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Fermat’s Theorem

(full statement)

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Introduction:

We undertake, in this article, the celebrated Fermat’s theorem, with the ambition of presenting a novel and original proof.

We will rely primarily on two implementations:

1 - One of them is widely known, namely the one that allows for the generation of primitive Pythagorean triples.:

$$\left(n^2 - m^2,\ 2\,nm,\ n^2 + m^2\right)$$

which yields:

$$4\,n^2\,m^2 = \left(n^2 + m^2\right)^2 - \left(n^2 - m^2\right)^2$$

Example : $(3, 4, 5) \Longrightarrow 3^2 + 4^2 = 5^2$

which yields :

$$3 = 2^2 - 1^2$$
$$4 = 2 * 2 * 1$$
$$5 = 2^2 + 1^2$$

Note: We will later present in this article another formulation, admittedly less elegant, but possessing highly useful properties.

2 - The second is a novel function and requires close attention from the reader, because despite its apparent simplicity, it possesses numerous properties essential to the problem under investigation:

- For odd integers, one may write:

$$\mathbf{n = 2m+1 \Longrightarrow f(n) = \frac{5\,n^2-29}{4}}$$

- And for the even integers, it will be:

$$\mathbf{n = 2m \Longrightarrow f(n) = \frac{5\,n^2}{4}}$$

Example :

$$3 \Longrightarrow f(n) = \frac{5 * 3^2 - 29}{4} = 4$$
$$5 \Longrightarrow f(n) = \frac{5 * 5^2 - 29}{4} = 24$$
$$4 \Longrightarrow f(n) = \frac{5 * 4^2}{4} = 20$$

Explanation:

If we have :

$$\mathbf{3^2 + 4^2 = 5^2}$$

then we will obtain:

$$\mathbf{4 + 20 = 24}$$

Let us note the perfect bijection between the set of integers in Pythagorean triples and the set constituted by their images under **f(n)**. Let us note that in the set consisting of these images, only the two elementary operations—addition and subtraction—can be utilized. Furthermore, these can only be applied in a restricted manner:

- Subtraction is limited to the integer images of odd integers; the smallest image is subtracted from the largest to yield the integer image corresponding to the even integer.

Example : **24 - 20** = **4**

- L'*addition* ne peut se faire qu'entre le plus petit et le pair.

Example : **20 + 4** = **24**

3 - We shall, at first, restrict our study to primitive triplets; composite triplets will be addressed, as a consequence, at the end of this article..

$$b^2 - a^2 == c^2 \implies (kb)^2 - (ka)^2 == (kc)^2$$

I – Pythagorean Triples Put to the Test of f(n)

Odd integers will be transformed as follows :

{{3 , 4} , {5 , 24} , {7, 54} , {9 , 94} , {11 , 144} , {13 , 204} , {15 , 274} , {17 , 354} , {19 , 444} , {21 , 544}}

Even integers will be transformed as follows :

{{2 , 5} , {4 , 20} , {6 , 45} , {8 , 80} , {10 , 125} , {12 , 180} , {14 , 245} , {16 , 320} , {18 , 405} , {20,500}}

First critical observation:

- The images under **f (n)** of the odd integers always terminate in **4**. Therefore, their difference will always yield an even integer ending in **0**.
- Now, the images under **f(n)** of even integers end either with **0** or with **5**. We must therefore exclude those whose images under **f(n)** end with **5**.

Significant consequence:

:

Only even integers that are multiples of 4 can be elements of Pythagorean triples.

The following two tables illustrate these properties

Let the Pythagorean triple be: :

(c , a , b) , such as : $b^2 - a^2 == c^2$

With the transformation function **f(n)**, we will write:

