PHYSICAL PRINCIPLES IN REVEALING THE WORKING MECHANISMS OF BRAIN

part TWO

by

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Abstract. This work, as a whole, is addressed to a wide range of scientists who, naturally, approach the research of the human brain from different points of view. Our own main point is that no matter of the angle of approach in the brain research, a scientist must be aware of the *physical possibility* of working of the brain. We describe such a possibility by modeling the brain as matter, active through an agent analogous to light: the brain behaves like a universe.

Consequently, in this instalment of the work, the emphasis is on quantization as an essential function of a universe, according to its concept. The Planck's quantization is taken as prototype in unfolding the concept. It is shown, *mathematically*, how the quantization works *for the charge*. Based on this physics, a scenario is described for the manner of communication in the brain. Specific conservation laws generated by the condition of quantization are provided, and a concept of interpretation is defined based on the idea of *dyons*, the particles invented by Julian Schwinger.

The subject is, obviously, not closed; there is more to follow from the work. However, an affine theory of a universe concept is proposed, as a kind of 'bonus' in closing this instalment of the work, just for the sake of illustration of some essential points of the general theory.

Keywords: Bohr-Kramers-Slater theory, hidden parameters, uncertainty relations defining the fields, Stroud's psychological moments, space of information, regularization procedure, fundamental analiticity conditions, the mind, the structure of mind, Barbilian quantization condition, physical time, psychological time, Hillis' scenario for brain working, message waves, cancel waves, public key cryptosystem, dyons, nervous matter, glia, DNA double helix, Chebyshev nets, de Sitter's charge background, two-fluids Maxwellian theory of electricity, Planck's differential equation, Louis de Broglie's light ray, Lorentz quantum, Bartolomé Coll's universal deformation, affine surfaces, Newtonian forces as statistics, Jeans' theory of forces

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1. Introduction

The previous instalment of the present work [(Mazilu, 2019); to be cited here as *I* from now on], although aimed at selecting a physics that, in our opinion, serves the physiology best – and, in fact, guided in broad lines by facts of a physiological nature – is exclusively physical and, justifiably we might say, mathematical. It was intended to help describing, or even constructing occasionally, the basic mathematical instruments of a physical theory of brain. This physical theory is based, in our case, on a concept of brain heavily drawing on physical cosmology: in short, *the brain is considered a universe*. The *concept* of universe is cogitated as having at least the differentiae of its only prototype we can have at our disposal – the universe of physics, *i.e.*, *the universe of our experience*. Along this line, we can declare that the present instalment of the work just continues the spirit of the first one, but adds, nevertheless, to the concept of universe some further differentiae based on facts of physiological and neurological nature, apparently manifested only in the brain. As it turns out, this kind of facts have, indeed, a great deal to add to the very concept of universe, and so, occasionally, we need to go on interpreting physics, according to the 'experience in the brain universe', as it were. The vacuum multiplicity and, in fact, the genuine quantization procedure, are two such important cases. Of course, we cannot forget our main goal with the present work as a whole: *understanding the physics of the brain*. In this respect we can say that our work just follows the history of a part of neurophysiology, namely, that part which unfolded along with the physics of 'thinking machines'.

Initially, the idea of brain as a universe was brought about by the close analogy between the cranial vault of the *human skull* – which contains the brain or, in fact, an essential part of it, the *cortex* – and the *classical Wien-Lummer enclosure* of experimental physics of light, containing a mixture of matter and light. It is this device, which served in the study of thermodynamics of light that led to the Planck's quantization and, as the brain *handles charges* during its functioning, the idea has occurred to us that the quantization of charge may be based on the same principles as the quantization of light, assuming that the charge may be taken as the analog of light. A conclusion, for the reason of which we where urged by the incidental electromagnetic nature of light. However, this assumption is not enough in order to conclude on the character of universe of the brain: its very analog – the Wien-Lummer enclosure – should be a universe, and this is the key to our thesis, which can be concluded from some other facts.

Indeed, the logical incentives that *a physicist* may invoke right away for this choice of the concept of brain are, we think, quite simple and, as a matter of fact, even intuitive: the man exists *as* a self-determined entity, *within* a self-determined entity – the physical universe we inhabit – and even behaves as such. To wit: physically speaking, the man exists *in a permanent gravitational field* just as the universe, and *under the permanent influence* of different physical fields in the universe of his existence, behaving just as 'individually', if we may use this expression, as the universe of his existence does. Now, based on some straightforward physical facts, we have concluded to *a prototype model of the universe* we inhabit: it must be physically modeled as a Wien-Lummer enclosure. Then, if an analogy exists, a Wien-Lummer enclosure cannot have as analog in man but… an enclosure, obviously. Along this line, one can reason that, since the cranial vault appears to be the only enclosure satisfying some intuitive requirements for such a physical thing, one can take its content, that is, the brain, as a universe.

The metaphor enthusiasts may recall the Blaise Pascal's adage: « Par l'espace l'universe me comprend et m'engloutit comme un point; par la pensée, je le comprends ». Along this line, we aim to show that 'la pensée' needs 'l'espace' too, not so much for 'comprising', like the universe we inhabit, but especially for 'comprehending'. It is the 'headquarters' of this space, *viz*. the brain, that are 'comprised' in a Wien-Lummer enclosure, just like a universe must be, according to the characteristics of the prototype concept we have at our disposal: the physical universe of our experience. But, one may rightfully ask, where the charge gets into this picture, and how? The theoretical physics indicates the feasibility of an idea that, if the man exists in the physical universe, then, from the general-relativistic point of view, he exists in a *background of charge*: the very same background in which the physical universe exists [(Mazilu, 2024), Chapter 3]. Furthermore, inasmuch as the light has, occasionally, an electromagnetic character, we concluded that this character is a result of the 'handling' the charges by the physical universe. On the other hand, the existence of the electroencephalograms and magnetoencephalograms, is surely due to handling the charges, which could have no other residence, but in the same background. Hence the idea of *brain as a universe* …

There are, nevertheless, essential differences in the existence of the two universes, which may bring some changes on both sides of the analogy. For once, it is striking that in the case of physical universe of our perception proper, the permanent physical fields in question appear to us, all of them, in such a way that they cannot be considered but unconditionally *internal* to the universe: we do not have any means to see them as … *external*. On the other hand, their counterparts from the case of brain, mainly appear as *external*: only some deep reflection, logically quite intricate indeed, based on solid physiological facts, may allow us to say that there are, among these physical fields, some that can be considered internal to the brain universe. And so, it happens that an essential incentive in deciding our choice of the concept, is the specific electromagnetic activity of the brain, which seems to be the only internal activity analogous to that of the physical universe of our perception, exerted, however, at different scales of *time* and *space*; moreover, in the brain, these last two categories must have quite different physical structures. In this respect the choice of concept may not be quite 'sufficient', if we may say so: the universe characterization may refer, just as well, *to the man as a whole*, for instance, as he is already considered mostly in the philosophical and poetical kinds of considerations. This part of the present work has also, among its tasks, the one of deciding, at least preliminarily, even on this issue.

1.1. The Brain as a Machine

It is, perhaps, harder for a neurophysiologist to assume such a way to conceive the brain than for a physicist, but, significantly enough, at least for *our* concern, the physiologists – again, perhaps only in our opinion, obviously! – were the ones who found the right way to envision what the universe, as a concept, might be. It all started with an idea that can qualify as having a subjective origin. There was a time, indeed, by the middle of the 20th century, when people among us with some specific freedom of thinking became focused on constructing thinking machines and, in doing this job, they could not look for inspiration but to what the life presented right in front of them, of course: *the human brain*. Fact is that in those times – as even today, for that matter, let us recognize! – the mankind did not know too much about *the physics of brain*, so that the people we are talking about had first *to figure out* how the human brain works. Obviously, they did not have too much to start with, and so, they needed first to construct a model of brain. Working to build something, by using a model that one has to build first, seems to do the same job twice, to say nothing of the questionable 'objectivity' of such a construction, which is practically based on a 'tautology'. But they started doing it anyway. As it turns out, none of the two jobs was unnecessary indeed: after all, *understanding* the nature is a job just as difficult to carry out as *copying* it technologically, perhaps even more difficult. And there is a solid reason for that: the background radiation of the physical universe was to be discovered a few decades later, so that no one at the time could think of an existing concept of a universe to comprise the brain. By the same token, an idea that the quantization can work for the charge in the very same way it works for the light, obviously could not be produced at that time.

However, *the understanding* came first. At that moment the understanding was imperiously necessary, for the mankind perceived, well ahead of scientists, as usual, a danger in the case: the brain-machine may become able to claim its place in the world, just as the man claimed his place in this world, and so prevail over the world just like the man does. This idea still dominates today the public conscience, and probably will last a long time, hopefully not forever. Thus, naturally, one of the main points to clear regarding the primacy of the man was his *technological non-reproducibility*, and the differentia targeted in this respect was, just as naturally, *the reasoning*. Unsurprisingly, the 'headquarters' of reasoning came among the first targets. Their location was, as the man realized even from the old times of Hippocrates of Cos, the noble descendant of the mythical Asklepios, in the head (Penfield, 1978). A question was then naturally aroused: *why in the head*?

The great american neurophysiologist Warren Sturgis McCulloch – one of the coryphaei in laboring for the initiation of cybernetics and construction of the concept of brain-machine [(McCulloch, 1949); see, however, (McCulloch, 2016) for a deeper documentation] – was always looking for a 'logic of the brain connections' [see (Arbib, 2000) for a thorough documentation on this essential kind of works of Warren McCulloch]. He was the one who answered the question above, and aptly we should say, by analyzing the physiological facts from the perspective of the concept of thinking machines. His answer to this fundamental question can be given along the conception rounded up by the middle of 20th century, namely that *the mind* is *essentially relational*: it involves the existence of ensembles of individuals – specifically, *neurons* – having connections among them (Hebb, 1949). Expressed in McCulloch's own written words by the way of a conclusion, this idea sounds:

> This brings us back to what I believe is the answer to the question: «Why is the Mind in the Head?» Because there, and only there, are *hosts of possible connections* to be formed as *time* and *circumstance* demand. Each new connection serves to set the stage for others yet to come and better fitted to adapt us to the world, for through *the cortex* pass the greatest inverse feedbacks *whose function is the purposive life of the human intellect*. The *joy of creating ideals*, new and eternal, in and of a world, old and temporal. *Robots have it not*. *For this my Mother bore me* – that *the mind* be in the head. [(McCulloch, 1950); *emphasis added*, *a*/*n*]

Profoundly philosophical expression, yet predominantly gracious and poetical, we have to admit! Typical, anyway, for thus great sage of an old school, of the kind hardly extant today. However, in this very spirit of our great sage of computer-coming era, and in keeping, actually, with his own philosophical note, we are tempted to update, and enhance in fact, this poetical roundup of the important work just cited. For this, we just have to take notice of the fact that the 'joy of creating ideals' may be prompted at an even higher 'commandment', as it were, addressed to the man.

This statement of ours is the result of a kind of inference springing up from a special perception of our experience, namely the perception of the physical universe *per se*. This perception created a distinctive awareness of the man, which was occasionally labelled *De Caelo* (*On the Heavens*) by the philosophers all along our history, starting with Aristotle, to name the most significant among them. A remarkable conclusion of this awareness was once expressed by the great Romanian philosopher Constantin Noica. Quoting:

> Do not forget that *God sent you in the world* in order *to replace* Him: *to assign meanings*, *to create*, *to take His beginning onward*. Make sure not to waste *your time*! [(Noica, 1937), *our translation and emphasis*, *a*/*n*]

According to this conclusion, if the man is 'purposive' at all in the world he inhabits, then this 'purpose' is established at a higher commandment, indeed, the highest our mind can imagine: *that of our own Creator*. Apparently, only by this higher authority Mothers themselves *must* bear us too. And only *physically* at that, since what the man, as a concept, should be, is in fact what *he becomes*: finding the 'mind' is his own task, to be accomplished along a life-time of living within physical universe. However, according to a physics like that implied by our birth, which, obviously, is carrying in more than one way the mark of the universe we inhabit, at times during our life this universe is overwhelmingly coming onto us, throughout the very mind. So that, our great philosopher is not long in indicating what would be the right 'attitude' of the 'mind' in this instance. Quoting, therefore, again:

> *Not about the astronomical worlds* has the man to trouble himself, and *not facing them ought him suffer his complex of inferiority*. He is not inferior with respect to something else, insofar as *there is not common measure between him and something else*. He is inferior (only) with respect to himself, if he does not stand on the steps of lucidity *high enough that he may have an appropriate self-awareness* [(Noica, 1937), *our translation and emphasis*, *a*/*n*].

Based on such ideas, it just occurred to us that the 'self', or the 'mind', may reflect the connection of the man with the universe he inhabits and, according to the conclusions of the analysis of Warren McCullogh just cited above, this differentia of the concept of human being – the self-awareness – makes all the distinction between man and machine, more specifically, between the man and the robot.

A question, though, legitimately arises, for any researcher, be him physicist, physiologist, or of some other persuasion: *is there any evidence that the man can be reduced to a robot*? Of course, we do not think of a reducing by coercion, which is, obviously, the 'rational' method that created slaves out of conscious men all along history, and still does this specific job even today for that matter. We are thinking of a *natural* degrading of the human brain to a machine, which involves the elimination of the 'headquarters of mind', thus reducing the man to a *robot*. And, sure enough, there is such a disease to do the job, historically labeled on occasion as a 'God-given disease': *the epilepsy*.

1.2. Wilder Penfield and the Physics of Brain

The remarkable Canadian neurosurgeon Wilder Graves Penfield has found physical reasons for the consciousness, or the mind, or the 'self', as we have called it above, starting from the direct observations of the manifestation of his epileptic patients (Penfield, 1978). He was using extensively *the electricity* in order to locate the centers of epileptic disruptions in the brain and thus eliminate them surgically, thereby alleviating the consequences of disruption. However, in doing this job, he and the team he directed, stumbled upon one of the most important discoveries of the 20th century that we cannot ascribe, for explanation toward an understanding, but exclusively to the science of physics: *the electricity can elicit hidden memories of our experience stored in the brain matter* (Penfield, 1938). Indeed, these memories were not of the nature of dreams, that may suggest an ancestral experience encrypted in the matter of brain, but facts of a verifiable common experience, which the subject could not know otherwise but only by witnessing them in a life scenario where he/she was participating at least observationally. This fact of experience of Penfield and his team, opened, in turn, an interesting odyssey, towards understanding the mind by studying the very human brain. Let us briefly illustrate this journey of our experience, since it was this particular experience that disciplined our research toward the present results. Quoting, again, this time from Penfield himself:

> These attacks of *epileptic automatism* show clearly the automatic, complex performance of *which man's computer is capable*. In an attack of automatism *the pacient becomes suddenly unconscious*, but, since other mechanisms in the brain continue to function, *he changes into an automaton*. He may wander about, confused and aimless. Or he may *continue to carry out* whatever purpose his mind was in the act of handling on to his automatic sensory-motor mechanism when the highest brain-mechanism went out of action. Or he follows a *stereotyped, habitual pattern behavior*. In every case, however, the automaton can make few, if any, decisions for which there has been no precedent. *He makes no record of a stream of consciousness*. Thus, *he will have complete amnesia* for the period of epileptic discharge and during the period of cellular exhaustion that follows. [(Penfield, 1978), pp. 38 – 39; *emphasis added*, *a*/*n*]

Starting from this idea of *robot* – an 'automaton' in his phraseology – Penfield guided his own specialty experience in order to construct a *concept of mind* that has to be added to machine in order to make a man out of it. From the conclusions of his practice, and with a research conducted deliberately, the essential ingredient of the theory became obvious and, furthermore, even theoretically describable from physics' point of view, within the concept of brain as a universe. Quoting again:

> And so I come to my final reconsideration: I worked as a scientist trying to prove that *the brain accounted for the mind* and demonstrating as many brain-mechanisms as possible *hoping to show how the brain did so*. In presenting this monograph I do not begin with a conclusion and I do not end by making a final and unalterable one. Instead, I reconsider the present-day neurophysiological evidence, on the basis of two hypotheses: (a) that man's being consists of one fundamental element, and (b) that it consists of two. I take the position that *the brain-mechanisms*, which we (my many colleagues and I all around the world), are working out, would, of course, have to be employed on the basis of either alternative. In the end I conclude that *there is no good evidence*, in spite of the new methods, such as the employment of stimulating electrodes, the study of conscious patients and the analysis of epileptic attacks, *that the brain alone can carry out the work that the mind does*. I conclude that *it is easier* to rationalize man's being *on the basis of two elements* than on the basis of one. But I believe that *one should not*

> *pretend to draw a final scientific conclusion*, in man's study of man, *until the nature of the energy responsible for mind-action is discovered* as, in my own opinion, it will be. [(Penfield, 1978), pp. 113 – 114; *our Italics here*, *a*/*n*]

There can never be 'a good evidence' on a way or another as long as we simply observe and conjecture, limiting ourselves to just logical connections, without an explanation from physics' point of view. *We* take the position that the understanding of those 'brain-mechanisms' is the thing missing here in discovering the Penfield's mind-action, and that 'energy responsible' was always under our eyes. Namely, that 'energy responsible for mind-action' *is the charge* or, more to the point, *whatever the cerebral cortex makes*, in the brain universe, *out of what we know as electric charge in the universe we inhabit*.

