 3 \cdot 2^{(1+1+2+1)} + 2^{(1+1+2+1+1)}$

As said $k - m = d_1 + \cdots + d_m$ then $d_m = k - m - \left(d_1 + \cdots + d_{m-1}\right)$ and if for a certain value of m if we find a value of r , which we indicate as r_m^i which depends on the values of $d_1^i, \cdots, d_{m-1}^i$, then from this value for $m+1$ we can obtain $r_{m+1}^j = 3 * r_m^i + 2^{\left(d_1 + \cdots + d_m^j\right)}$ with $1 \le d_m^j < d_m^i$.

Below is the algorithm code to generate the residue classes for $m \geq 1$:

```
get_dim(m)={my(log2_3=log(3)/log(2),X1=matrix(2,floor(1+m*log2_3)-m), d=0,dim=1);X1[1,1]=0;
if(m>=3,for(x=3,m,y=floor(1+(x-1)*log2_3);
for(i=1,y-x,for(j=i,y-x,if(i==1,X1[1+(d+1)%2,j]=1);X1[1+(d+1)%2,j]=X1[1+(d+1)%2,j]+X1[d+1,i]));
d=(d+1)%2);for(i=1,floor(1+m*log2_3)-m,dim=dim+X1[d+1,i]));dim;}
{m_max=9;dim_max=get_dim(m_max);N=vector(dim_max);R=matrix(2,dim_max);Dm=matrix(2,dim_max);
nr=0;m=1;log2_3=log(3)/log(2);kmm=floor(1+m*log2_3);c=1;N[1]=1;R[1,1]=1;Dm[1,1]=2;
print1("m = ",m,"    -    stopping time: ",kmm+m,"\nif n == ",N[1]," (mod 2^",kmm,")\ntotal residue classes:
",c,"\n\n");
for(m=1,m_max-1, kmm=floor(1+m*log2_3);mp1=m+1;kmmp1=floor(1+mp1*log2_3);
print1("m = ",mp1,"    -    stopping time: ",kmmp1+mp1,"\nif n == ");c1=0;
for(i=1,c,sdmm1=kmm-Dm[1+nr,i];for(j=1,Dm[1+nr,i]-1,rj=3*R[1+nr,i]+2^(sdmm1+j); c1=c1+1;
N[c1]=((2^kmmp1-rj)*lift(Mod(1/3^mp1,2^kmmp1)))%2^kmmp1;R[1+(1+nr)%2,c1]=rj;Dm[1+(1+nr)
%2,c1]=kmmp1-(sdmm1+j))); c=c1;N1=vector(c);N1=vecsort(N[1..c]);for(i1=1,c-1,print1(N1[i1],", "));
print1(N1[c]);nr=(1+nr)%2;print(" (mod 2^",kmmp1,")\ntotal residue classes:  ",c,"\n\n"))}
```

PARI/GP code of the algorithm

References

[1] L.E. Garner, On the Collatz 3n + 1 algorithm(2nd ed.), *Proc. Amer. Math. Soc., 82 (1981), pp. 19-22*