$$f(c) = \frac{5c^2 - 29}{4}$$

$$f(b) = \frac{5b^2 - 29}{4}$$

$$f(a) = \frac{5a^2}{4}$$

A = f (a) ; B = f (b) ; F = f (c)

```
Table[{c, a, b}, {m, 0, 5}, {k, 1, 5}]
table
Table[{F, A, B}, {m, 0, 5}, {k, 1, 5}]
table
```

```
{{{3, 4, 5}, {15, 8, 17}, {35, 12, 37}, {63, 16, 65}, {99, 20, 101}},
 {{5, 12, 13}, {21, 20, 29}, {45, 28, 53}, {77, 36, 85}, {117, 44, 125}},
 {{7, 24, 25}, {27, 36, 45}, {55, 48, 73}, {91, 60, 109}, {135, 72, 153}},
 {{9, 40, 41}, {33, 56, 65}, {65, 72, 97}, {105, 88, 137}, {153, 104, 185}},
 {{11, 60, 61}, {39, 80, 89}, {75, 100, 125}, {119, 120, 169}, {171, 140, 221}},
 {{13, 84, 85}, {45, 108, 117}, {85, 132, 157}, {133, 156, 205}, {189, 180, 261}}}
```

```
{{{4, 20, 24}, {274, 80, 354}, {1524, 180, 1704}, {4954, 320, 5274}, {12 244, 500, 12 744}},
 {{24, 180, 204}, {544, 500, 1044}, {2524, 980, 3504}, {7404, 1620, 9024}, {17 104, 2420, 19 524}},
 {{54, 720, 774}, {904, 1620, 2524}, {3774, 2880, 6654}, {10 344, 4500, 14 844},
  {22 774, 6480, 29 254}}, {{94, 2000, 2094}, {1354, 3920, 5274},
  {5274, 6480, 11 754}, {13 774, 9680, 23 454}, {29 254, 13 520, 42 774}},
 {{144, 4500, 4644}, {1894, 8000, 9894}, {7024, 12 500, 19 524}, {17 694, 18 000, 35 694},
  {36 544, 24 500, 61 044}}, {{204, 8820, 9024}, {2524, 14 580, 17 104},
  {9024, 21 780, 30 804}, {22 104, 30 420, 52 524}, {44 644, 40 500, 85 144}}}
```

And first, let us emphasize this decisive property of Pythagorean triplets: their image under the function **f(n)** always yields these endings (in one direction or the other):

(274 , 80 , 354) ou (4954 , 320 , 5274)

(1524 , 180 , 1704) ou (7404 , 1620 , 9024)

(12244 , 500 , 12744)

(198994 , 2000 , 200994)

Demonstration :

It suffices to construct a table of all possible combinations between the terminal digits of the integers **n** and **m**, since these two integers establish the necessary condition for any Pythagorean triple, as follows:

$$4\,n^2\,m^2 = \left(n^2 + m^2\right)^2 - \left(n^2 - m^2\right)^2$$

NB: let us recall that n and m must necessarily be one even and the other odd so that their sum yields an odd integer

Example :

- Termination of **n = 2** and termination of **m = 1** will yield us, for example:

c = **3** ⟹ 4 - 1

and:

b = **5** ⟹ 4 + 1

Which yields the following triplet: **(3 , 4 , 5)**,

- Termination of **n = 3** and termination of **m = 2** will yield, for example:

c = 5 ⟹ 9 - 4

and

b = **13** ⟹ 9 + 4

Which yields the following triplet :

(5 , 12 , 13)

- *Let us generalize:*

n is odd

m is even
Tables of Endings of **n** and of **m :**

n		n^2
1	⟹	1
3	⟹	9
5	⟹	5
7	⟹	9
9	⟹	1

m		m^2
0	⟹	0
2	⟹	4
4	⟹	6
6	⟹	6
8	⟹	4

Which yields:

$n^2 + m^2$	$n^2 - m^2$
1 + 0 ⟹ 1	1 - 0 ⟹ 1
1 + 4 ⟹ 5	1 - 4 ⟹ 11 - 4 = 7 or 21 - 4 = 17 ⟹ 7 ; etc.
1 + 6 ⟹ 7	11 - 6 ⟹ 5
9 + 0 ⟹ 9	9 - 0 ⟹ 1
9 + 4 ⟹ 3	9 - 4 ⟹ 5
9 + 6 ⟹ 5	9 - 6 ⟹ 3
5 + 0 ⟹ 5	5 - 0 ⟹ 5
5 + 4 ⟹ 9	5 - 4 ⟹ 1
5 + 6 ⟹ 1	5 - 6 ⟹ 15 - 6 = 9 ou bien 25 - 6 = 19 ⟹ 9 ; etc.

Endings of Pythagorean triples:

Let (**c , a , b**) be a Pythagorean triple with **a** as the even integer

The only possible units digits for c and b are the following, in either order:

(5, 7) ; (7, 5) ; (9, 1) ; (1, 9) ; (3, 5) ; (5, 3) ;

and **(1, 1) ; (5, 5)**

As illustrated by the following examples of Pythagorean triples:

(**3** , 4 , **5**) and (**5**, 12 , 1**3**) ; etc.

or : (**7** , 24 , 2**5**) ; and (3**5** , 12 , 3**7**) ; etc.

NB: Let us first exclude the Pythagorean triples generated by the pair (5, 5), as they are merely scalar multiples of arbitrary other Pythagorean triples.

Hence the table above :

(274 , 80 , 354)   or   (4954 , 320 , 5274)

(1524 , 180 , 1704)   or   (7404 , 1620 , 9024)

(12244 , 500 , 12744)

(198994 , 2000 , 200994)

II – The Fermat’s theorem

A - Puissance 4

As stated in the introduction to this article, we assume that the impossibility of Pythagorean triples raised to the power **4** has already been established by others.