Remarkably enough, Penfield also touched the problem from the perspective of that high command that involves the Creator of the world. For, as one can plainly recognize by following the overall pattern of the work of great sage, he has tasked himself with 'assigning meanings', as Constantin Noica would say, in the world of brain, and he succeeded in doing the task to a remarkably great extent. So he was most certainly entitled to the following closure of the last edition of his celebrated *Mystery of the Mind*, the book that represents what – again, in our opinion, of course – is the best scientific approach of an experimental 'study of consciousness', not only philosophically and at most logically, as usual in the physiological approach of the study of brain and mind, but even from the very point of view of physics:

> In ordinary conversation, the "mind" and "the spirit of man" are taken to be the same. I was brought up in a Christian family and I have always believed, since I first considered the matter, *that there was work for me to do in the world*, and there is a grand design in which *all conscious individuals play a role*. Whether there is such a thing *as communication between man and God* and *whether energy can come to the mind of a man from an outside source* after his death is for each individual to decide for himself. *Science has no such answers*. [(Penfield, 1978), p. 115; *emphasis added*, *a*/*n*]

Of course, we only have to agree again with the great sage. However, our agreement has a 'subordinate clause', as it were, in that we exclude 'death' from its 'protocol': as long as it is based on our experience, like the physics we practice, 'science has no such answers', indeed, for the death is beyond life and, consequently, in death *there is no experience*. That is, we aim to show, in this work as a whole, and particularly in this one of its instalments, that inasmuch as the 'life' is involved, the physics *can present evidence* of 'communication between man and God', by the very fact that the man, and implicitly the brain, *is a universe that subsists in the physical universe*.

One of the great realizations of Wilder Penfield, based on facts of solid experience involving *the electricity*, and being, therefore, of a manifest physical nature, is that if the learning process makes the top of the evolution tree out of man, that very fact is reflected in the existence of the *cerebral cortex*: it contains areas of brain *dedicated to learning*, and these areas exceed in magnitude – and, perhaps, differ in structure from – the corresponding areas in any animal [(Penfield, 1978), pp. 19 – 20]. Another great realization that, in our opinion, *must be assigned* to Penfield and his collaborators is the idea that the 'files of the memory', which define the consciousness, must be '*written*' *electrically*, once they *can be* '*read*' *electrically*, as Penfield routinely did. Incidentally, this was, as we already said, one of the greatest discoveries of his team of collaborators (Penfield, 1938). However, we found even a greater importance of the Penfield's work, in the fact that, on this account, he assigned a *positive job to physics* itself in the problems of brain. For, the physics, and only the physics, can have a saying in the description of the manner of working of the brain: the key discovery of Penfield is *that the memory can be activated electrically*. One could say that the *memory cannot be made by storage*, unless it is referring to facts of experience. Quoting:

> The demonstration of the existence of cortical "patterns" that preserve the detail of current experience, *as though in a library of many volumes*, is one of *the first steps toward a physiology of the mind*. The nature of the pattern, the mechanism of its formation, the mechanism of its subsequent utilization, and *the integrative processes that form the substratum of consciousness* – these will one day *be translated into physiological formulas*.
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> But when the day dawns, I surmise that men will still stand in doubt before the ultimate riddle – *What is the bridge between nerve impulse and thought*? *And what about a man*'*s soul*? [(Penfield, 1952), p. 191; *emphasis added*, *a*/*n*]

As one can see here, leaving momentarily aside the other important things in this conclusion of the great sage, of which, in fact, we shall be occupied later on, Penfield implicitly defines the essential problem of physics in the world of brain: to find *the bridge between nerve impulse and thought*. The 'day did not yet dawn', but what remains for us *is a problem of physics not physiology* that the great sage assigned to us. This problem will be touched in this work. And our natural-philosophical starting point in solving this problem is that *the nerve impulse arises somehow in connection with the motion of charge*, while *the thinking* itself is connected with *the very transformation of the charge*. Notice the distinction: *the motion* of the charge, *vs*. *the transformation* of the charge; they cannot ever be identical, and the differentia that makes the mind what it is, we presume, stays in the *transformation of the charge*. This is why we shall need to see what physics made out of charges, since the physiology seems to have stopped in the very beginnings of the history of electricity [see, for instance, (Gesell, Brassfield, & Lillie, 1954); in case there is a doubt regarding the actuality of the physical principles involved in this old work, the reader is invited to consult any modern compendium in neurophysiology; from the modeling perspective, though, of the present work, we can cite, for instance, (Keener & Sneyd, 2009)]

1.3. Some Tasks of Physics

It is, indeed, physics, the one and only among sciences that can give a solution to this problem, and the concept of brain as a universe seems to indicate the solution that follows in this part of our work. This solution is based on an observation that, we think, is evident even for a layman: any physical being – and by this we mean 'something' that moves and feeds by itself – *exists as a universe within the physical universe by the capability of living organisms to handle the electric charges*. However, as we mentioned before, in the spirit of modern theoretical physics we must add: what a layman cannot grasp, is that *these charges are created out of a background physical continuum in which any being*, and thus, implicitly, the brain, exists [(Mazilu, 2024); chapter 3]. Indeed, what seems to be physiologically obvious, is that the cranial vault plays, from physics' point of view, the part of a Wien-Lummer enclosure, however containing not light but, apparently, only matter. A question then arises: shall we conclude that *the brain matter has a component that is here analogous to light*? The issue of mind presented above, seems to indicate that this is, indeed, the case, and we shall pursue here this idea.

However, as we see it, this issue asks for a deeper understanding of the very process of quantization in physics: after all, a Wien-Lummer enclosure is the essential 'apparatus' of *studying the properties of light* in the presence of matter. Towards constructing a support of this statement, we have previously proved that the Planck's quantization procedure for light works also, *formally unchanged*, for the matter, indeed (Mazilu, 2022, 2023). Indeed, it operates based on *continuous probability distributions* in the case of matter, while the original Planck's quantization referring to light is operated based on *discrete probability distributions*, nonetheless, belonging to the *same* large class of statistical distributions. So, we were able to figure out that, *if* the original Planck's quantization was referring to the spectral energy of light, the quantization in brain must be referring to *some kind of distribution of the charge*, for, according to our experience, only the charge can move freely in matter. One can say that the matter in brain plays, with respect to charge, the same role that vacuum plays with respect to light. The cranial vault can be seen as a Wien-Lummer enclosure containing matter, and this matter is 'absorbing and creating charge', the way Planck's matter contained in the original Wien-Lummer enclosure is 'absorbing and creating light'.

The Penfield's discovery, then, can be taken as showing that the electricity, existing in man's body by some process, whereby the charge is 'absorbed' from the continuum background of the universe of our existence, and maintained in man – by the muscular activity, for instance, among other processes! We should not forget that this was the first kind of physiological manifestation of the electric charge. And, only based on this fact, we can infer today that the charge can be transformed *within the brain cortex* into that 'library of many volumes', wherein the man can learn from that *stored experience* of him, and proceed in building his *consciousness*. How this transformation may occur is – let us repeat once more – a matter of theoretical physics, for no other science is able to fathom it. And we cover, within this instalment of the present work, a few of different basic aspects of the modern part of this theoretical physics, just in order to see what it has made out of the charge as a concept. Consequently, we just *wish to turn* into a solid fact the anticipation of the great sage just mentioned, expressed at the end of the Chapter 2 of his last remarkable production of 1978:

> Can I discuss this mechanism understandably if I leave behind the technical phrase and speak the language of the unspecialized but educated man? I dare say Benjamin Franklin, founder of the American Philosophical Society, explained in easily understood excitement to the first members of that society how it is that electricity passes down the wet string of a kite. I wish I had been there. Perhaps *I would have understood the nature of this all-important wonder called electricity*. It seems to me that, somehow, *it is like the mind* in the sense that *one cannot assign to the mind a position in space and yet it is easy to see what it does and where it does it*.
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> Hans Berger, *the discoverer of electroencephalography*, when *he hoped* (*vainly*) *to record* the activity of the mind electrically, may have had in mind this similarity. [(Penfield, 1978), p. 10; *emphasis added*, *a*/*n*]

From the perspective of some three centuries since the times of Benjamin Franklin, and one century of modern theoretical physics, we can be assured that our great sage *had a brilliant intuition*. The case of Berger, however, illustrates another truth: if the man was created, indeed, if 'God created the mankind in *his own image*' then, surely, every man is in possession of truth, and he always acts under this impulse. However, when he needs to express this truth, he uses a language, which is never the proper language for that task. This is the moral of the old Tower of Babel story! Having only the experience at its disposal – the *sine qua non* condition of the existence of a language – the mankind needs science in order to uncover the truth. For, any synthesis – a 'Tower of Babel', as it were – needs communication among men, otherwise the mankind cannot act as a unit: in other words, they need a common language, which is not possible when having different experiences, *i.e.*, when 'scattered all over the Earth'. To wit: like Berger's, the Penfield's hope would have been 'vain' too. In order to really understand the nature of that 'all-important wonder', one has to understand first what the fact that 'one cannot assign a position in space' and yet 'one can see where and when', really means from physics' point of view. An 'unspecialized but educated man' cannot see this or, if he ever sees it, cannot realize its overwhelming importance! There is one single case in the history of science, that can teach us how this might happen, but to learn it takes a '*specialized* educated men', physicist or astronomer, for instance: *the case of Earth in the universe*. We cannot say *where* we are in the universe, by any means physical! And yet we do physics, which implicitly means 'seeing where and when'. Thus, if we take the brain as a universe, we might be able to describe 'what the mind does and where it does it', just like we do for the Earth in the universe we inhabit. We cannot 'record this activity' of brain, but, *as Penfield himself demonstrated*, the brain does it, *if the man witnessed it through his senses*! And this is what raises a big problem: why is it that the man has to 'be there' in order to be possible to 'record'?

As it turns out, in order to accomplish this task we just need to understand the right connection *between charge and quantum*. It is doubtful that the old Benjamin Franklin could teach us something of the kind, for even the modern theoretical physics is still in doubt on this issue, in spite of the progresses made in understanding the physiological charges (Gesell, Brassfield, & Lillie, 1954). To say nothing of the fact that, based on Franklin's ideas only, one could not fathom how the electricity can enter the realm of brain: there are great many basic aspects of the very *problem of electricity per se*, properly understood only long after Franklin. Those aspects are our tasks with this work. The order in which these basic aspects – and when we say 'basic' we understand basic from the physics' point of view – are presented, is the order in which they come to our wits, *i.e*., an *ad hoc* order.

2. Rational Basis for a Strategy of Approach

The basic concept in the theoretical description of a Wien-Lummer enclosure containing radiation and matter, is that of *ray of light*. In time, this concept raised some contradictions in connection with the universe of our experience, and if we are to use the Wien-Lummer enclosure as a model universe, we need to be aware of these contradictions. For instance, in no universe can the John Clarke Slater's physical principle – that any *atom* in a stationary state communicates *all the time* with *a priori* any other atom in a stationary state (Slater, 1924) – be truer than in the brain universe. That is, provided we are able to identify 'the atom' as a well-defined concept, specific to that universe. And in the brain physics, the best candidate to this part in a physical scenario seems obvious by physiology: *the neuron*. The physiological researches have detached the neuron as the fundamental structure that communicates perceptively with its likes in order to form what in physics can be taken as a ray of light. This is, in our opinion, the basis of a rational approach when doing physics in the brain universe. According to this view, if the charge plays, in the brain universe, the role we usually assign to light in the physical universe, the one we inhabit, then the neuron is, indeed, the best candidate for the analogue of an atom. The other side of this analogy, – that is, the side exclusively concerning the physics, in giving a basis to the Slater's principle – can then be formulated by saying that an atom proper must be understood as part of a ray of light, and this understanding cannot be accomplished but according to Planck's procedure of quantization. Let us pursue this idea for a while, in order to see where it leads, according to anatomical and physiological studies.

2.1. The Rays of Light and the Rays in Brain

If there is a message which the physics must extract, along the lines of Slater's principle, from the theory of optical rays, as these were conceived by Louis de Broglie (see I, § 2.2) and interpreted, for instance, by the introduction of luxons through Bartolomé Coll's theory (see I, § 2.1), then it can be expressed by the idea that the *rays must preexist* somehow in an optical medium. In this connection, the preconception always existed, that the ray must be constructed from a *propagation phenomenon* (of light, for instance, in the case of the physical universe) in a medium, conceived as a continuum. At least this was the spirit of physics in the times of Newton and Hooke, where the rays of light had to be *created* first, in order to be *studied*. By contrast, however, the brain universe presents a natural case demonstrating this statement: it is one of those distinctive characteristics of this universe – a differentia of the concept, as it were – *that the rays*, *in the form of neurons*, *are preexisting inherently*. There is no need to construct them, as we do for the rays in physical optics, in order to be studied: they are born, and even creep as living entities, growing synapses that serve for communication with their likes, and with the matter continuum in the form of *glia*, since in the spirit of our analogy, one can say that *the glia is the analog of physical vacuum*. All we have to do for understanding brain's functionality is to figure out how these already existing rays in the brain – which may be considered *elementary rays* in a general Newtonian view – connect with each other, in order to form the *physical rays* that support the transmission of information through them.

In other words, in the brain universe the physical problem of rays turns out to be exactly opposite to its counterpart from the classical physical optics. In this last case, we could have at our disposal a physical ray only *as a whole*, for only such a kind of rays were technologically possible to be created by human means or, putting it differently, according to the experience in the world we inhabit. The tools serving us in the process of creating rays of light are the classical opaque screens and holes through them, as described by the old Newton in his *Opticks*. Then, the physical structure of the light rays could not be thought but only in terms of some 'least parts', which we just called elementary rays above. An elementary ray is a fundamental structure of physics, that could not be but only *inferred deductively*: it cannot be constructed physically, for we do not have the means of doing it, the means at our disposal being very slow in their action. In brain, on the other hand, the elementary rays seem to preexist naturally, *as neurons*: in this universe it is the ray structure *as a whole*, not its components, that must be inferred from their existence, deductively. Quoting from Newton:

> *By the Rays of Light I understand its least Parts*, *and those as well Successive in the same lines*, *as Contemporary in several Lines* (*original emphasis on this statement*, *a/n*). For *it is manifest* that *Light consists of Parts*, both Successive and Contemporary; because in the same place you may stop that which comes one moment, and let pass that which comes presently after; and in the same time you may stop it in any one place, and let it pass in any other. For that part

> of Light which is stopp'd cannot be the same with that which is let pass. *The least Light or part of Light*, which may be stopp'd alone without the rest of the Light, or propagated alone, or do or suffer any thing alone, which the rest of the Light doth not or suffers not, *I call a Ray of Light*. [(Newton, 1730), Definition I, p. 4; *emphasis added*, *except as mentioned*, *a*/*n*]

Obviously Newton *systematizes* here the concepts connected with the light phenomenon according to human experience and technological possibilities of his times: notice, indeed, that he only insists on the definition of the light rays by ideal instantaneous processes of 'stop' and 'let pass', that in his times could only be realized by screens and holes in them. Then, the essential problem to Newton – and of the physics, in general, ever since his times – , was the *elementary structure of light itself*, *not the structure of a ray*, in terms of the least parts: these, however, were thought to be rays too.

In the brain, on the other hand, it appears that the modern physiology revealed quite a reverse situation apparently: the *elementary rays are provided* by the inherent construction of this universe. There is no need to 'chop a ray' in order to reveal them, it is 'already chopped', so to speak. Once again, this view is only permitted by a concept of brain as a universe. The problem of rays in the brain universe, by comparison with its equivalent in the physical universe, where the light carries the information is, indeed, exactly inverse when approached in the manner of the physical optics: the *existing* elementary rays must be interlinked within physical rays, by irreversible synaptic connections in 'successive' links, and by some other kinds of connections, 'reversible' this time, in 'contemporary' links, if it is to use the old Newtonian expression.

In spite of the fact that at the time when they were first conjecured, this latter kind of connections were deemed as being "less likely" [(Kubie, 1930); Figure 6], the theory of brain rays presented here will show that they are quite natural in the brain, as *manners of association* between neurons, with important physiological consequences. It is this process of constructing the brain rays that needs to be explained after all, if we are to explain the *consciousness* or, as one calls it regularly, *the mind*. In order to do this, we shall have to appeal, once again, to the practical philosophy of those people who constructed the modern 'thinking machines', in order to see what the condition of existence of mind asks for.

