On the decomposition of a vector using complex numbers

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July 4, 2025

Abstract

In this letter a theorem is stated on the decomposition of a vector with respect to two axes using complex numbers. This theorem may serve to develop exercises for education in mathematics and physics. In most physics exercises the vectors are given by their polar coordinates. Motivated by practical applications, we denote the vectors by their Cartesian coordinates. Cartesian coordinates provide a didactic aid to develop exercises which can be solved without the use of a calculator. The educational potential of the theorem is demonstrated by a numerical example.

1 Theorem

Let $\underline{F}$ be a given vector, $\underline{F_u}$ and $\underline{F_v}$ its components with respect to two given axes u and v, respectively. The directions of the u-axis and the v-axis are defined by the unit vectors $\underline{e_u}$ and $\underline{e_v}$, respectively. Then

$$\underline{F} = \underline{F_u} + \underline{F_v} = F_u \cdot \underline{e_u} + F_v \cdot \underline{e_v}$$

where F_u and F_v are the coordinates of the vector $\underline{F}$ with respect the axes u and v.

The coordinates F_u and F_v are obtained as:

$$F_u \coloneqq \frac{\mathrm{Im}\left(\underline{F} \cdot \underline{e_v^*}\right)}{\mathrm{Im}\left(\underline{e_u} \cdot \underline{e_v^*}\right)}, \qquad F_v \coloneqq \frac{\mathrm{Im}\left(\underline{F} \cdot \underline{e_u^*}\right)}{\mathrm{Im}\left(\underline{e_v} \cdot \underline{e_u^*}\right)}$$

with

$$\left|\underline{e_u}\right| = \left|\underline{e_v}\right| = 1$$

Remark: note that

$$\mathrm{Im}\left(\underline{e_v} \cdot \underline{e_u^*}\right) = -\mathrm{Im}\left(\underline{e_u} \cdot \underline{e_v^*}\right)$$

Proof

$$\underline{F} \cdot \underline{e_v^*} = \left(F_u \cdot \underline{e_u} + F_v \cdot \underline{e_v}\right) \cdot \underline{e_v^*} = F_u \cdot \underline{e_u} \cdot \underline{e_v^*} + F_v \cdot \underline{e_v} \cdot \underline{e_v^*} = F_u \cdot \underline{e_u} \cdot \underline{e_v^*} + F_v \cdot 1 = F_u \cdot \underline{e_u} \cdot \underline{e_v^*} + F_v$$

$$\Rightarrow \quad \mathrm{Im}\left(\underline{F} \cdot \underline{e_v^*}\right) = \mathrm{Im}\left(F_u \cdot \underline{e_u} \cdot \underline{e_v^*}\right) + \mathrm{Im}(F_v) = F_u \cdot \mathrm{Im}\left(\underline{e_u} \cdot \underline{e_v^*}\right) + 0 \quad \Leftrightarrow \quad F_u = \frac{\mathrm{Im}\left(\underline{F} \cdot \underline{e_v^*}\right)}{\mathrm{Im}\left(\underline{e_u} \cdot \underline{e_v^*}\right)}$$

The result for the coordinate F_v follows by replacing the axes $u \to v$ and $v \to u$ in the expression for the coordinate F_u.

□

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2 Example

Given:

$\underline{F} = 20 + 15i, \ \underline{e}_u = \frac{2+i}{\sqrt{5}}, \ \underline{e}_v = \frac{1+3i}{\sqrt{10}}$, see Figure 1

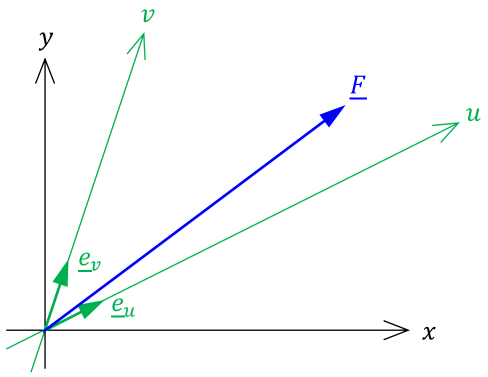

Figure 1: The given vector $\underline{F}$ and the given axes u and v.