Thus, for any natural numbers **a**, **b**, and **c**, it will always hold that:

$$a^4 \neq c^4 - b^4$$

However, we will examine the fourth power through our function **f(n)**, in order to draw conclusions that will assist us in demonstrating the impossibility for exponents greater than **4**.

Let us proceed as follows:

$$(c\,,\ a\ ,b\,) \implies a^4 == ?\ c^4 - b^4 \implies (c^2)^2 == ?\ (a^2)^2 - (b^2)^2$$

These “triplets” would therefore exhibit the same structure as Pythagorean triplets raised to the second power.

1 - Let us examine their transformation via the function f(n):

$$f\ (c) = \frac{5\,c^4 - 29}{4} = \frac{5\,(c^2)^2 - 29}{4}$$

$$f\ (b) = \frac{5\,b^4 - 29}{4} = \frac{5\,(b^2)^2 - 29}{4}$$

$$f\ (a) = \frac{5\,a^4}{4} = \frac{5\,(a^2)^2}{4}$$

Which yields the following excerpt:

```
{{{94, 320, 774}, {63274, 5120, 104394}, {18 757 74, 25 920, 23 426 94},
  {120 074 494, 200 000, 130 075 494}},
 {{774, 25 920, 35 694}, {243094, 200 000, 8840 94}, {5 125 774, 768 320,
  {43 941 294, 2 099 520, 65 250 774}, {234 235 894, 4 685 120, 305 175 774}
 {{2994, 414 720, 488 274}, {664 294, 2 099 520, 5 125 774}, {11 438 274, 6
  {857 186 94, 16 200 000, 1 764 476 94}, {415 188 274, 33 592 320, 684 976 59
 {{8194, 3 200 000, 3 532 194}, {1 482 394, 12 293 120, 22 313 274}, {22 313
  {151 938 274, 74 961 920, 440 344 194}, {684 976 594, 146 232 320, 1 464 1
```

Let us first note that in the vast majority of cases, these triplets exhibit this configuration (in one orientation or the other):

10k + 94

and **10k + 74**

In the same manner as these two cases:

(63274, 5120, 104394)

(19691194 , 81920 , 22313274)

However, no Pythagorean triple yields such a configuration, as demonstrated above (see illustration table).

On the other hand, this problematic situation is encountered here and there, since it also appears in higher-power triples **2 :**

(12007449 4 , 200000 , 130075494)

This is explained by the interplay of terminations and powers.

Terminaisons (c , a)	Terminaisons (c^2 , b^2)	Terminaisons (c^4 , b^4)
(5, 7)	(5, 9)	(5, 1)
(7, 5)	(9, 5)	(1, 5)
(9, 1)	(1, 1)	(1, 1)
(1, 9)	(1, 1)	(1, 1)
(3, 5)	(9, 5)	(1, 5)
(5, 3)	(5, 9)	(5, 1)
(1, 1)	(1, 1)	(1, 1)

In the end:

- For exponents of 2, only two pairs are found, in each direction.

(5, 9)
(1, 1)

- For exponents of 4, only two pairs are found, in each direction

4 fois (5, 1)
et 3 fois (1, 1)

It is therefore necessary to exclude the pairs (5, 1) and (1, 5), as they are not among the pairs constituting Pythagorean triples.

However, these are precisely the pairs that generated the terminations of the images (74, 94) and (94, 74), as shown in the following tables:

- **For the termination in 5**

```
n = 5 + 10 k ;
A = (5 n^4 - 29) / 4;
Table[{n, A}, {k, 0, 15}]
```

table

```
{{5, 774}, {15, 63 274}, {25, 488 274}, {35, 1 875 774},
 {45, 5 125 774}, {55, 11 438 274}, {65, 22 313 274},
 {75, 39 550 774}, {85, 65 250 774}, {95, 101 813 274},
 {105, 151 938 274}, {115, 218 625 774}, {125, 305 175 774},
 {135, 415 188 274}, {145, 552 563 274}, {155, 721 500 774}}
```

- **For the termination in 1**

```
n = 1 + 10 k;
A = (5 n^4 - 29) / 4;
Table[{n, A}, {k, 0, 15}]
```

table

```
{{1, -6}, {11, 18 294}, {21, 243 094}, {31, 1 154 394},
 {41, 3 532 194}, {51, 8 456 494}, {61, 17 307 294},
 {71, 31 764 594}, {81, 53 808 394}, {91, 85 718 694},
 {101, 130 075 494}, {111, 189 758 794}, {121, 267 948 594},
 {131, 368 124 894}, {141, 494 067 694}, {151, 649 856 994}}
```