Let us, therefore, take a large detour throughout the philosophy of Warren Sturgis McCulloch, who 'nailed', if we may be allowed to say so, the position of a physicist, in a way not too becoming, either for physicist or for the physics itself, expressing, however, the naked truth so bluntly. Quoting:

> By the term "mind", I mean *ideas and purposes*. By the term "body", I mean *stuff and process*. Stuff and process are familiar to *every physicist* as mass and energy in space and time, but ideas and purposes he keeps in the realm of discourse and will not postulate them of the phenomena he observes. In this I agree with him. But *what he observes is some sort of order or invariance in the*

> *flux of events*. Every object he detects in the world is some sort of regularity. *The existence of these objects is the first law of science*. To detect regularities in the relations of objects and so construct theoretical physics requires the disciplines of logic and mathematics. In these *fundamentally tautological* endeavors we invent surprising regularities, complicated transformations which conserve whatever truth may lie in the propositions they transform. This is invariance, *many steps removed from simple sensation but not essentially different*. It is these *regularities*, *or invariants*, *which I call ideas*, whether they are theorems of great abstraction or qualities simply sensed. The reason *for excluding them from physics* is that *they must not be supposed to be either stuff or process* in the causal sequences of any part of the world. They are *neither material nor efficient*. So, to my mind *Newton*, *Planck and Jeans sin* by introducing God as a sort of mind at large in the world to account for physical effects, like the action of gravity at a distance. [(McCulloch, 1950); *we used here the text from* (McCulloch, 2016); *our emphasis*, *a*/*n*]

Since the times when the great sage wrote these words, the theoretical physics ceased to be just 'tautological': the scale transitions came to 'the mind' with pressing necessities, showing that 'stuff and process' are not sufficient for physics, and we have to add to the 'invariance' a certain specific differentia: *the invariance to scale transitions*. This invariance came to characterize, in latter times, the theoretical physics. The 'ideas', even though 'neither material nor efficient', and therefore 'excluded', were always present in the natural philosophy, and, in the last part of the 20th century, they came to hit physics with the ram of necessities. What Newton, Planck and Jeans needed, in order not to 'sin', was an example to be given to physics, and, in contemporaneity, this example is the brain: *it is a universe*! Here the great sage found the right word. Quoting again:

> But, let us *compel our physicist* to *account for himself as a part of the physical world*. In all fairness, he must think to his own rules and show in term of mass, energy, space and time how it comes about that he creates theoretical physics. He must then *become a neurophysiologist* (*that is what happened to me*) but in so doing he will be compelled to answer *whether theoretical physics is something which he can discuss in terms of neurophysiology* [and *that* (*original emphasis*, *a*/*n*) is what happened to me]. To answer « no » is to remain a *physicist undefiled*. To answer « yes » is to become a metaphysician – or so I am told. [(McCulloch, 1950, 2016); *our emphasis*, *except as mentioned*, *a*/*n*]

As far as we are aware, though, neither our great sage himself, nor anyone else has really discussed *physics in terms of neurophysiology*: the discussion was always rather in reverse. In spite of this state of the facts, 'that is *actually* what will happen to us' here! However, we were compelled to learn more, for we had no one 'to tell us' anything in order 'to become a metaphysician': rather, our firm conviction is that one can remain 'physicist undefiled' and still say a « yes »! The theoretical physics is, indeed, something 'that *must* be discussed in terms of neurophysiology', for the brain is a universe, 'part of the physical world', as McCullogh says. A neurophysiologist cannot be expected to figure out something like that, for he has no concept of a universe, except for that built by physicists. Here, however, the trouble is lurking. Quoting, again:

> But is that just? The *physicist believes entropy to be somehow in or of physical systems*. It may or must increase with time. But *it is neither material or efficient*, in fact it is a number, namely the logarithm of the probability of the state. It is, therefore, a measure of the disorder of an ensemble – or a collection of systems. Now Norbert Wiener has proposed that *information is orderliness* and suggests that we define it as a negative entropy, the logarithm of the reciprocal of the probability of the state. Let us, for this argument, *accept his suggestion*. Ideas are then to be *construed as information*. *Sensation becomes entropic coupling* between us and the physical world, and our interchange of ideas, *entropic coupling among ourselves*. Our knowledge of the world, our conversation – yes, even *our inventive thought* – are then limited by the law that *information may not increase on going through brains*, or computing machines. [(McCulloch, 1950, 2016); *our emphasis*, *a/n*]

Fact is that, just about the times when McCulloch wrote the above words, the physicists were struggling to bring in their field that 'invariance many steps removed from sensations' in the name of which Newton, Planck and Jeans could be labeled 'sinners' (Jaynes, 1957). The ideas of McCulloch's circle were proved to be generating a solid principle *of logical descent* – the principle of maximum information entropy – which helped building a mathematics able to encompass the *a priori* principles of a *natural-philosophical descent* (Jaynes, 1968). The mathematics of these principles is just naturally coming to us (Jaynes, 1973), and even those facts 'material and efficient' serving the physics right, turned out to be of the same logical descent (Jaynes, 1988). In the name of this principle, people even unearthed the old idea of statistical spacetime (Calmet & Calmet, 2004).

These facts meant for us that the "mind" must be constructed on a physical support, or "body", made of 'stuff and process', as these are imagined by those old people who, in the name of "mind", constructed such "bodies": *the cyberneticians*. To their construction, we just had to add the right physical principle that made those bodies work, namely *the electricity*.

So, along this line, we just found that the scenario which we need to follow in the description of the physical structure of brain is the one first imagined by William Daniel Hillis, another one of the coryphaei of the construction of thinking machines. It is taken from one of his first works dedicated to the subject [(Hillis, 1981), p. 8]. This scenario gives the fundamentals regarding the functionality of modern 'thinking machines', in terms of their necessary constitutive elements, generically called *cells* by Hillis. What we propose here, is to use these fundamentals for the most refined thinking machine: the brain. They will give us the starting logic of construction of the societies of cells. To this end, we just present *a rephrasing* of Hillis' scenario, again, enabled by the concept of universe as applied to brain, since it seems to us that it is the most rational scenario that the Penfield's concept of *automaton* allows for. The obvious *replacements of terms* made by us in the original Hillis' text [see also (Hillis, 1984)], are then due only to physical requirements imposed by the differentiae of *our* concept of universe, in order to be applied specifically for the case of brain. Here is the excerpt, with our due replacements, from which we shall then extract the necessary natural philosophy allowing the *construction of a physics* of the brain universe:

> Unconnected *cells* can *establish connection* by a mechanism called *message waves*. This works in the manner that follows.
>
> Assume a cell C_1 needs to get *a pointer* to the cell C_2: obviously there is no correlation as yet between the two, or at least so we may be permitted to assume for a starting situation. C_1 can get such a pointer by broadcasting message waves *through the network*, searching for C_2. Each one of such messages in the wave *contains the address* of C_1, as the originator of the wave. The wave is propagated through network *via individual cells*: each cell of the network forwards the wave to its neighbors, *incrementing or decrementing the backpointer appropriately*. When the wave reaches the cell C_2, this one *sends its address back*, using the address of C_1 specified in the wave message. C_1 then *sends out a second wave*, to cancel the request which, obviously is still spreading through network. This *cancel wave propagates instantaneously*, *overtakes the original request*, and *prevents it from spreading further*. [(Hillis, 1981), p. 8; also (Hillis, 1984), p. 217; *our adaptation and emphasis*, *a*/*n*]

Obviously, the cells in the case of brain can be assimilated, for once, with the *neurons*: apparently they are the elementary rays in this universe – we do not say that they are the 'least parts', as Newton said. For, these least parts may involve the agent transmitting the information, just as in the classical case of light, and communicate along physical rays formed *ad hoc* by synaptic connections in the brain, in order to transmit *the charges that carry messages*. If, for instance, a synapse receives different quantities of electricity, it forwards this electricity, "incrementing or decrementing it… appropriately".

The above scenario still require some addendums of a physical nature, which need to be conceptually ascertained. First, there should be, according to this Hillis' scenario, *two kinds of waves* to consider in a universe, in general, as differentiae of the concept of universe: the ones *propagating through the existing network of rays*, that physically carry the messages, and the ones *propagating* '*instantaneously*', *i.e.*, in our understanding, with a velocity well beyond the possibilities revealed by any experience in the universe. These last waves carry messages too, but these messages only serve for 'cancelling' the message transmitted by the waves propagating through the network of rays. The network category of waves are, therefore, waves of the *de Broglie type*, associated with fictitious particles of the 'luxon' type which, in context, can be assumed to have the structure of the *groups of waves*. In the universe of our experience, *the waves* of such a group are known, indeed, to have speeds beyond that of light. It all depends, in fact, on the definition of the time interval. Obviously a kind of interaction must be conceived *between the network of rays* carrying messages proper, *and the cancel waves*, which, propagating 'instantaneously', cannot be envisioned, according to our experience, *but as propagating within the background*: the *vacuum* in the physical universe we inhabit, the *glia* in the case of brain.

On this occasion too, the neurophysiology can be considered ahead of physics' thinking, with two concepts that pressed upon it with the necessity of facts demanding explanation. The first of these was the concept of *current source density*, that made its appearance from the necessity of physical explanation of the fields of electromagnetic nature that go along with the brain activity [see (Tenke & Kayser, 2012) for a significant (obviously, from our point of view!) discussion and further references on the idea of current source density]. Basically, the concept of current source density, has brought to light the necessity of the concept of interpretation in the brain universe, that took, to us at least, a specific form in connection with a quantal description of this universe, in Planck's initial acceptance. Namely, it seems to confirm the idea, essential for Planck's quantization procedure, that the elementary physical structure of the matter in a universe is *a dipole*. For once, this means that the brain must be treated, indeed, as a universe, which is our central idea here in the first place. However, from physiological point of view, in order to establish a source of field, the dipole image needs to be completed with *the motion of the charges* making the currents that generate the fields [(Bazhenov & Al., 2011), see especially the Figure 3 of that work, and compare it with the Figure 1 of (Tenke & Kayser, 2012)].

Two neurophysiological phenomena can suggest the concept of Planck's resonator here – the necessary physical structure of a physical universe – and these are the *volume conduction* and the *field closure* (Tenke & Kayser, 2012). Both of them are nowadays subjects of intense physiological research, which, we should mention expressly, is almost exclusively conducted as being founded on the classical Maxwell's equations (Gratiy & Al., 2017). Both phenomena are suggested here as coming naturally in the form of *cancel waves* as accepted above and, respectively, the *action of stopping the information* of the message waves.

In this connection, let us admit that we cannot imagine a process that *stops* the waves, in general, that is, stopping with *no further physical consequences* whatsoever. Nevertheless, fortunately we must say for now, this should not be the case, at least not in the reality of the brain universe: what is actually stopped is not the wave *per se*, *but the message it carries*. And, theoretically at least, we can imagine another scenario, that does not need stopping at all: the cancel waves, which can be interpreted as waves propagating in the manner described by Louis de Broglie [(Mazilu, 2020), §2.1], that is, *through volume conduction* only, according to neurophysiology, are only *altering the information* content of a message wave, anywhere they meet that wave *within the network*. So, a message wave can propagate through the physical structure as long as it lasts, but with the address changed for instance, or with the information it carries distorted: the message wave simply *becomes useless as the carrier of a specific information*. Let us stop here, for just a moment, in order to reveal a distinctive property of exclusive physiological nature, necessary to be considered and understood properly.

For a machine, therefore talking pure physics as it were, the change in address or the distortion of information, as described above, may not be of consequence: the signal carried by wave is simply useless, and 'enters the background', so to speak, in the form of a noise. However, from a neurophysiological point of view, like in the case of brain, the propagation of such a signal may turn unfortunate, for an information still exists in it, of course, even if distorted, and may be perceived as such somewhere in the brain, *leading to neurological ailments*. Such an ailment was, as a matter of fact, the reason for introducing the 'circling waves' mentioned before (Kubie, 1930). Again, we shall present such a theory of distortion in this part of the work. But first we need to draw some conclusions helping in understanding the physics itself, for the physics has also undefined concepts here, contrary to the prevailing contemporary prejudice served by – and serving itself only to – technology. To wit: the information carried by a light ray, cannot be stopped but only along with the ray itself, which may turn out to be impossible on occasions, or by a *physical process of jamming*, which is always possible. It is this last process that we have in mind here.

As we have already said, the Hillis' scenario asks, obviously, for an already established *system of rays* in a universe, 'grown by the universe itself', as it were. While, therefore, these rays are assumed to be existent by the very nature of the brain, we need an explanation regarding the way to be seen as analogs of the regular light rays of the universe we inhabit. On the other hand, the brain has also an already established background: *the glia*, playing the part of the vacuum from the case of our standard universe, and described as an optical medium of the kind of Maxwell fish-eye, serving for propagation [(Mazilu, 2024), § 1.2]. The system of rays makes a network that serves for supporting the *message waves*, as we said, while the background serves in supporting the propagation of the *cancel waves*. These last waves are propagating 'instantaneously', with the concept of *instantaneous* still to be defined, but which, according to theoretical physics, must be described by an equation of propagation like the d'Alembert one, used by Shpilker (1984) when he has defined a reference frame for the receiver of the propagating signal (see I, §5.5).

On the other hand, if a motion must be at any rate associated with this propagation in a process of interpretation, then it cannot be but a kind of *zitterbewegung* process (Barut, 1978), of the type once imagined by Joseph John Thomson for the charges (Thomson, 1920, 1924) and used in modern times by Richard Feynman in the construction of a remarkable variant of the wave mechanics [see (Mazilu, 2024), §5.1]. In general, though, we have to deal here with what theoretical physics currently describes as *tachyonic motions*. Everything depends on the definition of 'instantaneous', which, in the most general case can be physically defined in connection with the concept of an 'instant'. At least this is how we take it: the space comprising the locations and motions which occur in an instant, according to our experience. We found a way of defining the instant, in general, *i.e*., in *any experience*, as explained based on the classical Newtonian differential calculus, which can be easily extended even to a *fractal calculus* [(Mazilu, 2024); for a preliminary definition of the 'instant' see also I, §5.1]. Geometrically, this definition contains the current physical case of *instanton*, but also covers the psychological case of *moment*, to be explained immediately.

The physics' point we want to make here, can be easily understood if we are willing to reconsider the classical theory of light rays, by admitting that this theory is – like any physical theory, actually – based on *a purely fictitious experience* and, when referring to the brain universe, we need to recall and recognize this fact. Indeed, in classical optics the rays are 'experimentally made' *instantaneously*, by a physical procedure of Newtonian type involving, for instance, screens and holes in them, as we said. In specific 'technical terms', such a construction involves *a priori an infrafinite scale of time*: the Newtonian rays are constructed by limiting *instantaneously* the bundles of light using the holes in screens, and chopping them *instantaneously* with opaque screens. In view of the fact that such operations are, *practically speaking*, *impossible*, the Newtonian rays, themselves, are essentially fictitious things, indeed. However, from the point of view of the concept of universe, this is only a property of the rays in the universe we inhabit.

In brain, on the other hand, the rays are 'almost given', if we may say so: the neurons are growing in a finite time, existing permanently afterwards, *a priori transfinitely* in time, even though this is an ideal case. They are changing as long as they exist, for they need to grow or lose synapses in order to connect with each other, according to momentary necessities, in network rays – to use Hillis' locution – a process that *can be imagined* as instantaneous too, even though it is hardly instantaneous. However, in this case one has to 'reconstruct the experience', as it were, which involves an *inverse process* when compared to that of physical optics: while in our regular experience the screens and holes in them are provided, helping Newton in constructing elementary light rays, in a 'brain experience' the elementary rays are provided, from which we must imagine how to construct the rays of a network. More to the point, for the brain universe the very experience is imaginary. And, in such an imaginary experience, we have to physically describe, and understand of course, a process of 'growing of synapses' and, moreover, a process of 'connection between rays'.

So, to conclude here, there is a kind of duality between the two universes, involving *the experiences within them*, for these experiences are necessarily different under the general concept of a universe. One can say that the neurophysiology endeavors to build a reliable experience in the brain universe, in order to make possible the construction of a physics of the brain based on this experience. Again, this way a logic emerges for manipulating the time scales: we cannot neglect *improving physics* based on physiology, on account of the very same argument that we use when claiming to *improve the physiology* based on physics! And this means exactly to fulfil the prescription of Warren McCulloch. The neurons are formed anatomically, and the synapses are grown by a *process of learning*: this is why we also need to describe *physically* the *process of growing*, as we said.

Speaking of improving physics, based on a fictitious experience in the brain, we have a case in the history of physics completely analogous to Hillis' when it comes to the type of processes described above. This analogy is particularly obvious when considering the difference between *propagation along rays* – which can be assimilated to a motion, by a special interpretation – and the *propagation in a space*, where the interpretation must be taken *cum grano salis*, so to speak. In order to settle the logic of the Hillis' argument, we feel the urge of describing this scenario right away, on a historical example in physics, insofar as it brings some control in establishing the right concept of universe. Thus, let us talk now about the *Compton effect*.

2.2. A Suggested Perspective on a Case in Theoretical Physics

The previous approach based on the Daniel Hillis' scenario, currently taken in consideration only for some all-purpose aspects of a natural-philosophical character, will be, of course, also detailed mathematically in order to make us understand the physics of brain. At this moment, however, we think worth our while unveiling some details of the same nature for the historical case of *Compton effect* in physics. This effect is one of the big cases in the theory of quanta, understood as physical entities, that enticed our intellect into considering the cosmological approach of the natural philosophy, in order to include the brain under the general concept of universe. First, it should be, obviously, mentioned the Louis de Broglie's *work on the concept of light ray* [see (Mazilu, 2020), and the related works cited there]. This specific work of the great theorist completed, in our opinion, both phenomenologically and theoretically, the classical theory of light, eventually promoting – again, we have to confess, in our opinion – the holography as the all-encompassing phenomenon of the physical optics. Let us get into some specific details, in order to explain this statement.

Louis de Broglie approached the problem of quantum in the case of light merely working on the equation of propagation of light in homogeneous mediums, – specifically, the d'Alembert equation – assuming, however, the *a priori* existence of the light rays in the physical form of *capillary* '*conduits*' within a medium that supports the propagation of light. Then he succeeded in showing that the diffraction phenomenon, *viz*., *the essential distinctive characteristic of the light phenomenon*, was entirely compatible with the corpuscular theory, provided the density of corpuscles serving for *the interpretation* of light is represented by the *square of the amplitude of the optical signal* along the physical ray [see (Mazilu, 2020), introduction to Chapter 4 and §5.1, for the concept of interpretation]. This is pretty much what de Broglie could do along the idea of interpretation of the light: his outstanding proposal of equivalence between quantal energy carried by a wave and the energetical content of the inertial mass still remained an assumption. And the Compton effect was one of the strongest arguments sustaining it, for no other *theoretical reason* of it could be found among the principles of physics of the time.