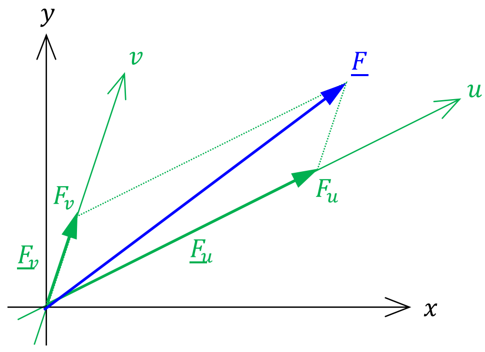

Figure 2: The coordinates F_u and F_v and the corresponding components $\underline{F}_u$ and $\underline{F}_v$.

Applying the theorem

The coordinate F_u, see Figure 2:

$$\underline{F} \cdot \underline{e}_v^* = (20 + 15i) \cdot \left(\frac{1+3i}{\sqrt{10}}\right)^* = \frac{(20+15i)\cdot(1-3i)}{\sqrt{10}} = \frac{65-45i}{\sqrt{10}} \quad \Rightarrow \quad \mathrm{Im}\left(\underline{F} \cdot \underline{e}_v^*\right) = \frac{-45}{\sqrt{10}}$$

$$\underline{e}_u \cdot \underline{e}_v^* = \frac{2+i}{\sqrt{5}} \cdot \left(\frac{1+3i}{\sqrt{10}}\right)^* = \frac{(2+i)\cdot(1-3i)}{\sqrt{5}\cdot\sqrt{10}} = \frac{5-5i}{\sqrt{5}\cdot\sqrt{10}} \quad \Rightarrow \quad \mathrm{Im}\left(\underline{e}_u \cdot \underline{e}_v^*\right) = \frac{-5}{\sqrt{5}\cdot\sqrt{10}}$$

$$F_u = \frac{\mathrm{Im}\left(\underline{F} \cdot \underline{e}_v^*\right)}{\mathrm{Im}\left(\underline{e}_u \cdot \underline{e}_v^*\right)} = \frac{\frac{-45}{\sqrt{10}}}{\frac{-5}{\sqrt{5}\cdot\sqrt{10}}} = \frac{-45}{-5} \cdot \sqrt{5} = 9\sqrt{5}$$

The component $\underline{F}_u$, see Figure 2:

$$\underline{F}_u = F_u \cdot \underline{e}_u = 9\sqrt{5} \cdot \frac{2+i}{\sqrt{5}} = 18 + 9i$$

The coordinate F_v, see Figure 2:

$$\underline{F} \cdot \underline{e}_u^* = (20 + 15i) \cdot \left(\frac{2+i}{\sqrt{5}}\right)^* = \frac{(20+15i)\cdot(2+i)}{\sqrt{5}} = \frac{55+10i}{\sqrt{5}} \quad \Rightarrow \quad \mathrm{Im}\left(\underline{F} \cdot \underline{e}_u^*\right) = \frac{10}{\sqrt{5}}$$

$$\underline{e}_v \cdot \underline{e}_u^* = \frac{1+3i}{\sqrt{10}} \cdot \left(\frac{2+i}{\sqrt{5}}\right)^* = \frac{(1+3i)\cdot(2-i)}{\sqrt{5}\cdot\sqrt{10}} = \frac{5+5i}{\sqrt{5}\cdot\sqrt{10}} \quad \Rightarrow \quad \mathrm{Im}\left(\underline{e}_v \cdot \underline{e}_u^*\right) = \frac{5}{\sqrt{5}\cdot\sqrt{10}}$$

$$F_v = \frac{\mathrm{Im}\left(\underline{F} \cdot \underline{e}_u^*\right)}{\mathrm{Im}\left(\underline{e}_v \cdot \underline{e}_u^*\right)} = \frac{\frac{10}{\sqrt{5}}}{\frac{5}{\sqrt{5}\cdot\sqrt{10}}} = \frac{10}{5} \cdot \sqrt{10} = 2\sqrt{10}$$

The component $\underline{F_v}$, see Figure 2:

$$\underline{F_v} = F_v \cdot \underline{e_v} = 2\sqrt{10} \cdot \frac{1+3i}{\sqrt{10}} = 2 + 6i$$

Note that

$$\underline{F} = 20 + 15i = (18 + 9i) + (2 + 6i) = \underline{F_u} + \underline{F_v}$$

3 Acknowledgement

The authors acknowledge the support of Alexandra Jacobs and Harald Theunissen both affiliated with the Department of Engineering of Zuyd University of Applied Sciences.