However, the triplets whose residues are (1, 1) yield, when raised to the fourth power, the residues of their images under **f(n)** as:

(94 , 94)

That which is "compatible" with certain Pythagorean triples raised to the second power. That being said, in degree **2**, it is most often encountered as the image under f(n) of the following pairs:

(1 , 9); (9 , 1) ; (9 , 9)

Nevertheless, there are indeed cases where it is also found among the images of the pair **(1, 1)**. It is precisely this exceptional situation that we shall elucidate, in order to demonstrate that it is impossible for both odd terms in these triplets to simultaneously be perfect squares, and thus to potentially yield (subject to the parity constraint) a Pythagorean triple elevated to the fourth power.
To this end, we shall employ the second formula discussed at the outset.

2 - General Expression of the Triplet Formula (a, b , c) :

$$a = (1 + 2k) * (1 + 2k + 2m)$$
$$b = 2m * (1 + 2k + m)$$
$$c = (1 + 2k) * (1 + 2k + 2m) + 2m^2$$

We obtain exactly the same primitive triples as with the well-known formula cited above.

Note :

Despite its apparent complexity, it is very straightforward to comprehend.

Explanation of its genesis:

Step 1:

- Let us consider an example that will serve to illustrate the following steps of our demonstration.

Let the triplet be:

(11 , 60 , 61)

61 - 11 = 50

We can write:

$$50 = (11 - 1)*\left(\frac{11 - 1}{2} \right)$$

- An other example :

Let the triplet be:

(7 , 24 , 25)

25 - 7 = 18

We can write:

$$18 = (7 - 1)*\left(\frac{7 - 1}{2} \right)$$

Step 2 :

Let us now consider the triplets generated from the root triplet.

(11 , 60 , 61)

We'll have :

(11 , 60 , 61) ; (39 , 80, 89) ; (75 , 100 , 125) ; (119 , 120 , 169) ; (171 , 140 , 221) ; etc.

- Note 1 :

All exhibit an identical difference (**50**) between the two odd integers that constitute them.

- Note 2 :

The initial elements of these triplets are written as follows:

11 = 1 * 11

39 = 3 * 13

75 = 5 * 15

etc.

The third terms are obtained by adding **50** to the first terms.

11 + 50 = 61

13 + 50 = 89

75 + 50 = 125

etc.

3 - Final Demonstration

Although the proof has long since been established, we do not refrain from presenting this new approach, if only because it is rendered entirely straightforward by this new formulation. While this demonstration can be conducted for all triplets, we have seen, thanks to the function f(n), that only triplets terminating in 1 require this further clarification.

Let us suppose that we are given a Pythagorean triplet in the form:

$$(b = 1 + 10k \ , \ a \ , \ c = 1 + 10k + 50k^2)$$

And suppose that **b** is a perfect square. And suppose that **c** is also a perfect square (it could be shown that this is impossible, but let us proceed with our reasoning).

In this case, we must be able to write:

$$a = \sqrt{c^2 - b^2}$$

Which gives us:

$$a = 10 * k * (1 + 5k)$$

No integer solution is possible, except for the case:

$$k = 0$$

Which gives us: $(b = 1 \ , \ 0 \ , \ c = 1)$

Which is ruled out by hypothesis.

Vhus, this entire laborious - yet straightforward - reasoning demonstrates that it is impossible to have Pythagorean triples in the fourth power.

2 - Study of Triplets with Even Powers Greater than 2

Here again, the transformation function **f(n)** demonstrates that all powers of the form **4*s** yield a scenario identical in every respect to that of the fourth power. This is understood because these exponents have exactly the same endings.