An examination of historical environment at the time when de Broglie demonstrated his optical statements, allows us the conclusion that they can be taken as parts of a right natural philosophy, which the theoretical physics has never been able to appropriate correctly (Mazilu, 2024). According to our discussion above, the existence *a priori* of the rays is a fact in the brain, just as the volume conduction is, and so we can talk of a 'physical optics' in the brain universe, explaining the physiology's construction of the specific experience. Provided, of course, the light from the 'message waves' of physics is somehow replaced by the charge: for what we know, or can infer, the *charge must be the carrier of information in the brain universe*. Obviously, from physics' point of view we can talk of a potential only in the case of the 'cancel waves', propagating through the background, so that the connection between potential and charge is tantamount to *a connection between the cancel waves and message waves*. But, let us see what really was this historical environment at the time of Louis de Broglie.

As we just said, that environment was closely associated with the Compton effect. After the relatively rapid scientific community's appreciation of de Broglie's by now classical idea of associating a frequency with a particle, thus making, as de Broglie used to say, 'a wave phenomenon out of the classical material point', this effect was the only one allegedly connecting *directly* a quantum of light with a particle, in a conservation law of the classical type. The need then occurred to build "a more adequate picture of optical phenomena than previously existed" (Slater, 1924) for the interaction between light and matter, and the general idea was, apparently, that the basis of such a construct must be the *Bohr*'*s correspondence principle*.

A significant work of theoretical physics or, better yet, of *a modern natural philosophy*, which can be considered – by its intent at least, even if we neglect the consequences – as ranking equal to that of Newton's, from this point of view, thus appeared at the time [(Bohr, Kramers, & Slater, 1924); the so-called *BKS theory*]. That work, was elaborated on the ground of thesis of John Clarke Slater mentioned by us earlier, in the introduction to this chapter. It was destined to connect the completely new Bohr's ideas on the relation between the stationary states of fundamental matter structures, with the frequency associated to a quantum. Quoting:

> Any atom may, in fact, be supposed to *communicate with other atoms all the time it is in a stationary state*, by means of *a virtual field of radiation*, originating from *oscillators having the frequencies of possible quantum transitions*, and the function of which is to provide for *statistical conservation of energy and momentum by determining the probabilities of quantum transitions*. *The part of the field originating from the given atom itself* is supposed to *induce a probability* that that *atom lose energy spontaneously*, while radiation from external sources is regarded as inducing aditional probabilities that *it gain or lose energy*, *much as Einstein has suggested*. The *discontinuous transition* finally resulting from these probabilities *has no other external significance* than simply *to mark* the *transfer to a new stationary state*, and *the change from the continuous radiation* appropriate to the old state to that of the new. [(Slater, 1924); *emphasis added*, *a*/*n*]

As one can see, the Bohr's postulates on the manner in which radiation and matter interact have left a deep impression of something 'unexplainable' on the scientific community: *a sudden event*, if we may be allowed to use an expression which can be taken by some as a pleonasm – after all, one certainly can think that an event, at least in space and time, cannot be but sudden – *had to be understood physically*. This needed a special natural philosophy, in order to incorporate the new facts into physical thinking. In our opinion this natural-philosophical character of the BKS theory explains its much-discussed influence on the physics that followed: after all, it succeeded to fulfil *this* purpose. For once, Slater's conception can be taken as equivalent to that of Planck's original procedure of quantization, in that it appeals to 'virtual oscillators'. If his idea could not come from the very original Planck's procedure – it was referring, in fact, to isolated events independently of any concept of a universe, which, in context, entered the play only much later, through the works of John Archibald Wheeler and Richard Feynman – *in hindsight*, at least, we can associate it with a classical procedure of regularization of the Kepler motion, whereby the virtual oscillators play a significant part [see (Mazilu, Agop, & Mercheș, 2021), Chapter 3]. And the regularization is capable, in turn, of explaining the Kepler structure at any space scale, in an appropriate time scale, which thus can be taken as an essential property of any world.

Fact is that the main thesis put forward by the work of Bohr, Kramers, and Slater, namely that of "statistical conservation of the energy and momentum", was rejected *experimentally* right away by Walther Bothe and Hans Geiger in Germany, and respectively by Arthur Compton and Alfred Simon in the USA. The history of this rejection is rich, and we may be allowed to defer the case to a later, dedicated work. For now, however, we just have to take heed of a general feeling that the case was not 'closed' at all, actually, by that rejection from the year 1925, a fact for which we can even give the logical reason just mentioned: an event *belonging to a universe* must be explained only *as such* – that is, as an event *per se* – and this kind of an explanation has never occurred in physics, even today. Everything in matters of principles was left to quantitative decisions at the time, in view of the almost naturalized principle of referring everything to experiment, thus leaving, in turn, the door open to subjectivity and, of course, to doubt. To wit: after ten years of 'closed case' in the BKS matters, Robert Sherwood Shankland reopened it, based on new experimental data (Shankland, 1936). What we are interested in here is only *the theoretical intervention* destined to 'save the day', as it were, on this occasion, from which we extracted conclusions according to the character that we already assigned to the BKS work: natural-philosophical guidance for the modern approach of physics.

First, we need to show just how the Compton effect came around on this occasion, and this could be done easily, in view of the already existing concept of *wave surface* in physical optics (Stoner, 1925). The Rutherford atom was already in place as a physical structure at that time, so an atomic transition was naturally conceived as a result of the interaction of the wave surface with the electron from the physical structure of the atom. This directs our process of thinking, right away, towards a Compton effect: *the light produces a new electron from an old one*, *in an encounter that changes its frequency and direction of propagation*. This encounter can be physically described by the *exact* laws of conservation of energy and momentum, as Arthur Holly Compton has shown for the first time ever, based on the Planck's idea of quantum, just about the time when Louis de Broglie introduced his fundamental thesis to the public awareness (Compton, 1923). This demonstration was plainly done in a classical manner of approach, inasmuch as the light was cogitated in terms of some classical particles – only later baptized by Gilbert Newton Lewis as *photons* (Lewis, 1926c) – which was one of the first genuine *cases of interpretation* in the history of the wave mechanics, counterpart of the Einstein's case of interpretation from 1905 in the case of quantum mechanics. And so the trouble started brewing, with the attempts of understanding the physical process based on the existing concepts. Mainly targeted in the case were the concepts of optical physics. Quoting:

> In whatever form radiation exists in free space there can be no doubt that *matter can only gain or lose radiant energy in complete quanta*. If as a result of a *quantum switch* an atom loses an amount of energy *E*, *this energy must then be present in free space in a form*, and *with properties which must be deduced from the behavior of matter surrounding the emitting source*. From this it is concluded that radiant energy travels with a velocity *c*, and that a quantum emitted may be reabsorbed as a whole. The question at issue is whether *quanta emitted from atoms spread spherically*, *with power to collapse at one point*, *or travel linearly*. [(Stoner, 1925); *emphasis added*, *a*/*n*]

As we said, the physics involved in such demands is left for a future occasion of development, recommending, for the moment, the original works to the interested reader. Here we just have to notice that in the quantization issue, *the very Planck's procedure was conveniently forgotten*. It was swamped, so to speak, within the new current of ideas which sprung from Einstein's suggestion of 'gas of quanta' from 1905, based on the Wien's limit of the Planck's law of radiation (Einstein, 1965). No one could see how a quantum, having the properties of a particle – as practically demonstrated by the already established, at the time, Compton effect – especially the property of moving in a straight line, can spread spherically. So much the more, no one could see how a spherical wave – if we even forget about the quantum as a particle – can be stopped by a process of absorption in one single point in space vhere the wave arrives at a given time.

The trouble progressed around this argument, for which the BKS theory promoted Slater's statistical idea, thus allowing immediate rejection (Bothe & Geiger; Compton & Simon, 1925): *the conservation of energy and momentum cannot be statistical*, *it is an exact law*, *just like in the classical case*. Mention should be, however, made that the experiments that led to rejection were *coincidence experiments*, involving, therefore, *the concept of simultaneity* in some way. We see here a concept of universe involved, along with that of simultaneity, and if we take heed of this indication, then the Planck's procedure of quantization gets into play with a possible logical explanation of the facts [see (Mazilu, 2023, 2024), *passim*]: Planck's background optical medium is a Maxwell fish-eye medium, whose fundamental 'virtual structure' is not simply an oscillator, but a dipole involving, somehow, the electricity and magnetism. In the historical case of Planck, it was a purely electric dipole. This fundamental physical structure started reclaiming its right place in theoretical physics only in the second half of the 20th century under the influence of idea of *asymptotic freedom* [see, for instance (Callan, Dashen, & Gross, 1978); see also (Schwinger, 1969), which is an instrumental work in understanding the asymptotic freedom, as we shall see later on]. From geometrical point of view, the Riemannian structure of this background involves special geodesics on spheres, so that the straight propagation between the two charges of the dipole is governed by a statistics, indeed, but of the kind of those involved in the physical theories of hidden parameters. The light itself can be seen as a continuum characterized by a Maxwell tensor whose measurement involves *two fundamental statistics* that can be tied up with the forces aroused by light in the optical medium [(Mazilu, 2024), §§6.3–5]

Nevertheless, the situation ought not to come even to this, since Einstein's quantization is not a proper quantization in the Planck's sense: for once, it *is not referring to the complete radiation*. Indeed, we just have to recall again that, in getting his suggestion, Einsten was using the particular situation where only the Wien's radiation law was valid, and so the association of a quantum with the properties of a particle may be taken as doubtful from the start. To wit: the thermodynamical equilibrium, that allowed Einstein's conclusion, is only incidental, but such an incident *does not* endorse the further conclusion that the quantum can be a virtual particle moving along the geodesics of the optical medium, thus being capable of determining events. Again, it is our opinion that this was the case with the BKS theory, for which the Robert Shankland's experiments opened a new wave of ideas of which we, personally speaking, availed in full: it is just a *new* natural-philosophical methodology.

The logic of such a theory, as connected to the phenomenology of the Compton effect, is well presented succintly in a note of Rudolf Peierls issued on the occasion of Shankland's work (Peierls, 1936), which we recommend for a 'global' view of the situation, as it were. However, the theoretical hypotheses triggered by this incident are those that interest us most, especially for their methodological value: on such a critical occasion the spirit is always capable of issuing ideas out of the routine way of thinking, instituted by academies and endorsed, as such, for public use and debate. The first one among physicists, who opened a longer line of discussions with some conclusions of this sort, was Paul Dirac, who considered the Shankland's data as "more accurate" than those of (Bothe & Geiger, 1925) and (Compton & Simon, 1925), which led initially to the rejection of the BKS theory, just as soon as it was put forward in 1925. This was one of those conclusions based on quantitative experimental data. Adding to this the general aspect of the theoretical physics at that epoch, which even had to invent a particle – the neutrino – in order to save the law of conservation of energy – then in a great danger due to the experimental results regarding the behavior of particles desintegrating with the participation of the *beta*-processes – Dirac concluded that Shankland's experimental results actually *confirm* the BKS theory. Quoting:

> The above experimental results suggest that we take as the starting-point *in our reformulation of atomic theory* the assumption that *energy and momentum are conserved in atomic processes in which the velocities concerned are all small compared with the velocity of light*, but *are not in general* conserved in processes involving large velocities, including radiative processes. This assumption is all the more plausible on account of the fact that *the present quantum mechanics*, *with its conservation of energy and momentum*, forms a satisfactory theory only when applied non-relativistically, to problems involving small velocities, and loses most of its generality and beauty when one attempts to make it relativistic. In this way, we see that we can retain the whole of the present non-relativistic quantum mechanics, and we see the need for a *profound alteration in current theoretical ideas*, involving a departure from the conservation laws, before we can hope to get a satisfactory relativistic quantum mechanics. [(Dirac, 1936); *emphasis added*, *a*/*n*]

And, as well-known, the great sage did for physics just as he says in this excerpt: 'profound alterations'. We took from this line of Dirac's thinking, a moral regarding the essentials of his method of linearization of the d'Alembert equation – basically the spin and isospin formalism – and applied it only to charges, by the way of mathematical principles that led to special relativity. The Dirac's 'profound alteration' thus stays here precisely in the fact that the motion of charges in matter is akin that motion which, through the work of Erwin Schrödinger (1930 b) came to be known as *zitterbewegung*. This idea was advanced for the first time two years before the appearance of Schrödinger's work, by Gregory Breit (1928) in the United States.

However, just forgetting the quantitative considerations, a first subtle point of change – along the lines of a natural philosophy – in the theoretical views, straightforwardly enticed by Shankland's experimental results, was suggested by Evan James Williams, who directed the attention of physicists upon the uncertainty principle of Heisenberg (Williams, 1936). As the experiments in question were *coincidence experiments*, 'if the results of Shankland were correct', and

> … assuming *with Dirac* that *they mean that there is no coincidence in time between the* '*scattering*' *by* and *the recoil of*, *an electron*…[(Williams, 1936); *emphasis added*, *a*/*n*]

one had to either decide or assume that both *the electron of recoil* and *the secondary quantum occur in the same event*. For this reason Williams adopted a remarkable hypothesis:

> If it is certain that a scattered radiation field has originated within a *certain volume* δv, *and nowhere else*, then there must have been a scattering particle in that volume at the time concerned. [(Williams, 1936); *emphasis added*, *a*/*n*]

We see the rationality of this hypothesis in the fact, only implicit in it, that to a moment of time marked by an event – the moment of scattering – *there should be a volume* associated with the simultaneous possibilities for this event. Moreover, if this is the right meaning of the uncertainty principle, the very moment of time associated to an event becomes itself a random variable in a certain time range. Pending the concept of interpretation, this is the essence of the modern idea of gauging of the forces that stay at the foundation of the classical mechanics (Berry & Klein, 1984). As we see it, under the same circumstances, the same should be true for the idea of quantum, considered from the perspective of the wave mechanics (Berry, 1984). Fact is that there should be a dichotomy in the case of the concept of quantum – *quantum as quantity vs quantum as physical object* – on which we shall turn a little later here. So, based on his hypothesis, Williams proved that Shankland's results were contradicting the uncertainty principle, and concluded cautiously, only with a recommendation:

> *As the results of earlier work are not in accordance with those obtained by Shankland*, it is desirable that *further information be obtained regarding the conservation of energy and momentum in individual quantum processes*. [(Williams, 1936); *emphasis added*, *a*/*n*]

However, having in mind the observation right above, and the very discussion of Arthur Holly Compton on the occasion of the classical work that introduced the *Compton effect* to our awareness (Compton, 1923), we conclude that that 'certain volume' of Williams is of essence: *it is the* '*volume of a quantum*' so to speak, where the quantum must be understood here under the rational concept of a physical object only once put forward in physics by Hendrik Antoon Lorentz (Jeans & Al., 1913), on which, as promised, we shall return immediately with a few details.

Talking of coincidence of events and relativistic individual processes, one just can guess that the relativity should be involved in such necessary changes, as those professed by Paul Dirac. Obviously, Niels Bohr had himself a word to say in this case, and he did it, placing the culprit on the 'atomistic nature of electricity', which at that time, as even today in fact, *could not be put in connection with the Planck*'*s idea of quantum*:

> As the fundamental relations *between the wave and particle aspects* of light and matter *can be expressed in full conformity with the relativity principle*, the still unsolved difficulties of quantum electrodynamics, emphasized by Dirac in connection with this discussion, can scarcely be attributed to any *incompatibility between the foundations of quantum theory and relativity theory*. *The root of these difficulties* may rather be looked for in *the atomistic nature of electricity*, which is as *foreign to classical physical theories as the quantum of action itself*. The rational incorporation of these different aspects of atomic problems in a comprehensive theory will probably claim *entirely new points of view*, taking the essentially atomistic structure of all

measuring agencies into consideration; but at the moment *there would seem to be no reason to expect that this would involve any real departure from the conservation laws* of energy and momentum [(Bohr, 1936); *emphasis added*, *a*/*n*].

Thus, we have, for the first, *and only time in history*, explicitly expressed an implicit overwhelming theoretical problem here, which could not be solved even today: *is the* '*atomistic nature of electricity*' *connected to the quantization*? As we shall show here a little later, if the quantization is taken in the form originally established by Max Planck, and strictly according to Planck's original procedure, this is, indeed, the case: *the atomistic nature of charge can be taken*, indeed, *as a consequence of the quantization*, which is a law controlling any universe, the *only* law of nature, in our opinion, for it is enticed by the fact that *we are* 'compelled to account for ourselves in the universe', as Warren McCulloch would express it. No doubt about that! Moreover, this 'atomistic nature, like the quantum itself' is not at all 'foreign to classical physical theories'. As a matter of fact, it was with us from the very beginning of the modern physics, only we have not realized it as such. Speaking just on behalf of the classical mechanics: the Newtonian forces, going inversely with the square of distance, are an expression of this law. On behalf of the wave and quantum mechanics, on the other hand, we can state a deep reason of this state of the knowledge: *it was not recognized that the quantum concept contains a fundamental gnoseological dichotomy*.

But, let us continue with the present streak of discussion: the suggestions around BKS theory followed their own paths, for once by the observation of the obvious fact that the Robert Shankland's experiments were essentially different from the previous ones of Bothe & Geiger and Compton & Simon, in a phenomenologically explicit way: while Shankland used Gamma-rays, the previous experiments used just X-rays. Moreover, the original Compton experiments that established the famous effect (Compton, 1923) were referring to a special type of matter: the 'light elements', for the data on the scattering of X-rays. The idea was then advanced that the two photons – that of X-rays from 1924, and the one of Gamma-rays from 1936 – must be different as particles, whence the difference in the results. As far as we are concerned, the situation has remained at this stage ever since, but some new suggestions on this occasion have been aroused and submitted to our theoretical awareness. For once, that meant, – again, in our views – that the 'Lorentz's quantum' should be entirely realistic concept.

Félix Cernuschi, to continue our list of suggestions related to the BKS theory, has advanced an equivalent idea to that of different photons, but having strong ties with the conception of 'particle as a universe'. This was a case detailed later on by Alphonsus John Fennelly in the form of the existence of a 'Gödelised hadron', as he calls it, which is actually a kind of instanton, once it is connected with a Riemannian-geometrical structure (Fennelly, 1974). Cernuschi advanced the opinion of an *excited state of the electron*, or *of a particle*, in general, for that matter. Quoting:

> We shall admit that *the simultaneity* and *the conservation of energy and momentum* apply to elementary processes, but, *for energies higher than a million volts*, *the electrons may be excited at the moment of collision*. Our knowledge on the elementary particles is for the moment quite insufficient, and probably a few unusual experimental results, which seem that are not confirming the principle of conservation of energy, might have an explanation in accordance with this principle, given *a modification of our ideas on the nature of elementary particles*. Some experimenters have found, for instance, different values for the mass of neutron; it is possible that these differences are not altogether due to experimental errors, but are produced due to different states of excitation of the neutrons. Regarding the electrons, there is not, for the moment, any experimental fact proving that they *cannot have excitation states*. We, therefore, find useful giving an explanation to the *Shankland*'*s phenomenon*, without assuming the non-validity of the principle of conservation of energy, which was so fruitful for Physics, *starting from the hypothesis of the excitation states of the electrons* [(Cernuschi, 1936); *our translation*, *emphasis added*, *a*/*n*]

And Cernuschi gives, indeed, an explanation along the lines suggested by Peierls. If such a case may occur, then the question is aroused *as to the charge associated with the excited state of the electron*, for which we need to provide a theory. This idea seems to be very old in the theory of electricity: for instance, to Poincaré, it appears in the form that the charge is the expression of certain kind of *deformation of the matter* characterized by a *constrained equilibrium* up and above the usual mechanical equilibrium of the *neutral matter* [see (Poincaré, 1890), Chapter II: *The Hypotheses of Maxwell*]. It is in this deformation that we see the appropriateness of the phenomenon of 'incrementing or decrementing the backpointer', requested by a philosophy of constructing the physics of brain as the one enticed through a Hillis' scenario of communications.

At this point, an old idea of Sir Joseph John Thomson (1920) came to our mind, in order to fill in just naturally for some important missing points of physics: the potential energy to which Cernuschi's observations are referring is, according to Thomson, connected to the *kinetic energy of the interpretative particles* of the instanton representing an electron, which move over the space of instanton at the speed of light. Therefore, Cernuschi's idea turns out to have a natural philosophical basis: if the speed of light is a universal constant, then the potential depends on the inertial characteristics of the interpretative particles over the space ocupied by electron. However, it may turn out that some kind of anisotropic relativity of Boltyanskii type should be in force in such a universe [see (Mazilu, 2024), §5.3]. In this case, the gravitation field enters the scenario, by regulating a kind of Yang-Mills structure of the universe [see (Mazilu, 2024), §6.5]. In any case, Félix Cernuschi's observations have opened, for us at least, the mathematical possibility of describing the general concept of a universe according to Fennelly's idea of a 'Gödelised hadron' and, moreover, to figure out how these universes interact. These issues will be touched in due time in the present work.

A final suggestion, due to Banesh Hoffmann, Allen Shenstone, and Louis Turner, concentrates again on the *concept of coincidence in time* and, implicitly, that of *simultaneity*. However, they drew attention on a possible misunderstanding regarding the experimental methodology and, in our opinion, this suggestion calls for a statistical theory of the concept of time, just as the Williams' observations do. As we shall see later on, the statistical theory in question is connected to the idea of *psychological moment*. Quoting:

> One important aspect of the matter has received but scant attention, however. In so using a photon counter, which is set off principally by the recoil electrons coming from the wall of the tube, *one is assuming coincidence in time between the arrival of the photon and the ejection of the electron*. This is practically *equivalent to the assumption of the correctness of the simple theory*. Thus, *the instrument used in the experiment is used in a manner which presupposes one particular outcome of the experiment*. To be sure, if there were a *time lag* between the arrival of a photon and the ejection of a scattered electron which varied in random fashion the experiment could give no true coincidences for a double reason – the *failure of coincidence both in the scatterer and in the photon counter*. [(Hoffmann, Shenstone & Turner, 1936); *emphasis added*, *a*/*n*]

This conclusion obviously implies, as we just said, a time statistics, which, again, can be developed in connection with the concept of charge, and we shall do it here according to the idea that the events are 'particles' whose ensembles offer the interpretation of a universe. The bottom line is that this interpretation respects in all details the definitions once put forward by Hertz on the occasion of his revision of the principles of mechanics [see (Hertz, 1899), p. 45]. To wit: a particle, as defined by Hertz, is a *means of correlating* two positions in space at different times. The wave mechanics only added the fact that in order to do this job, a particle needs first to *indicate a position*, and this is an essentially random event on the space of an instanton.

One final word of justification: why must we consider this historical case? Having in mind the concept of brain as a universe, the answer is quite simple: the BKS theory indicated that, as a physical structure, *the atom of Rutherford is an elementary ray*, *just like a neuron*. The analogy can be even taken as an equivalence: these 'devices' work the very same way. There should be, therefore, a good profit, for both the physiology and physics, when further delving into this analogy. This requires a certain setting up of the Planck's concept of dipole, but from the general point of view of a classical phenomenology of the charge, as once presented by Ernst Katz (1965). This point of view would logically imply not only the consideration of classical dipoles of the electric-electric or magnetic-magnetic type, but also dipoles of magnetic-electric or electric-magnetic types, usually known in theoretical physics as *dyons* (Schwinger, 1969). For the moment, therefore, it is worth saying a few words regarding the cosmological character of the problem of charge, in order to 'pave the way', so to speak, of that general theory we have in mind.

2.3. The *Status Quo* of Relativity, Charge, and Quantum

The substance of our last work dedicated to a historical and logical justification of the *concept of instanton* (Mazilu, 2024), can be best understood considering some 'words of the wise', as it were, that can be found in a remarkable Solvay Report dedicated to the physical structure of the universe we inhabit (Gold, 1958). It appears that what the physics did for cosmology is only a reconsideration of the ideas regarding charges. These ideas would have to be superimposed on that 'symmetrical view' of the world, that cosmology constructed according to the Maxwellian theory of electromagnetism. Certainly, such a state of the case reminds us of the Niels Bohr's observation on the fundamental part played by the charges in a right theoretical physics. Quoting:

> *Symmetry with respect to the sign of the electric charge* and *with respect to mirror reflection* was thought until recently to be separatelly obeyed by all the laws of physics. Now the discovery of the non-conservation of parity in weak interactions implies that *the laws are not invariant to a mirror reflection alone*. *A certain* « *handedness* » *is shown to be resident in elementary particles*. Since a right-handed screw becomes a left-handed screw when viewed in a mirror, such a particle *will be transformed into something different by reflection*. But this does not force us to believe that nature is not mirror-symmetrical, for there may be a complete symmetry between matter and anti-matter. For every right-handed particle of matter, there may be a left-handed one of anti-matter having all the same properties, but possessing the *opposite sign of electric charge* or *magnetic dipole moment*. I suppose that most physicists now would regard this as the most likely situation. *If symmetry is preserved only with respect to the combined operation, then this could really be understood best if charge had a geometrical representation*, *possessing a* « *handedness* » [(Gold, 1958), *emphasis added*, *a*/*n*]

This is what we are after here: some 'handedness' to be attached to the charge, for the charge has never been defined otherwise in connection with the matter, but always with some 'strings attached'. One can even say that, with Paul Dirac, such 'strings' became a theoretical necessity (Dirac, 1931). But, let us go on with our discourse, in order to find the logical stream of facts.

We have analyzed the foundations of special theory of relativity, in order to find the ontological reasons for the existence of the Lorentz's transformation [see (Mazilu, 2024), §2.2]. Sure enough, that old reasoning contains a hint on the connection between the charges and the Lorentz transformation. It is the space coherence of the matter, which, daily manifest in our experience, became critical for physics, once the Michelson-Morley experiment brought the undeniable evidence of the fact that the ether is not 'dragged' by matter. We should even say that it *cannot be dragged*: according to the de Sitter's view of the general relativity of Einstein's, there is a background continuum, allowing us to think of charges in a specific way [(Mazilu, 2024), § 3.3]. That is, the charges are existent potentially everywhere in the universe, regardless of whether the matter exists everywhere or not, and regardless of any scale of space or time. However, continuing the idea of physics here, according to Poincaré, the Lorentz's reasoning for introducing his famous transformation, was that the equations of electromagnetic ether are not changed by the motion. Whence the conclusion that a physical system in motion through ether must be the accurate replica of the same system at rest. But this conclusion, in our opinion, must be seen also as *an invariance of the internal forces* responsible for the space coherence of the physical systems with respect to the motion represented by Lorentz transformation, *viz*. the Newtonian forces.

Fact is, however, that the Einstein's theory of relativity is strictly referring to the physics that can be deemed as 'one-sidedly' constructed, so to speak, *i.e*., based exclusively on the properties of transformation *of the Maxwellian electromagnetic continuum*. The forces of cohesion of the matter were left, in this construction, to be described by the same Maxwellian electromagnetism, but as a case apart. Since the charge in the universe we inhabit seems to be perceived only in *a priori countable* portions, our imagination could not go as far as considering such possibilities as the transformation of charges *per se*. So, that case which the Maxwellian electromagnetism had to describe, was conducted only in the manner imposed by the space-time transformation, with an *a priori* type of invariance of the charges: *invariance of the measured quantities*. Whence the impossibility of association of the charge with the quantization procedure, thereby entertaining the idea that the charge cannot be tied up with the quantization, a fact noticed as an essential impediment in physics at large by Niels Bohr, as shown in one of the excerpts above. In our opinion, though, observations like those of Thomas Gold's one, of a cosmological character, brought by us in the excerpt right above in this section, compel us to considerations of *continuity* of charge, – taken in the mathematical sense of *cardinality*, as opposed to *countability* – thus placing this physical magnitude on a par with the color in the Riemann's views of multiply-extended quantities [(Riemann, 1867 a); see, for an explicit presentation of this point of view, (Schrödinger, 1920)]. This continuity of the charge, which seems to be the systematic case in a universe like the brain, allows us to construct a geometry of the essential physical magnitudes governing the world we live in – mass and the charges (see I, §§4.2, 4.3 and 4.4) – and, in fact, in any other world where the essential law of existence is that of Planck's quantization. This conclusion appears as a differentia of the concept of universe, and, as a physical property of our experience, it seems to be encountered chiefly within the world of brain.

The necessity of existence of Hillis' network of rays, has been demonstrated in such a world, in the form of an idea of *simultaneity* of existence of matter structures, just as required by Slater's thesis of the BKS theory. It is, for physics at least, an attractive notion that this idea could not be presented but in the form used by Louis de Broglie in the description of the optical rays, therefore involving only the light. So, the notion became 'a fact', and this fact 'closed the door', as it were, for any other possibility of construction of the rays, in spite of the other facts existing in physics for a long time. We are thinking specifically to the well-known situation from the case of physical charges, whereby the necessity of networks has long been realized under the pressure of technological needs, in order to describe the moving charges. To wit: the Kirchhoff's laws describing the *electric nets* have appeared, with their consequent explanation of the Ohm's law. Not even the necessities of physiological nature for describing the *electrocardiograms* were able to 'open' that door again (Gabor & Nelson, 1954), in spite of the overwhelming physical necessity, so that the concept of ray in the problem of interpretation still remained a fiction, exactly as it was for Newton once, and as Louis de Broglie, for that matter, presented it. This situation had the important consequence that what was a natural fact had to be 'declared' as hypothetical. Quoting:

> I am going to make the contrary *assumption* (namely) *that an atom never emits light except to another atom*, and to claim that *it is as absurd to think of light emitted* by one atom regardless of the existence of a receiving atom *as it would be to think of an atom absorbing light without the existence of light to be absorbed*. I propose to eliminate the idea of mere emission of light and substitute the idea of *transmission* (*original emphasis*, *a/n*), or a process of exchange of energy between two definite atoms or molecules. Now, if the process be regarded as a mere exchange, *the law of entire equilibrium*, which I have recently advanced, *requires us to consider the process as a perfectly symmetrical one*, so that we can no longer regard one atom as an active agent and the other as an *accidental and passive recipient*, but both atoms must play coordinate and symmetrical parts in the process of exchange. [(Lewis, 1926 a), *emphasis added*, *except as mentioned*, *a/n*]

Obviously, the general optical notion of propagation of light, whereby the ray cannot be fathomed as an existent physical object, has imprinted in our mind an attitude that makes this very philosophy a matter of assumption. Simply put, the *physical* absence of natural light rays in the universe we inhabit, and the idea that the atom was corporeal, are the obvious sources of hypothetical scenario from the excerpt above. However, let us just think of what physics has brought instead: *the propagation*. According to physics, the propagation must be described in spacetime, and according to this description the 'length' of the 'path of light' between two bodies is null: the *proper time* of light is always zero in special relativity.

Quite as obvious, however, in the case of brain, taken as a universe, a *fictitious experience*, this time, would tell us that this is, indeed, the case: the physiology realized, for a long time now, that the rays are formed by linking neurons together, either 'series' or 'parallel', like in the electrical circuits – there is a zero proper length between them – and this is the way of constructing Newtonian rays in this universe, whereby the light is replaced by charge. Had the brain been accepted as a universe, the physics might have had the chance of an analogy regarding the idea of ray a long time ago, perhaps long before Einstein; but this was not the case. In fact, such an analogy – however with a capillary tube this time – was the whole idea of the ray optics constructed by Louis de Broglie (1926), but even it seems to need some reconsideration, resorting on an old idea of Riemann, whereby the Lorentz's theory of electrical matter can be electrodynamically explained by a continuous variation of charge [see (Mazilu, 2024), especially §1.5].

Along this path of evolving of ideas, people went as far as considering *a mixture of field and quanta*, these last being taken in the sense of a meaning assigned to them by the Compton effect (Tolman & Smith, 1926). However, the necessity of waves occurred, in a place where, in our opinion, a *Lorentz quantum* for the structure of the field was meant, but the physics did not have such a notion. Quoting:

> In conclusion the writers would like to state a possible method of regarding the conflict between the wave theory and the light quantum theory which seems heuristically adequate for describing the present facts. Since *both theories contain elements of truth* let us regard waves and light quanta as *both* (*original emphasis*, *a*/*n*) *present in a radiation field*. The energy is carried by the quanta and they move in straight lines except when deflected or absorbed and then they *obey at least statistically the laws of the conservation of energy and momentum*. The waves carry no appreciable energy *but provide the signaling system* by which in accordance with the laws of interference the atoms can "know" whether or not they are allowed to interact with an oncoming quantum. On the basis of this point of view the action of light can occur only at those places and those times where both the wave theory and the quantum theory would permit such action. The facts of interference show that quanta cannot act unless the wave theory permits, while the experiments of Compton and Simon (1925) show that radiation can act only at those *places* (*original emphasis*, *a*/*n*) where *the quantum theory would predict the passage of a light quantum*, and the experiments of Bothe and Geiger (1925) show that radiation can act only at those *times* (*original emphasis*, *a*/*n*) when *the quantum theory would predict the presence of a light quantum*. Each theory thus restricts the predictions of the other. [(Tolman & Smith, 1926a); *emphasis added except as mentioned*, *a*/*n*]

We think that a clear idea surfaces from the cosmological approach of the problem of charge and quantization from these proposals, which we try to develop in the present work: *the simultaneous existence of the rays and waves*. While waves must be described by the mathematical methods of background propagation, the rays are connecting the material structures, and need to be conceived in the manner indicated by Louis de Broglie, just about the same time with the work from which we took the above excerpt (de Broglie, 1926 b, c). Of course, we need to be a little more broad-minded in regards of the idea of vicinity of physical structures, and this seems to be the whole moral of the special relativity. The case of de Broglie's theory here just shows what was missing in the scenario of Tolman and Smith: a demonstration of the fact that 'the interference and diffraction phenomena are understandable by the corpuscular theory of light', as de Broglie expressed it (1926 b).

One explanation we owe to our reader, who may have asked, and many times by now: what in the world is the thing we keep calling a *Lorentz quantum*?! We reproduce here the original definition, which is actually *the only* natural-philosophical characterization of a quantum ever, made by Hendrik Antoon Lorentz at a meeting from 1913 of the British Association for Advancement of Science, and published in a celebrated report by James Hopwood Jeans. Quoting:

> Now it must, I think, be taken for granted, that *the quanta can have no individual and permanent existence* in the ether, that *they cannot be regarded as accumulations of energy in certain minute spaces flying about with the speed of light*. This would be in contradiction with many well-known phenomena of interference and diffraction. It is clear that, *if a beam of light consisted of separate quanta*, which, of course, ought to be considered as mutually independent and unconnected, *the bright and dark fringes to which it gives rise could never be sharper than those that would be produced by a single quantum*. Hence, if by the use of a source of approximately monochromatic light, we succeed in obtaining distinct interference bands with a difference of phase of a great many, say, some millions, of wave-lengths, we may conclude that *each quantum contains a regular succession of as many waves* and that *it extends therefore over a quite appreciable length in the direction of propagation*. Similarly, the superiority of a telescope with wide aperture over a smaller instrument, in so far as it consists in a greater sharpness of the image, can only be understood if each individual quantum can fill the whole object-glass.
>
> These considerations show that *a quantum ought at all events to have a size that cannot be very small*. It may be added that, according to Maxwell's equations of the electro-magnetic field, an initial disturbance of equilibrium must always be propagated over a continually increasing space. [Lorentz, in his contribution to (Jeans & Al, 1913), pp. 381 – 382; *emphasis added*, *a*/*n*]

Notice here, indeed, the natural-philosophical definition of this *concept of quantum*, but *as a physical object*, not as a quantity: it is accounting for both the facts of experience in the world we inhabit, as well as the theoretical facts of physics and optics. It is the only concept of quantum thus defined, for no other concept acounts for so many facts. And we are using it as guidance in the brain-universe, in order to describe the behavior of charge. Inasmuch as currently the quantum is understood in physics as a mere quantity of energy, and, on the other hand, the de Broglie's idea involves an *ensemble of waves* – in particular, a group of waves – which is in fact, a Lorentz quantum as presented right above, we will reserve this name for the *physical object* in question. This way the confusion will at least be diminished, if not eliminated altogether: we simply have to accept that the concept of quantum has a fundamental *dichotomy*, as we said before. The de Broglie's idea of associating a wave with a particle must therefore be seen as the necessity of having an interpretation for a Lorentz quantum.