Following the model of this example for the eighth power, in comparison with the power **4**

```
F1 = (5*c^4 - 29) / 4;
B1 = (5*b^4 - 29) / 4;
A1 = (5*a^4) / 4;

F2 = (5*c^8 - 29) / 4;
B2 = (5*b^8 - 29) / 4;
A2 = (5*a^8) / 4;

Table[{F1, A1, B1}, {m, 0, 5}, {k, 1, 3}]
table

Table[{F2, A2, B2}, {m, 0, 3}, {k, 1, 3}]
table

9]= {{{94, 320, 774}, {63 274, 5120, 104 394}, {1 875 774, 25 920, 2 342 694}},
  {{774, 25 920, 35 694}, {243 094, 200 000, 884 094}, {5 125 774, 768 320, 9 863 094}},
  {{2994, 414 720, 488 274}, {664 294, 2 099 520, 5 125 774}, {11 438 274, 6 635 520, 35 497 794}},
  {{8194, 3 200 000, 3 532 194}, {1 482 394, 12 293 120, 22 313 274}, {22 313 274, 33 592 320, 110 661 594}},
  {{18 294, 16 200 000, 17 307 294}, {2 891 794, 51 200 000, 78 427 794}, {39 550 774, 125 000 000, 305 175 774}},
  {{35 694, 62 233 920, 65 250 774}, {5 125 774, 170 061 120, 234 235 894}, {65 250 774, 379 494 720, 759 466 494}}}

0]= {{{8194, 81 920, 488 274}, {3 203 613 274, 20 971 520, 8 719 696 794}, {2 814 844 238 274, 537 477 120, 4 390 599 317 394}},
  {{488 274, 537 477 120, 1 019 663 394}, {47 278 574 194, 32 000 000 000, 625 308 016 194},
   {21 018 906 738 274, 472 252 497 920, 77 824 613 014 194}}, {{7 205 994, 137 594 142 720, 190 734 863 274},
   {353 036 920 594, 3 526 387 384 320, 21 018 906 738 274}, {104 667 422 363 274, 35 224 100 536 320, 1 008 075 114 867 594}},
  {{53 808 394, 8 192 000 000 000, 9 981 156 536 394}, {1 758 010 772 794, 120 896 639 467 520, 398 306 016 113 274},
```

The proof is therefore identical to that for exponent 4, since for any exponent **4*p**, we can write : $(n^{p})^{4}$, and thus must necessarily lead to the same outcome:

And thus it is impossible to have Pythagorean triples raised to that power 4*s

C - *study of Pythagorean triples of degree 3*

Is it possible to have a triplet **(b, a, c)** such that:

$$a^3 \; = ? \; c^3 - b^3$$

Let us examine the following table, which provides the images under the function **f (n)** :

```
b = 2*r + 1;
a = 2*s;
Table[{b, (5*b^3 - 29) / 2} , {r, 1, 10}]
table
Table[{a, (5*a^3) / 2}, {s, 1, 20}]
table
```

Out[156]= {{3, 53}, {5, 298}, {7, 843}, {9, 1808}, {11, 3313}, {13, 5478}, {15, 8423}, {17, 12 268}, {19, 17 133}, {21, 23 138}}

Out[157]= {{2, 20}, {4, 160}, {6, 540}, {8, 1280}, {10, 2500}, {12, 4320}, {14, 6860}, {16, 10 240}, {18, 14 580}, {20, 20 000}, {22, 26 620}, {24, 34 560}, {26, 43 940}, {28, 54 880}, {30, 67 500}, {32, 81 920}, {34, 98 260}, {36, 116 640}, {38, 137 180}, {40, 160 000}}

Let us first note the following: for any even integer b, the following always holds:

f (b) = 10*t

It would therefore be necessary for the images **f(b)** and **f(c)** to both be:

- Either in the form of: **10*r + 3**
- Either in the form of : **10*s + 8**

This enables us to deduce two interleaved sequences, thereby establishing that all odd integers necessarily belong to one or the other.

Suite n°1 ⟹ f (n) = 3 + 4* k

Suite n°2 ⟹ f (n) = 5 + 4* k

Our triplets must necessarily be represented as follows:

(3 + 4* k , 10*t , 3 + 4* k1)

(5 + 4* k) , 10*t , 5 + 4* k1)

We shall now demonstrate that, within such a configuration, it is impossible for any triplet to yield

$$a^3 \; = ?? \; c^3 - b^3$$

Warning:

We have emphasized that the images under (f(n)) of all even numbers raised to the power **3** are expressed as **10*t,** and now let us add this very important particularity for what follows:

All these images 10*t are multiples of 20 $\Longrightarrow$ which now yields: **10*t**

It is therefore necessary that the differences between the images under **f(n)** of the odd integers also be divisible by **20**, if one hopes to find Pythagorean triples in the context of cubes.

This yields the following: in what follows ***f (n) = 3 + 4* k*** , Odd terms must be spaced apart by **8** or an integer multiple thereof.

Thus, the components of the triplets must be expressed as follows:

c = 3 + 4k

b = 3 + 4k + 8r

Example :

c = 3 $\Longrightarrow$ b = 11

c = 3 $\Longrightarrow$ b = 19

etc.

And :

c = 11 $\Longrightarrow$ b = 19

c = 11 $\Longrightarrow$ b = 27

etc.

The table below clearly illustrates our points.:

```
a = 2 s;
b = 3 + 8 * r;
B = f (b)  = (5 * b^3 - 29) / 2;
A =  f (a)  = (5 * a^3) / 2;
Table[{b, B}, {r, 0, 10}]
table
Table[{a, A}, {s, 0, 10}]
table

= {{3, 53}, {11, 3313}, {19, 17133}, {27, 49193}, {35, 107173}, {43, 198753},
   {51, 331613}, {59, 513433}, {67, 751893}, {75, 1054673}, {83, 1429453}}

= {{0, 0}, {2, 20}, {4, 160}, {6, 540}, {8, 1280}, {10, 2500},
   {12, 4320}, {14, 6860}, {16, 10240}, {18, 14580}, {20, 20000}}
```

1st remark: :

To be eligible as a Pythagorean triplet, the difference between two odd integers must always yield a terminal digit of 0.