In order to better grasp here the concept of interpretation by the way of example, with the tangible example of the light itself, let us turn again to Gilbert Newton Lewis, who defined the photon *not as a quantum*, specifically in order to avoid observations like those of Lorentz above, but as an *interpretative particle*, going as far as sketching the properties of the ensemble offering the interpretation of light. Quoting:

> Had there not seemed to be *insuperable objections*, one might have been tempted to adopt the hypothesis that we are dealing here *with a new type of atom*, an *identifiable entity*, *uncreatable and indestructible*, *which acts as the carrier of radiant energy* and, after absorption, persists as an essential constituent of the absorbing atom, until it is later sent out again bearing a new amount of energy. If I now advance this hypothesis of a new kind of atom, I do not claim that it can yet be proved, but only that a consideration of the several objections that might be adduced shows that there is not one of them that can not be overcome.
>
> It would seem *inappropriate to speak of one of these hypothetical entities as a particle of light*, *a corpuscle of light*, *a light quantum*, or *a light quant*, if we are to asssume that *it spends only a minute fraction of its existence as a carrier of radiant energy*, while the rest of the time *it remains as an important structural element within the atom*. It would also cause confusion to call it merely a quantum, for later it will be necessary to distinguish between the number of these entities present in an atom and the so-called quantum number. I therefore take the liberty of proposing for this *hypothetical new atom*, which *is not light but plays an essential part in every process of radiation*, the name of *photon* (*original emphasis here*, *a/n*) [(Lewis, 1926c); *emphasis added*, *except as indicated*, *a/n*]

This definition of the photon encompasses just about any case of interpretation we happen to know. It can be summarized with the locution just presented above: *a photon is the interpretative particle of the ensemble representing a Lorentz quantum*. This statement is valid in light as well as in the matter. It is simply following from a Hertz-type natural philosophy, wherein a Lorentz quantum can properly represent a material point whose material particles are the photons. On the other hand, one needs to take notice of the fact that the Lorentzian definition of the physical object called quantum is in accordance with the very definition adopted by Bernhard Riemann on the occasion of his celebrated dissertation on the grounds of geometry (Riemann, 1867 a):

> Definite *portions of a manifoldness*, distinguished by a mark or by a boundary, *are called Quanta* (Clifford's translation, *emphasis added*, *a/n*).

One can see, therefore, that the quantum is classically conceived as a physical object having a space and time extension, rather than being a simple material particle: properly speaking, according to physics' experience, it can have space extension but no time extension or *vice versa*, it can have time extension but no space extension. These concepts must be extended by our intellect, beyond the meaning bestowed upon them by our senses. Taken as such, however, the Lorentz quantum has all the properties of *a Hertz material point*, as we just said. Indeed, what counts in the Hertz's definition, is the concept of cardinality of the particles, which became so critical with the idea of quantization, that Gilbert Newton Lewis had to account for it in his definition of the photon from the excerpt above: we must distinguish "between the number of these entities present in an atom and the so-called quantum number".

2.4. The Library of Current Experience and Its Volumes

Let us explore, in this section, the concept of a *fictitious experience*. Apparently, Wilder Penfield was closer to physics than any physiologist could ever be, obviously by the very nature of his work in the capacity of a brain surgeon. His different proposals may not have the desired clarity from physics' point of view, partly because the physics itself had not, in those times – apparently, still does not have even today, for that matter – the right principles to deal with the observed manner of working of the brain. In order to assume a position from which we may be able to avail of the depth of Penfield discoveries, we still need to remain in the fields of physiology, psychology, and psychiatry, where definite conceptual closures were possible just based on experimental observations, with no physical explanation whatsoever. Any physical explanation here is related to *that* notion of experience necessary in the construction of a concept of universe. And, except for the experience of the mankind regarding the universe within which it flourished, any other such experience is fictitious, as we said. Occasionally, this created confusion, as in the case that follows.

The distinguished psychologist John M Stroud, was one of the participants to the earliest efforts of constructing the thinking machines along with Warren McCulloch and his circle. This circle of the illustrious neurophysiologist included notable names, such as Norbert Wiener, Claude Shannon, and John von Neumann, who are well-known through their important parts played in the creation of modern cybernetics. However, regardless of the higher achivements of this remarkable accumulation of brains, we need to take due notice of the fact that it was John Stroud who has found what we think is the right word in the problem of those 'files of memory' mentioned by Wilder Penfield. To wit: these 'files' are what the modern theoretical physics has called *instantons*, a definition that we have adopted with a slight change in the currently accepted physical meaning, only in order to cope with a concept of universe derived from the Planck's procedure of quantization [see the discussion of the cosmological point of view in (Mazilu, 2024), Chapter 1, and particularly in the Chapter 6 of the work]. In a word, our instanton, just like Stroud's 'moment of psychological time' is defined by a *condition* *of simultaneity*. Further on, this condition of simultaneity allows a remarkable mathematical description, just as it did in the case of Einsteinian relativity in its both instalments – special and general [(Mazilu, 2024), Chapters 2–3]. However, it is interesting to take Stroud's story from the very beginning, since it contains a 'strong note', as it were, demanding the intervention of physics almost explicitly.

John Stroud was mainly interested in the *psychological problem of vision*, involving, obviously, what can be taken as a whole space, 'encompassed', as it were, by the human eyes. By the middle of 20th century, it was obvious to mathematicians and physicists alike, that *the visual space*, as it is defined in psychology, was not at all Euclidean, but rather a Riemannian space of the type of Maxwell fish-eye optical medium (Luneburg, 1950, 1964). Within modern cosmological views, this conclusion is natural, and comes as no surprise: the man perceives the light – the carrier of information in the universe we inhabit – by its rays in a Riemannian environment called Maxwell fish-eye [(Mazilu, 2024), §§1.1 and 1.2]. However, what comes as quite surprising in Stroud's approach of the problem of vision, is the *position of the man as a spectator in his universe*, which allows, in Stroud's opinion, a special validation only through the *concept of surface*. A more involved mathematical reflection on this construction of Stroud's, based this time on physics, will show that the man cannot live but in what may be described as a Riemannian environment, and this is actually determined by the *existence of charges* (see I, §§5.3–4). Let, therefore, start our account of this story as we usually do, by quoting from Stroud:

> The concept of a finite but unbounded surface may be a bit awkward, but it can be arrived at in some such fashion as the following: *Imagine standing inside a crystal sphere where the only information was on the surface of the sphere*. It is a finite but unbounded surface *as far as the information is concerned*. You, as a "normal" individual may have the concept of inside and outside bounds in a space of three dimension of extension – and if it is *a literal sphere not too far away* you *may be able to place quite accurately the inside and outside bounds* of the sphere. However as far as useable information goes, if you are doomed to remain precisely in the center of this sphere, *the inside-outside bounds are meaningless*. Imagine now that *the ability to conceive of an inside and outside were lost to you. The information on the sphere is quite unaltered to you*. What *you call a length* I may call *either a length on the sphere or an angle*. What *you call the angle* between two lines *I also will call an angle*. [(Stroud, 1948), p. 1; *our emphasis*, *a*/*n*]

Two important statements strike us right away in this excerpt. The first one is explicit and regards the position of man in the Stroud's imaginary sphere: *this is exactly the natural position of the man under the skies of his physical world*. In this respect the sphere of Stroud is not at all imaginary, but only undecided, indeed: 'as far as the information is concerned' it is a 'finite but unbounded surface', like Stroud would say, because 'we are, in fact, doomed', to use the Stroud's own expression, by our very existence to 'live and act in the center'. For once, let us recall that this state of the man, of existing within some ‘meaningless inside-outside bounds’, has actually decided the fate of the modern science and, implicitly, the fate of humankind: the Kepler laws of the sky could not be discovered otherwise, and with them the whole modern physics would not be here as we have it today. Notice, therefore, in the spirit of this last observation, that *a measure of time* is involved here, emanating from the very definition of that ‘undecided surface’ of the humankind – being, as it were, a ‘technological’ result that involves the whole civilization – which determined the modern concepts of time and forces: it is *the measure of time given by the Kepler*’*s second law* [see (Mazilu & Al., 2015), especially §III].

A second striking statement of Stroud’s is somewhat implicit in the above excerpt. Namely, one can say that this ‘ability to conceive inside and outside’, *effectively* ‘*lost*’ *by us*, due to our condition – truthfully speaking, we never had it, to begin with – became critical by generating the consequences of a real loss. The physical interpretation of this fact was given by Albert Einstein, who in 1917 stirred one of the most important moments of the modern physics: the Einstein-de Sitter debate, which, in our opinion, liberated the general relativity from some unnecessary canons contained in Einstein’s axiomatics [see (Mazilu, 2024), especially Chapter 3]. One of those canons was Einstein’s thesis that the metric should be determined by the presence of matter. Willem de Sitter demonstrated that, in fact, this is not the case, and thus we got the possibility to introduce *the idea of a continuous charge background* (*loc*. *cit*. *ante*, §5.2).

On the other hand, speaking of *moments* in general, it is remarkable that John Stroud follows, through his definition of the visual space, exactly the pattern of a concept of time left *undefined* by Einstein in the construction of his first cosmological model, based specifically on the precepts of general relativity [(Einstein, 1917); see also *loc*. *cit*. *ante*, §5.2]. Quoting:

> My prior studies of *the moment* can very profitably be rephrased in these new terms as *a study of the temporal aspects of the space of visual information*. Personally, I like this phrasing of the problem much the best. Speaking in such ambiguous terms as “consciousness” and “experience” always annoyed the hell out of me anyhow. The *general construct* of “the space of information” and its subordinate divisions strikes me as a *much more suitable system of constructs*.
>
> The *space of information* is an *hypothetical construct with no history*. One can admit he knows nothing precisely about it as yet, without seeming stupid. One can easily propose concrete and definitive experimental studies of its nature, which is not the case with such *ambiguous constructs* as “consciousness” or “experience”. Above all it does not drag in by the tail of implication all the sense and nonsense in our non-scientific culture about these basically *non-scientific constructs* “consciousness” and “experience”. [(Stroud, 1948); *our emphasis*, *a*/*n*]

Hopefully, what may appear as non-academic expressions in this excerpt will be pardoned by the lenient reader – the work from which we took it was not intended for public anyway, but we find it as being of reference in what we intend to do here and, therefore, we are ready to take the whole blame – so that, with no further ado, we go for the substance of its message.

For once, it is not quite clear for us (we actually miss some of his essential works) what Stroud understands by 'consciousness' and 'experience' in order to label them 'non-scientific constructs': all we can say, *according to our notions*, is that in this case they are not what is meant to be by their names, *because they are inventions*, and hope that Stroud had this fact in his mind. On the other hand, we can just assign the cause of Stroud's expression to that ever-existing 'non-scientific culture', that 'drags by the tail of implication all the sense and nonsense' which can easily be associated with them, even in cases when they are what *we say* they are: that is, *fictitious*. Sure enough, though, we take them here under the definitions of Penfield, which appear to us as sufficiently precise to work with, involving the 'two elements' in making the man: *the brain* on one hand, as a physical structure of support, and *the mind* on the other hand, as an upholder too, but, this time, of the *relationships between the physical elements of the brain*. We take that this should be the 'library of experience' invoked by Wilder Penfield, and perhaps Stroud was, like many psychologists for that matter, a partisan of 'one element' theory of the man's being: after all, this may explain as well Penfield's 'two elements' preference, not just Stroud's 'non-scientific' attribute.

Be it as it may, fact is that Stroud associates the *space of information* with the *concept of moment*, based on the fact that *the physical time* (*i.e.* the *time of motion* in physics), and *the psychological time* (*i.e.* the space of information, *in the visual field*) are *not* the same. Quoting:

> For our present purposes, we can restate the hypothesis (that *the psychological time and physical time are not the same thing*, *a/n*) in terms of easily apprehended analogies, and so stated it is far from terrible, formidable or abstruse notion. Fundamentally, it says no more than that *we make our artifacts very much like ourselves*, scarcely a profound idea, since if we did otherwise our artifacts would be singularly useless to us. In a motion picture *nothing moves in the physical sense*. The "motion" in a motion picture is satisfactory because, at the level of data processing of which we are aware, our visual system processes visual inputs in similar *logical blocks*. *I called these logical blocks* "*moments*", *and they are directly analogous to the frames of motion pictures*. So long as we look at at least *one frame of a movie per moment of psychological time*, the movies move most satisfactorily. To have two or more frames per moment of psychological time makes them less afflicted with flicker. Whereas *physical time is a continuous variable*, *psychological time is a discrete variable*. There are approximately ten moments of psychological time for every second of physical time, though there may be more; as many as twenty or less, as few as five.

> There are, of course large periods of physical time for which there is no corresponding psychological time, when we are unconscious and when we are in deep sleep. A lifetime is roughly 20 billion moments. [(Stroud, 1967); *our emphasis*, *a*/*n*]

In order to understand the meaning *we attach* to these statements, we feel necessary to reiterate an explanation regarding *our* concept of universe.

This concept is constructed based on the prototype we have at disposal as physicists, that is the universe of our habitation or, as we called it, the physical universe. In a little more elaborated manner, scientifically established according to our experience – the only real one from among the existing experiences – this habitation has *defined* the inherent *finite scales* of space and time, and of anything else within our world. In their turn, these scales *are scientifically defined* only by the extension of the space we occupy, and by our possibility to move *within this space*. One of the essential characteristics of our existence is that we can never be aware of the place of this space in the universe: we cannot say by any means, scientific or any other kind, where we are located in the world we inhabit. Scientifically stated, for instance in a statistical inference about our position in the universe, we cannot *specify* our location in the universe. This is a physical fact that led to the special relativity in the first place. But, for the mathematical necessities, of the general relativity to take a known example, the locations in the universe must be described as having *imaginary* coordinates proper [see (Mazilu, 2020), §2.5]. In spite of all appearances, especially those coming from some 'non-scientific culture', such descriptions enhance significantly our 'consciousness', however not the 'experience'.

We close the case on John M Stroud with one of his achievements that gave us the possibility of some mathematical constructs representing the physical situation of the man: *the role of the concept of surface*. The modern physics started from the experience connected to a surface: the Earth's surface, a place where the 'experience' makes real sense, no matter what Stroud would have to say on this issue [see (Mazilu, 2024); follow especially the §5.5]. As we said, is only our guess that his characterization of 'non-scientific construct' is referring to an imagined experience. However, this reference is made just like in the case of Galilean experience that built the modern physics. It is, indeed, 'non-scientific', just because we do not have an established concept of universe in order to make the imagined experience 'scientific', by analogy with the Galilean experience of our prototype universe. Then one can understand better the great merit of Stroud, who connects the moments with the concept of surface, more precisely with what we call nowadays *topological surface*. Quoting again:

> I can now vary the shape and location of your crystal sphere at will, *provided I keep the angles you call length*, constant, and *the angles you call angles* constant from your point of view. All your information remains unchanged, *plotted on a finite but unbounded surface*. But to further specify the surface is meaningless. Whatever one of the infinite number of transformation

I work on your information sphere, so long as they belong to the family of transformations which maintain for you the constancy of the angles you call length, and the angles you call angles, it is the same information to you. Thus it becomes meaningless to say that your information surface has any form in tridimensional extension.

What one can say is:

That you can have only such information as can be represented on a surface.

That your information surface has no bounds.

Now visual information *as it comes from each eye field* is also only such information as can be plotted on a surface *with two dimensions of extension* and *three or more of intension*. (*An intensive dimension is a dimension which can have any values over its range – but only one value at any one point in the space of extension at which it appears*). It comes in pairs of "sheets" with bounds, which we may consider plane sheets (for there is a transformation) by which information in bounded surfaces, regardless of their shape in three space, can be transformed to a bounded plane figure with no alteration of the relative information contained in the data sheet in and of itself. The "fact" that the sheet is a sheet in a threespace is not information directly plotted on the sheet itself. [(Stroud, 1948); *our Italics*, *original underline*, *a*/*n*]

It is not hard to see, especially after reading this excerpt, that Stroud touched the essence of the concept of quantum as it was described by Lorentz in the excerpt above from (Jeans & Al., 1913). As we said, that characterization of the quantum – making a 'Lorentz quantum' out of it – is, in fact, the only realistic description of this concept ever: it is only based on that part of our experience created by experiments. In this instance, the quantum is akin Hertz's material point, with its particles 'associating a position in space at a time, with another position in space at another time' [(Hertz, 1899), p.45], but not 'without ambiguity'. As we shall see here, the ambiguity is represented statistically, in a remarkable way, according to Planck's procedure of quantization.

In the excerpt above, the definition of *intensive dimension* is quite significant: such a dimension can have *any values over its range*, but only *one value at any one point* at which it appears in the space of extension. However, the universe we inhabit has as characteristic the fact that the charge can have only *some* values in *some* points, and we need to raise awareness about this incidental property of this universe. Going a little ahead of us, we think that *the prototype of such an intensive dimension is the charge*, indeed, in that it allows for a Feynman interpretation, but in general, *i.e.*, with a Stroud addendum in its definition [see (Mazilu, 2024), §5.1, and the works cited there]. This general property is, again, a consequence of the procedure of Planck quantization. Let us, this time, get into the essential details connected with this statement.

2.5. Louis de Broglie's Light Ray: the Lorentz Quantum in Action

We have long realized that there is no direct theoretical-physics' demonstration of the de Broglie's formula of energetic equivalence between a quantum and a material point [see (Mazilu, 2020), and the works cited there, for a proper documentation]. The story we have made earlier in this work out of the BKS case, seems to point out exactly what the culprit is in this case: a wrong notion of the quantum, suggested by the de Broglie's original formula. We dare to advance the thesis that the right concept of a quantum is that of a *Stroud moment*, *viz*., an instanton [(Mazilu, 2024), *passim*] and the differential-geometrical theory of surfaces offers us one strong incentive for this conclusion: the quantum of a Louis de Broglie light ray must be a *Lorentz quantum* of the kind described in the excerpt given above from (Jeans & Al, 1913). In concluding this part of our work (see Chapter 9), we will propose a physics based on the differential geometry of affine surfaces, that may be able to offer enough incentives in order to be considered as a fundamental tool in theoretical physics at large.

Meanwhile, however, we must stay with what we already have from physics. As we said, the Louis de Broglie's light rays must preexist in a universe, like in the particular case of the brain-universe, where it exists in the form of the network rays supporting the *message waves*, according to Hillis' philosophy as applied to a Stroud moments. Such a ray dwells within a 'bath' of *cancel waves* of a Maxwell fish-eye medium, just like a radio set which dwells in a 'bath of radio waves'. And the cancel waves measure the extension of the 'least parts of the ray', extended in different directions. Their physics meets the procedure of de Broglie's construction of the light ray, and this fact can be explained according to the idea of quantum, as outlined by Lorentz in his intervention from 1913 reproduced by us in the excerpt from the § 2.3 above. Only, we have to add to this image the fact that, within a Lorentz quantum, the waves, oriented chaotically, must be considered plane waves, from reasons to be explained later.

In order to justify this, we have to recall two essential points realized by the theoretical physics in the problem of measurement. One of them is as old as the realization that the electricity compels us to start describing processes too, rather than mere motions. It was best summarized by Josiah Willard Gibbs toward the end of the 19th century. Quoting:

> We must distinguish… between the *actual electrical displacements*, *which are too complicated to follow in detail with analysis*, and which *in their minutiæ elude experimental demonstration*, and the *displacements as averaged for spaces* which are *large enough to smooth out their minor irregularities*, but *not so large as to obliterate to any sensible extent those more regular features of the electrical motion*, which form the subject of optical experiment. These spaces must therefore be *large as measured by the least distances between molecules*, but *small as measured by the wavelength of light*. We shall also have occasion to consider similar averages for other quantities,

> as electromotive force, the electrostatic potential, etc. It will be convenient to suppose that *the space for which the average is taken is the same in all parts of the field*, say *a sphere of uniform radius having its center at the point considered*. [(Gibbs, 1883); *Italics ours*, *a*/*n*]

As far as we can see, the theoretical arena of physical events, according to Gibbs' idea, appears as a *matter of scale*. The 'displacements' must be averaged, but not quite arbitrarily: they must be averaged in such a way as to provide the quantities to be recorded as motions "which form the subject of *optical* experiment". This requirement places constraints on the space regions of averaging, and Gibbs characterizes these regions, quite precisely we should say, from a phenomenological point of view: they should be '*large*, when measured by the distances between molecules', however, '*small* when measured by the wavelength of light'. Consequently, if by some physical procedure we discover the possibility of a certain representation for a physical quantity, then the averaging procedure must consider the whole evolution of the arena itself of averaging. Only this can explain the consistency of the whole physics. To accomplish this task, Gibbs even prescribes a geometrical form of the averaging space: 'a sphere having the center in the point considered'. For the moment, its essential dimension is just a 'uniform radius'.

Let us, nevertheless, not forget the two essential conditions of carrying out the Gibbs program: the first one is the *existence of the particles* within the spaces of averaging, between which *the distances can be measured*, and the second one is the existence of the waves passing through this space, *whose wavelength can be measured* somehow, in order to be compared with the distance between particles. As far as we are aware, the physics tackles these conditions only incidentally, that is, *not as fundamental physical requirements*, to say the least. And we found even a reason for that: there is an inadequate concept of quantum dwelling in physics, even when this quantum is taken in the quantitative sense, *i.e*., not as a physical object. This observation brings us to the second one of the two essential points of the theoretical physics in the problem of measurement, as revealed by Gibbs. This time the point is just made by us, based, however, on a certain perspective in the history of physics.

One of the main tasks of Louis de Broglie, all along his life we can say, was that of interpretation of the wave phenomena, especially those regarding the light. The inception point of this constant preoccupation of de Broglie was that fateful 1926 year of physics, when he realized the essential need of the new physics (de Broglie, 1926 b, c): to put in accordance the concept of interpretation, which involves particles, with the optical phenomenon of interference, involving exclusively waves [see also (Mazilu, 2020), *passim*]. Two principles can be extracted from the de Broglie procedure to deal with the interpretation: the first of them amounts to the fact that, when dealing with the interpretation, the physical optics must be treated as *a static condition for the amplitude of representation of the phenomena*. The second principle, amounts to the fact that a physical ray is essential in the process of interpretation, inasmuch as *it provides a way of dealing with this statical condition*. Namely, along the ray, and referring to the 'point considered as center', the propagation is instantaneous, in the sense of a simultaneity of events. In other words, there is a sphere in the Stroud's sense, the center of which is in the position of the interpretative particle, that must be treated 'statically', as a *region of simultaneity* of events.

Now, the Lorentz's quantum can be introduced into this picture, in order to explain the de Broglie's 'region of simultaneity', as it were, in the following manner. We have to recall that, in constructing the statistic that 'legitimately', if we may be allowed to use the word, represents the radiation law, there is a point during the unfolding of the theory, demanding the explicit consideration of the plane waves: they are necessary in the construction of *a temperature-independent statistics of frequencies*. The plane waves are thus brought into picture, first of all because they represent best a single frequency and, on the other hand, since they provide a space view of the frequencies, as *three-dimensional vectors* (Jeans, 1905). Then, this last property of the frequencies allows the geometrical construction of an *a priori* measure of their *magnitudes*, to be used in the final statistics that gives the correct radiation law. Based on this *a priori* measure of the *space of frequencies*, the theoretical physics then goes as far as constructing, based on purely classical principles, the only *theoretical* law of radiation of physics at the closing of 19th century and the beginning of 20th century – the *Rayleigh-Jeans law*. The failure of this law to describe the whole spectrum of radiation, was thus the main incentive of the experimental researches on the light thermodynamical properties, leading to the Planck's radiation law.

Thus, we can see a Lorentz quantum more generally, as a portion of space attached to the interpretative particles along any position of a de Broglie light ray, where this particle can be found with 'equal probability' in any point. This is a condition that Hertz himself forgot in his definition of the particle: for him, a particle is just a means 'to associate to a position in space at a time, another position in space at another time, *with no ambiguity*'. But this task is impossible if the particle *does not first indicate* a point in space, *and such a job cannot be done without ambiguity*. Think of a star: it seems to indicate a point in space, but not unconditionally. The Sun, for instance, is a star according to our experience: it *produces* light and heat. However, it does not indicate a position in space, but rather a region of space occupied by its matter. One can 'locate' the Sun 'in such and such' region – or 'house', using the old language of astrologers – but not in a point. The indication of a point in space is a matter of scale: the Sun is too close to us in order to be able to indicate a position. Likewise, a Lorentz quantum is a portion of space where a particle can exist, *a priori* with equal probability, containing a 'chaotic ensemble' of plane waves which propagate in any direction; and each of these plane waves has its own frequency. The values of frequency of such waves can be assumed to form a continuum, upon which we have to construct a distribution of frequencies. How shall we proceed to do that? It is the idea of interpretation that will help us. Namely, if we are to provide a frequency to a particle associated to a Lorentz quantum, that can be none of the frequencies of its constitutive plane waves. Neither will it depend on the direction of the waves.

In order to get a glimpse at the position of plane waves within the framework of a Lorentz quantum, we must go back in time, and recall the Augustin Fresnel's fundamental natural philosophical principles in the physics of light. The physical optics of Fresnel involved the plane waves as the fundamental entities of the theory, for the very same reason invoked in constructing the right radiation law: these waves are, aparently, the only waves making possible a rational definition for the frequency. So, let us reproduce here the *fundamental* principles of the Fresnel's physical optics, taken, just for the sake of transparency, in the formulation of the illustrious Henri Poincaré. After explaining at length the Fresnel's theory of light, Poincaré concludes to the following:

> This is, in a nutshell, *the theory of Fresnel*. It is in every respect in conformity with the experimental laws; but we notice that it rests upon *two hypotheses* demanding a closer examination. These two hypotheses can be enunciated as:
>
> 1° The *elastic force aroused by the motion of a plane wave* is *independent of the direction of the plane of wave*, it *depends only on the direction of vibrations* of the molecules, and *is proportional to the force developed by an isolated molecule*, the *other molecules from the plane of the wave remaining at rest*.
>
> 2° The only *effectual component of the elastic force* is the *component parallel to the wave plane*.
>
> The first of these hypotheses, which Fresnel vainly tried to justify, *is entirely arbitrary*, but nothing precludes its acceptance … [(Poincaré, 1889), §151, pp. 229 – 230; *our translation and emphasis*, *n*/*a*]

In hindsight, this summary of the Fresnel's theory – a summary of this conciseness Fresnel himself did not present with such a clarity [see (Fresnel, 1827) for certification], and this is the main reason we chose the Poincaré's formulation – shows that it actually contains all the details Planck needed in developing his theory of quantization. A short analysis of some of these details of the excerpt above will elucidate our statement.

First of all, Poincaré makes it clear that the Fresnel physical theory already assumes an interpretation in place – all classical theories assume an interpretation, as a matter of fact – and that the waves propagating through the continuum thus interpreted, are always plane. The first hypothesis of Fresnel then says that the force 'aroused by the motion of the plane wave' does not depend on its direction (of propagation), but only on 'the direction of vibration of the molecules'. Obviously, the interpretation, here, does not involve just particles as such, *but particles in motion*: they are 'molecules' of a continuum supporting the propagation. When the plane of the wave reaches one of these molecules, it is already in motion along a certain direction in space, and the wave inflicts a change of this motion but in a direction from its plane. It is this way that the force aroused by a (plane) wave in a medium becomes 'effectual', and can be used in an expression of the second principle of dynamics, in order to determine the motion, and implicitly a 'dynamics', in the classical sense, for the fields. Then, may we ask, why two hypotheses, while a single one will do just fine? The answer, as we see it, is in the last observation of Poincaré from the excerpt above: the first hypothesis 'is entirely arbitrary, but nothing precludes its acceptance'. This would mean that the hypothesis in question is ranking – in realm of physics, of course – equal to the axiom of parallels staying at the foundation of any geometry, or to the axiom of choice and the continuum hypothesis staying at the foundation of any mathematical theory of continua: it is a truth that one cannot prove by any means, however, one cannot do without it.

Fact is that, the appearance of Maxwell's theory of electromagnetism compelled the theoretical physics of the end of 19th century to an intensive effort to 'translate' the Fresnel's physical theory of optical light in 'electromagnetic terms', as it were [see, for an advised guidance on this issue, (Poincaré, 1901), especially the *4ème Partie* of the work, which is concentrated on the Larmor's contribution to electrodynamics]. The recognizable point at issue in this enterprise was the construction of an appropriate *transport theory*, in order to cope with the kinematics of continua. This is our understanding of the issue, with the benefit of a distant perspective, as it were, since at the epoch we are talking about it could not be recognized explicitly as such. For instance, this issue determined the appearance of the Lorentz theory of electric matter [see (Lorentz, 1892); also (Poincaré, 1889, 1901) for a detailed documentation]. As a matter of fact, the theory of transport phenomena would just get to its inception point at the epoch of Poincaré, with the appearance of the fundamental work of Osborne Reynolds on the atomism in continua (Reynolds, 1903). We just understand – and explain, of course – the issue in connection with the concept of Lorentz quantum, as presented in 1913 (see, for its definition, the § 2.3 above).

A Lorentz quantum can be conceived as a 'chaotic mix' of plane waves of all directions of propagation and frequencies, covering a certain portion of space. It is not necessarily interpreted, but when we try to interpret it we need to recall the problems encountered by old Fresnel. Associating a particle to such a chaotic mix, has necessarily two faces, not just one as in the previous excerpt: on one hand, the wave *arouses forces*, and that even in the absence of the particle. On the other hand, only *after these forces are already aroused can they act* '*dynamically*', as it were. This is the deep meaning of the first hypothesis of Fresnel, delineated in the excerpt above from Poincaré: the interpretive particle is not necessarily the physical particle upon which the 'aroused force' acts. Thus, problems arise when associating a temperature to a frequency, as the Wien's displacement law would expressly ask [see (Planck, 1988), Chapter IV]

In understanding this statement, we need, again, the help of the great sage of modern physics who was, and, obviously, still is and will certainly continue to be, Henri Poincaré. He was predominantly interested not in physics *per se*, we might rightfully say, but rather in the theoretical status of the *mathematical principles* of a natural philosophy as a whole, capable to complete the classical natural philosophy of the Newtonian physics. Thus, to him, the connection between electrodynamics and mechanics is always to be approached from this point of view. Quoting, therefore, for an essential illustration in the definition of the coordinates:

> We have not considered until now but only *the isotropy in the plane of the wave* (*our emphasis*, *a/n*), and that does not cover for all the conditions imposed by the symmetry of the medium.

Indeed, our medium being isotropic, the previous equations *must remain the same no matter of the orientation of the plane of the wave* (*our emphasis*, *a*/*n*); moreover, they must not change when one replaces the system of axes by a symmetrical one with respect to the origin, because the medium is not only isotropic without symmetry (…) but it is *isotropic* and *symmetrical*.

We are thus compelled to distinguish two kinds of coordinates:

1º *The vectorial coordinates* which I shall call X_k, Y_k, Z_k, and which will be the components of a vector; and,

2º *The scalar coordinates* which I shall call T_k, and which will be totally independent of the choice of the axes.

The introduction of these scalar coordinates must not astonish us. In fact, let us justify their use by a mechanical image. Consider a pulsing sphere of Bjerknes, susceptible, furthermore, of a motion of translation; well, in defining the situation of system we need four coordinates: the radius of sphere and the Cartesian coordinates of the center of sphere, which are vectorial coordinates. [(Poincaré, 1901), § 438; *original emphasis*, *a*/*n*]

For the moment we may consider the 'sphere of Bjerknes' as just a plain 'sphere', nothing more: no need to know exactly what it is from a physical point of view. The important fact is that we are mainly accustomed with the idea of *vectorial coordinates*, but the great sage appears to have been well aware of the fact that we have to admit the existence of some other coordinates, which represent parameters not having but an implicit connection with the idea of vector. The general idea appears to be incorporated in the modern mathematics as the *theory of modules*. However, the example brought here by Poincaré, in order to illustrate the idea of *scalar coordinates*, can be taken as the epitome of the case: the scalar coordinates must reflect the *dimensions of spaces containing matter*, in the form of some physical structures, of course. The idea is that such spaces can have geometrical forms described by surfaces, just like a Bjerknes sphere, which is described as a … sphere. Our contention is that these spaces are of service even if they do *not containing such matter*. The Lorentz quantum can be taken as such a space which, void of matter like the vacuum, supports the propagation of the waves.

The obvious point of connection of the Poincaré new natural philosophy with the old Newtonian natural philosophy, stays explicitly in *the fact that the isotropy in the plane of the Fresnel wave* must be considered separately from *the isotropy of the medium supporting the waves*. However, between the two there should be some connection, once some equations 'must remain the same no matter of the orientation of the plane of the wave', and that connection is a matter of physics. This is an 'assignment' left by Newton, indeed, in order to be completed in a future natural philosophy, and Poincaré does nothing but just formulates precisely the problem we have to solve, in concordance with the new phenomenology which includes the electromagnetism. The essential discovery of Newton can be briefly summarized as follows: *the physical ray* is not a plane figure, as Hobbes presented it first, and Hooke used it in creating the first rational theory of the colors of light. If it is to attach the color as a quality to the light, then the physical ray is not a plane figure in the longitudinal plane through the direction of propagation, but it is rather a space figure, having a finite volume, as the experiments on light show. If the light carries many colors, for instance if it is white, then it is only *transversally heterogeneous*, *i.e.*, it is no more axially symmetric, *although not as a geometrical shape*, but with respect to color as a parameter of heterogeneity. Taken as such, a light ray is, indeed, composed of axially symmetric color-homogeneous rays, that can be exhibited as an elongated spectrum, for instance when passing the light ray through a prism. This is, in broad strokes, the Newtonian conclusion of the celebrated prism experiments, and the ground for Newton's discussion on colors, which later on, during 19th century, generated the physical theory of light spectrum, and implicitly led to the theory of quanta [see (Mazilu, Agop, Gatu, Iacob, Butuc, Ghizdovat, 2016), especially § 2]. So, we can rightfully say that the property of isotropy in the plane of a light wave is, indeed, an assignment left by Newton to be properly clarified for the upcoming natural philosophy.