Furthermore, the images of odd integers under f(n) require that the penultimate digit is also even.

example : **2**0 ; 1**6**0 ; 5**4**0 ; etc.

This therefore compels us to refine the classification by proceeding as follows:

(f (b) - 3) /10

et : **f (a) /10**

Hence the following pivotal table:

```
a = 2 s;
b = 3 + 8 * r;
B = f (b)  = (5 * b^3 - 29) / 2;
A = f (a) = (5 * a^3) / 2;
Table[{b, (B - 3) / 10}, {r, 0, 20}]
table
Table[{a, A / 10}, {s, 0, 20}]
table
```

```
]= {{3, 5}, {11, 331}, {19, 1713}, {27, 4919}, {35, 10717}, {43, 19875},
   {51, 33161}, {59, 51343}, {67, 75189}, {75, 105467}, {83, 142945},
   {91, 188391}, {99, 242573}, {107, 306259}, {115, 380217}, {123, 465215},
   {131, 562021}, {139, 671403}, {147, 794129}, {155, 930967}, {163, 1082685}}

]= {{0, 0}, {2, 2}, {4, 16}, {6, 54}, {8, 128}, {10, 250},
   {12, 432}, {14, 686}, {16, 1024}, {18, 1458}, {20, 2000},
   {22, 2662}, {24, 3456}, {26, 4394}, {28, 5488}, {30, 6750},
   {32, 8192}, {34, 9826}, {36, 11664}, {38, 13718}, {40, 16000}}
```

2d remark :

- Once again, the representations of the even numbers all yield a congruence modulo 10 :

- The last digit = 0 ⟹ (0 , 10 , 20 , etc.)
- The last digit = 2 ⟹ (12 , 22 , 32 , etc.)
- The last digit = 6 ⟹ (4 , 14 , 24 , etc.)
- The last digit = 4 ⟹ (6 , 16 , 26 , etc.)
- The last digit = 8 ⟹ (8 , 18 , 28 , etc.)

These five sequences share a common ratio of 10.

⟹ All the numbers in this table are expressed in modulo **10** congruence :

a + 10k

where **a** is an even integer between 0 and 8 (inclusive of 0 and 8)

- Once again, the images of the odd integers yield a congruence **40**:

All the odd integers in this table are arranged into four interleaved sequences, according to their terminal digits. Here they are in the order they appear in the table:

- The last digit = 5 $\Longrightarrow$ (3 , 43 , 83 , etc.)
- The last digit = 1 $\Longrightarrow$ (11 , 51 , 91 , etc.)
- The last digit = 3 $\Longrightarrow$ (19 , 59 , 99 , etc.)
- The last digit = 9 $\Longrightarrow$ (27 , 67 , 107 , etc.)
- The last digit = 7 $\Longrightarrow$ (35 , 75 , 123 , etc.)

These five sequences share a common ratio: 40

$\Longrightarrow$ All the numbers in this table can be expressed in congruence modulo 40:

b + 40k

where **b** is an odd integer between **1** and **9** (inclusive)

- ***3rd stage :*** *proof*

1 - Let us begin with an example.

Consider the sequences generated by **27** and by **3** :

Since :

- the terminal value of the image of **27** under **f(n)** is : **9**
- and the terminal value for **3** is : **5**

9 - 5 = 4 $\Longrightarrow$ $\Longrightarrow$ the terminal value for the image of the even number must be 4. The table above shows us that it is the sequences generated by 6 that will yield this terminal value.

Therefore :

$$A = (27 + 40 (r + s))^3 - (3 + 40 s)^3$$

Which gives us:

$$\mathbf{A = 8 (3 + 5 r) \left(819 + 2280 r + 1600 r^2 + 3600 s + 4800 r s + 4800 s^2\right)}$$

Let us set:

$$p = (3 + 5 r)$$

$$B = \left(819 + 2280 r + 1600 r^2 + 3600 s + 4800 r s + 4800 s^2\right)$$

Since **8 = 2^3** , and since B is odd regardless of the values of r and s, for **A** to be a power of **3**, it is necessary that **(3 + 5r)** be odd (and thus r must necessarily be even).

(NB: The case where (3 + 5r) itself is a perfect cube does not alter the proof; we find ourselves in precisely the same situation.

Let us proceed:

$$B = k^3 (3 + 5 r)^2$$

Thus, we can write:

$$A = 8 * (3 + 5r) * k^3 (3 + 5r)^2$$

$$\Longrightarrow \quad \mathbf{A = 2^3 * k^3 (3 + 5 r)^3}$$

Now, let us consider the third element of the triple, namely, the even term:

6 + 10t

Which gives us:

$$(6 + 10 r)^3 = \mathbf{2^3 \ (3 + 5 t)^3}$$

Let us denote:

$$\mathbf{q} = (3 + 5 t)$$

We can therefore state the following equality:

$$k^3 (3 + 5 r)^3 = (3 + 5 r)^3$$

$$\Longrightarrow \quad \mathbf{k = \frac{q}{p} = \frac{3 + 5 t}{3 + 5 r}}$$

Two possible solutions :

First case : t = r + a

$$\Longrightarrow \quad k = 1 + \frac{5 a}{3 + 5 r}$$

which is impossible in ℕ

2d case : t = r

in this instance, one should be able to write:

A =

$$8 (3 + 5 r) \left(819 + 2280 r + 1600 r^2 + 3600 s + 4800 r s + 4800 s^2\right)$$

$$\Longrightarrow \quad A = 8 (3 + 5 r)^3$$

Is it possible to have:

$$\mathbf{A - 8 (3 + 5 r)^3 == ??? \ 0}$$

The answer is no:

$$2267 + 10800 r + 15975 r^2 + 7875 r^3 + 9120 s + 32400 r s + 24000 r^2 s + 9600 s^2 + 24000 r s^2$$

This quantity is always positive, even when **r = 0** and **s = 0**, because we will always have: 2267

Thus, we see that in every scenario, it is impossible to construct a Pythagorean triple raised to the third power with the set of all odd integers composed of:

{ 3+40 s , 6 + 10t , 27+40 (r+s) }

The same holds true for all other sequences whose difference in endpoints is 4.

As shown in this table:

```
Terminaison du nombre pair : 4;

(43 + 40 (s + r)) ^3 - (11 + 40 s) ^3 // Factor
(35 + 40 (s + r)) ^3 - (19 + 40 s) ^3 // Factor
(27 + 40 (s + r)) ^3 - (3 + 40 s) ^3 // Factor
(43 + 40 (s + r)) ^3 - (19 + 40 s) ^3 // Factor
(59 + 40 (s + r)) ^3 - (27 + 40 s) ^3 // Factor
(6 + 10 t) ^3 // Factor
```

Out[40]= $8\,(4+5\,r)\,\left(2443+3880\,r+1600\,r^2+6480\,s+4800\,r\,s+4800\,s^2\right)$

Out[41]= $8\,(2+5\,r)\,\left(2251+3560\,r+1600\,r^2+6480\,s+4800\,r\,s+4800\,s^2\right)$

Out[42]= $8\,(3+5\,r)\,\left(819+2280\,r+1600\,r^2+3600\,s+4800\,r\,s+4800\,s^2\right)$

Out[43]= $8\,(3+5\,r)\,\left(3027+4200\,r+1600\,r^2+7440\,s+4800\,r\,s+4800\,s^2\right)$

Out[44]= $8\,(4+5\,r)\,\left(5803+5800\,r+1600\,r^2+10\,320\,s+4800\,r\,s+4800\,s^2\right)$

Out[45]= $8\,(3+5\,t)^3$

None of the following cases can yield an integer:

$$k = \frac{q}{p} = \frac{3+5\,t}{4+5\,r} \quad ; \quad k = \frac{q}{p} = \frac{3+5\,t}{2+5\,r}$$

Notes :

Note -1 :

In the example under consideration, it might be objected that **(3 + 5r)** itself could be a perfect cube.

However, this does not affect the argument:

- Recall that the integer **r** must necessarily be even, as previously emphasized.
- Recall that only integers ending with the digit **7** can yield a perfect cube whose terminal digit is **3**.

Accordingly, we may thus express it as follows:

$$(3+5\,r) = (7 + 10\,u)^3$$

And so the proof is exactly the same as for the example previously analyzed.

Note - 2 :

There is no need to repeat the same procedure for the second sequence discussed above:

Suite n°2 ⟹ f (n) = 5 + 4* k

we arrive at precisely the same impossibility.

The same applies to the other four possible remainders : { 0 , 2 , 6 , 8 }
There is no need to display the tables for each possible remainder here; the current one suffices to demonstrate that we encounter analogous cases to those already examined, and that the proofs proceed in an identical manner :

```
Pour trouver 2;
(19 + 40 (s + r)) ^3 - (11 + 40 s) ^3 // Factor
                                         factorise
(67 + 40 (s + r)) ^3 - (35 + 40 s) ^3 // Factor
                                         factorise
(35 + 40 (s + r)) ^3 - (3 + 40 s) ^3 // Factor
                                        factorise
(43 + 40 (s + r)) ^3 - (19 + 40 s) ^3 // Factor
                                         factorise
(51 + 40 (s + r)) ^3 - (27 + 40 s) ^3 // Factor
                                         factorise
(2 + 10 t) ^3 // Factor
                 factorise
```

Out[36]= $8\,(1+5\,r)\,(691+1960\,r+1600\,r^2+3600\,s+4800\,r\,s+4800\,s^2)$

Out[37]= $8\,(4+5\,r)\,(8059+6760\,r+1600\,r^2+12\,240\,s+4800\,r\,s+4800\,s^2)$

Out[38]= $8\,(4+5\,r)\,(1339+2920\,r+1600\,r^2+4560\,s+4800\,r\,s+4800\,s^2)$

Out[39]= $8\,(3+5\,r)\,(3027+4200\,r+1600\,r^2+7440\,s+4800\,r\,s+4800\,s^2)$

Out[40]= $8\,(3+5\,r)\,(4707+5160\,r+1600\,r^2+9360\,s+4800\,r\,s+4800\,s^2)$

Out[41]= $8\,(1+5\,t)^3$

- ***4 phase*** *: conclusion*

Therefore, it is impossible to obtain a Pythagorean triple for the exponent 3.

4 - Study of Odd Powers Greater than 3

Let us first observe once again that these exponents yield precisely the same sequences as the exponent **3** (owing to identical terminal digits).

as shown in this table :

$$n = 2m + 1$$

```
Table[{n, (5*n^3 - 29) / 2}, {m, 1, 5}]
table
Table[{n, (5*n^5 - 29) / 2}, {m, 1, 5}]
table
Table[{n, (5*n^7 - 29) / 2}, {m, 1, 5}]
table
Table[{n, (5*n^9 - 29) / 2}, {m, 1, 10}]
table
Table[{n, (5*n^11 - 29) / 2}, {m, 1, 5}]
table
```

```
{{3, 53}, {5, 298}, {7, 843}, {9, 1808}, {11, 3313}}, etc.

{{3, 593}, {5, 7798}, {7, 42003}, {9, 14 7608}, {11, 40 2613}}, etc.

{{3, 5453}, {5, 19 5298}, {7, 205 8843}, {9, 1 195 7408}, {11, 4 871 7913}}, etc.

{{3, 49193}, {5, 488 2798}, {7, 10 088 4003}, {9, 96 855 1208}, {11, 589 486 9213}, etc.
{{3, 44 2853}, {5, 12 2070298}, {7, 494 331 6843}, {9, 7 845 264 9008}, {11, 71 327 917 6513}},
```

Note: let us bear in mind that:

$$\frac{5\,n^{2r+1}}{2} = 10k$$

And thus it is necessary that the odd terms possess the same ending. The proof is exactly analogous to that for the power **3**, utilizing the same intercalated sequences.

Suite n°1 $\Longrightarrow$ (3, 7, 11, etc.)

Suite n°2 $\Longrightarrow$ (1, 5, 9, etc.)

In conclusion :

It can therefore be concluded that there do not exist any Pythagorean triples with odd exponents.

5 - Study of even powers greater than 2, which are expressed as:

$$k^{2*(2k+1)}$$

Are there Pythagorean triples that can be represented in this form:

$$a^{2*(2k+1)} = ?? \; c^{2*(2k+1)} - b^{2*(2k+1)}$$

The answer is negative, since none exist, as they can equally be expressed as follows:

Let us define:

$$A = a^2 \; ; \; B = b^2 \; ; \; F = c^2$$

We can write:

$$A^{(2k+1)} \neq F^{(2k+1)} - B^{(2k+1)}$$

General conclusion

We have not addressed the case of multiple triplets derived from primitive triplets, since, ultimately, the discussion reduces to the primitive ones.

Thus, Fermat's theorem is established by means of this novel approach, which makes extensive use of the transformation function **f(n)**, about which I will elaborate in a forthcoming article on a related topic.

To conclude this article, let us dare, nonetheless, a slight smile: if this demonstration proves to be correct, I cannot help but think that perhaps it was this very approach that the great Monsieur Fermat had in mind. And beyond the painstaking nature of this exposition of the celebrated theorem of the illustrious Monsieur Fermat, let us bear in mind that, ultimately, it can be presented in such a straightforward manner that one might be tempted to say that, while it could not quite have fit in the margin of a letter, still—it would have required only marginally more space to encapsulate, as we may do ourselves when the appropriate occasion arises...

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Besançon, 24 June 2025