Assume, now, a Lorentz quantum: a space filled with plane waves, chaotically propagating in all directions. Assume further, that these waves give us the chance of measuring the size of space directly. For instance, this chance may be given by the fact that a wave from among those of this chaotic mix becomes suddenly a stationary wave; or it is reflected instantaneously, and then received, as in the operational definitions of special relativity. As long as the man cannot intervene to record and assess these circumstances, they remain, indeed, chances. Having at our disposal some characteristics of these waves, the essential of which is the frequency – after all, this is the '*raison d'être*', as it were, of the plane waves ever since the Fresnel's times – we can come to the estimation of the dimensions of this space. From a statistical point of view, three such chances should be just enough for the estimation of the mean extension of the Lorentz quantum, and the standard deviation of this extension (Ferguson, 1978). This assumes an *a priori Cauchy distribution* of this statistical variable over the 'population' of the Lorentz quanta, which can be explained by its 'universality'.

Indeed, the property of one-dimensional Cauchy distribution of a continuous statistical variate, is that it cannot have reliable estimators but only for the first order moment, *i.e.*, for the *mean*. In the statisticians' jargon, it has no finite moments of order higher than one. Applied to the physical structures, this property is coping with the corresponding one of the matter formations in the universe we inhabit: bodies of our experience, stars, planets, nebulae, etc, can have any space extension. That is, these matter formations do have an unlimited spectrum of dimensions in any direction in space. This observation started to be extended with no restrictions, in the world of elementary particles, but soon enough our intellect was forced to change our mind. The key point of that change was the concept of quantum, especially by its differentiae embodied in the Lorentz quantum.

Practically, this meant that the man was compelled, by the physical existence of the microcosm, to admit that the free space should be submitted to the same restrictions as the space filled with matter. For, once the microcosmos has entered physics, the man could speak no more of an atom as being that permanent, simple, ‘uncuttable structure’, if we may be allowed such an expression, of the initial definition adopted in the old by sages such as Leucippus and Democritus. Anyway, we like to think that the idea has been perpetuated *objectively* in the natural philosophy and, starting with the pertinent speculative extensions of Epicurus [see, in particular (Furley, 1967); also, one can consult the exquisite 2018 Oxford Edition of Diogenes Laertius’ *Lives of the Eminent Philosophers*] it broke out into modern physics as a necessity to correlate the ‘atoms with the voids’, if it is to use the old expressions. The Planck’s quantization carries in modern times, at least in our opinion, the mark of such objectivity. According to this opinion, the old philosopical speculations, just as the new ones belonging to the natural philosophy instituted by Newton, may be taken as only *the appearance* of an objective fact, which, as any appearances, entertain much discussions [see, for instance (Makin, 1989), for a significant discussion on the topic of atom, obviously made possible by the fact that the physics of the 20th century has just shown that there is not a ‘partless’ atom in the universe, thus adding a new *differentia* to the *old concept* of Greek atom]. These discussions, however, do not add very much to the *existing* concept, the one that arises in physics, which will benefit from them only through a proper mathematics. And *thus*, that mathematics acquires a distinctive task: it must help discerning between the apparent and the essential.

An example of such a kind of discernment stays in the Poincaré’s idea above of scalar coordinates. According to this, the Fresnel force aroused by the wave in a medium (‘void’, in the old Greek’s philosophy expression) ‘must be parallel with the plane of that wave, no matter of the direction of propagation’. The most obvious case of such a representation would be the wave-plane normal, obviously, when it coincides with the propagation direction. This, however, is not always the case, which explains the addition to the hypothesis of the requirement that this force ‘depends only on the direction of vibration of the molecules’. How are we to understand this clause?

The explanation may come like this: *when in matter*, a Lorentz quantum always covers an ensemble of molecules, in a scenario like that of Gibbs above. However, from physical point of view, there is a reciprocal to this assertion, saying that, on the space of a Lorentz quantum, while moving, a molecule can be under the action of the ensemble of waves of that quantum. The Poincaré formulation of Fresnel principles does not cover this requirement, but it is instrumental to the physics of waves. It says that the ‘vibration of a molecule’ must have, in general, *i.e.*, if the propagation direction does *not* coincide with the normal to the wave plane, two different components: one in the plane of wave, the other normal to this plane wave. Both of these are, obviously, to be calculated from the condition that the molecule is ‘located’ at the intersection of the planes of the waves containing it *simultaneously*. Furthermore, we translate the Poincaré formulation of the Fresnel’s principles in a modern form inspired by a definition of the quantum measurement of half-spin (Schwartz, 1977): this molecule ‘measures’ somehow the fields of forces, by *always producing forces in the plane of the waves*. The result of the measurement is independent of the direction of the wave. If, as in the case of half-spin, we represent mathematically this measurement by a 2×2 matrix, then the following basic mathematics apply to this situation.

Denote (θ, φ) the angles of *orientation of the plane waves* in the space occupied by the Lorentz quantum, and assume that such a quantum is described by continuous ensembles of the two variables, in each and every point of this quantum. According to the quantum-mechanical recipe of measurement, the quantities to be assigned to a 'moving Fresnel molecule' in such a 'measurement process' are the two eigenvalues of a 2×2 matrix. Using the Fresnel principles, they can be considered as the forces aroused by light upon the molecule, and just like in the case of spin they must be independent of the angles of orientation of the plane wave which produces them. Therefore, the 2×2 matrix representing this 'quantum measurement process' needs to have its four entries *dependent on the angles of orientation of wave planes* within the space of Lorentz quantum. On the other hand, the eigenvalues of the matrix, which according to Fresnel's principles are the outcomes of such measurements within the space of the quantum, *must be independent of these angles*. In cases where the eigenvalues are real, the matrix needs to be Hermitian, a fact which fits perfectly into quantal precepts regarding the idea of measurements: transposed, it should give its complex conjugate. Such a matrix can be constructed along the following lines.

It is known [see (Schwartz, 1977), §3], that the null-trace matrix, whose square is identity 2×2 matrix, given by:

$$\boldsymbol{Q} \stackrel{\text{def}}{\equiv} \begin{pmatrix} \cos\theta & \sin\theta \cdot e^{i\varphi} \\ \sin\theta \cdot e^{-i\varphi} & -\cos\theta \end{pmatrix} \tag{2.5.1}$$

has the eigenvalues ± *1*. This can be simply verified by direct calculation, even without recurring to the principles of quantum measurement. Then the matrix:

$$\boldsymbol{M} \equiv q \cdot \boldsymbol{1} + p \cdot \boldsymbol{Q} = \begin{pmatrix} q + p\cos\theta & p\sin\theta \cdot e^{i\varphi} \\ p\sin\theta \cdot e^{-i\varphi} & q - p\cos\theta \end{pmatrix} \tag{2.5.2}$$

where ***1*** is the identity matrix while p and q are real numbers, has the eigenvalues ($p \pm q$). So, this matrix is to be assigned as algebraical representative of the molecule associated to a Lorentz quantum in a position within its space. Its eigenvalues do not depend on the direction of propagation of a plane wave inside the Lorentz quantum, but, according to Fresnel's principles may depend on the state of the molecule within the space of this quantum. This state is dictated by the very position of the Lorentz quantum itself, decided along a de Broglie ray for instance. It is important, therefore, to know what kind of forces p and q may be. This shall be one of our main tasks here.

This description of the quantum measurement seems to serve best the Fresnel physical optics, inasmuch as it fits perfectly within its very principles: the mathematical formalism actually sanctions the formulation of those principles. It would plead, therefore, for the association of a 2×2 matrix with an interpretative 'molecule', for which association, however, we need to exercise a little precaution: the matrix (2.5.2), for instance, embodies *mixed properties* in its entries. Indeed, these depend on the orientation of the plane wave that arouses the forces on a molecule within the Lorentz quantum serving for measurement, as well as on the state of the very molecule on which the force of the wave is 'aroused' by that plane wave. It is this last 'part' of the entries of the matrix that would rightfully qualify for the association in question. Therefore, at this stage of the mathematical theory association of a matrix with the molecule – in general, with a material particle of the ensemble serving for the interpretation of the matter – is quite vague, to say the least: we do not know how to describe the simultaneous action of an ensemble of plane waves on the same molecule.

However, we just take this opportunity in order to describe a general 2×2 matrix in terms of its *actions*, which will be of great help later on, when we shall go deeper into mathematical theory. Such a matrix, say of entries (α, β, γ, δ), arranged in the form of a square table:

$$\boldsymbol{M} \stackrel{\text{def}}{=} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \tag{2.5.3}$$

can have, as well known, two main *algebraic* actions: *linear*, on two-dimensional vectors – *kets*, *i.e.*, matrices with two lines and one column – and *homographic*, on a one-dimensional range. These actions produce invariants, and the matrix can be seen as providing a connection between these invariants. First, the linear action produces eigenvectors: directions along which the matrix acts by simple multiplication. The multiplication factors corresponding to the two eigendirections are *the eigenvalues* of the matrix – $\lambda_{1,2}$ say – the roots of the algebraic equation:

$$\lambda^2 - \text{tr}\boldsymbol{M}\cdot\lambda + \det\boldsymbol{M} = 0,\ \ \text{tr}\boldsymbol{M} \stackrel{\text{def}}{=} \alpha + \delta,\ \ \det\boldsymbol{M} \stackrel{\text{def}}{=} \alpha\delta - \beta\gamma \tag{2.5.4}$$

Secondly, the homographic action of the matrix leaves the ratios of the *eigenvectors*' *components* unchanged, *i.e.*, these ratios, – $e_{1,2}$ say – are the roots of the quadratic equation:

$$\gamma e^2 - (\alpha - \delta)e - \beta = 0 \tag{2.5.5}$$

These last roots are usually known as *fixed points* of the homographic action of the matrix, and we adopt here this name for them. Now, the equations (2.5.4) and (2.5.5) allow us to write the entries of the matrix strictly in terms of these elements of the algebraic actions of the matrix. They produce the linear system of four equations for the four entries of the matrix:

$$\lambda_\sigma = \gamma\cdot e_\sigma + \delta,\ \ \lambda_\sigma\cdot e_\sigma = \alpha\cdot e_\sigma + \beta,\ \ \sigma = 1,2 \tag{2.5.6}$$

which, when solved, produces in turn the matrix (2.5.3) as

$$(e_1 - e_2)\cdot\boldsymbol{M} \stackrel{\text{def}}{=} \begin{pmatrix} \lambda_1 e_1 - \lambda_2 e_2 & (\lambda_2 - \lambda_1)e_1 e_2 \\ \lambda_1 - \lambda_2 & \lambda_2 e_1 - \lambda_1 e_2 \end{pmatrix} \tag{2.5.7}$$

Notice, that the matrix thus written is properly characterized, for we have:

$$(e_1 - e_2)^2\cdot\det\boldsymbol{M} = (e_1 - e_2)^2\cdot\lambda_1\lambda_2 \tag{2.5.8}$$

Also notice the following property of linearity involving the elements of the two actions in the decomposition

$$(e_1 - e_2)\cdot\boldsymbol{M} \stackrel{\text{def}}{=} \lambda_1\cdot\begin{pmatrix} e_1 & -e_1 e_2 \\ 1 & -e_2 \end{pmatrix} + \lambda_2\cdot\begin{pmatrix} -e_2 & e_1 e_2 \\ -1 & e_1 \end{pmatrix} \tag{2.5.9}$$

In other words, the eigenvalues are the ‘linear homogeneous coordinates’ of a 2×2 matrix, in a *linear pencil of matrices* generated by two singular matrices whose entries depend exclusively on the fixed points. Let us insist a little longer on this result, in order to get some conclusions regarding the association mentioned above, with a ‘molecule.

In the form given in equation (2.5.9), such an association seems to be already in place: the eigenvalues are ‘homogeneous coordinates’ of the matrix in a linear pencil of matrices, whose ‘reference frame’ is given by two singular matrices depending exclusively on the fixed points. It will serve our purpose to express these fixed points in terms of the directions of propagation of a plane waves within a Lorentz quantum. If the eigenvalues are ($p \pm q$), as before in equation (2.5.2), then the definition (2.5.9) becomes:

$$\boldsymbol{M} = p \cdot \boldsymbol{1} + \frac{2q}{e_1 - e_2} \cdot \begin{pmatrix} \dfrac{e_1 + e_2}{2} & -e_1 e_2 \\ 1 & -\dfrac{e_1 + e_2}{2} \end{pmatrix} \tag{2.5.10}$$

By comparison with (2.5.2), we get the identifications:

$$e_1 + e_2 = 2 \cdot \cot(\theta) \cdot e^{i\varphi}, \;\; e_1 \cdot e_2 = -e^{2i\varphi} \tag{2.5.11}$$

which produce the following fixed points depending on the orientation of the plane waves within the Lorentz quantum serving for measurement:

$$e_1 = \cot\left(\theta/2\right) \cdot e^{i\varphi}, \;\; e_2 = -\tan\left(\theta/2\right) \cdot e^{i\varphi} \tag{2.5.12}$$

Also, we need to mention for future reference, that in terms of the direction of propagation (θ, φ) of the plane waves within a Lorentz quantum, the so-called *Lorentzian metric*, is the regular metric on *the unit sphere*:

$$4 \cdot \frac{de_1 \cdot de_2}{\left(e_1 - e_2\right)^2} \equiv (d\theta)^2 + \sin^2(\theta) \cdot (d\varphi)^2 \tag{2.5.13}$$

Thus, we are able to conclude here with a great hope, regarding the possibility of solving our problem of association of a matrix to a particle, based on the Fresnel principles, as formulated above. First of all, in terms of the directions of propagation of the plane waves within a Lorentz quantum, this ‘physical object’ behaves like a sphere, possibly like a Stroud’s sphere, to be attached to any interpretive particle. The metric (2.5.13) indicates some property of measurability of the ensemble of possible directions of propagation of plane waves over the space of a Lorentz quantum. This measurability shall be described here in terms of a continuous group. The property needs to be, likewise, extended for the eigenvalues ($p \pm q$), in the sense that the overall ‘result of measurement’ produced on the space of a Lorentz quantum shall be described with the help of some statistics over the positions of particles at the moments of ‘arousing of the forces’.

2.6. Feasibility of a Physiological Concept of Ray

The present section would have its right place at the end of Chapter 1, but we preferred to insert it here being more meaningful after the arguments from this chapter, especially after the presentation of the concept of ray from § 2.1, and the discussion that ensued afterwards. Indeed, what we intend to show here is that the physical properties of the *nervous matter* have already made possible an idea of 'nervous ray', so to speak, even independently of its structural elements, *the neurons*. Phenomenologically speaking we are, in the physics of brain, exactly in the same position we were four centuries ago, with Newton, in the case of the optical physics. We have a clear advantage, though, insofar as we are certain in regards of the *physical existence* of those elementary rays that Newton struggled to make logically understandable in the case of the physics of light, by an imaginary scenario. Thus, a concept of ray in the nervous system was possible right after the discoveries of Luigi Galvani and Alexander Humboldt on the electricity of nervous matter, even starting by the end of the 18th century and the beginning of the 19th. For such a concept, the subsequent discovery of the neuron by Santiago Ramón y Cajal can count as *evidence of the existence of elementary rays in brain*, which, in turn, indicates a physical structure of the nervous ray. No doubt, the old habit of 'cutting the rays' was not entirely lost by transition between optics and physiology, but on this occasion it brought to light what we think is one of the most important properties of the 'nervous rays'.

As shown in the §§ 1.2 and 1.3, Wilder Penfield was entirely taken by the undeniable existence of the mind, a fact that, apparently, made his life highly enjoyable, to a degree that not too many people on this Earth can hope to reach. For, it is one thing to be a believer – if you are lucky enough, this kind of conscience structures your life into a much desired equilibrium in living it – but it is an entirely different thing if, as a scientist, you find *the scientific reason* in order to believe. This is a realization that gives one a sense of highest possible fulfilment, an the satisfaction of having participated to the life of the entire universe, to be part of the Creator, so to speak, as the great philosopher Constantin Noica would say. Quoting, on behalf of Wilder Penfield, this realization gives one a kind of 'legitimacy':

> Since every man must adopt for himself, without the help of science, his way of life and his personal religion, *I have long had my own private beliefs*. *What a thrill it is*, then, to discover that *the scientist*, *too*, *can legitimately believe in the existence of the spirit*! [(Penfield, 1978), Chapter 21, *emphasis added*, *a*/*n*]

Penfield seems to have realized, even as a scientist, that 'God sent us in this world to replace Him', as Noica would say, and 'he did not forget this', so that 'he did not waste his time'. We cannot see a common believer capable of such a high achievement!

The great sage raised the mind at the rank of spirit of God, and this idea grew out from *his own studies* of the brain. It was, certainly, such a kind of studies the ones which aroused, by the end of 19th century and the beginning of the 20th, that great philosophical problem that occupied Wilder Penfield so much and so deeply: *should the mind be a common production among the functions of brain*, *or it should stand apart*, *as a structure by itself*? Quoting:

> If there are *two elements*, then *energy must be available in two different forms*. There is *a force that is made available through neuronal conduction* in the brain. Is there force that is available to *the mind*, *which has no such circuits*? Can chemical action in nerve cells result in brain action on the one hand and in mind action on the other? Electricity *was first revealed* to science while it was *being conducted along the nerves* of living organisms. *Physicists might well consider our questions seriously today*, *if only out of gratitude*! [(Penfield, 1978), Chapter 20, § d; *emphasis added*, *a*/*n*]

As the existence of the mind is undeniable, that phrase of Penfield: 'the mind has no circuits' shows plainly that if it is a structure at all, it *cannot be a physical structure*. So, we cannot talk about 'the energy of the mind', in the first place, no matter what position we adopt, dualistic like Penfield, or not. But for that structure, he has put his faith in physicists anyway, and the physicsts are then bound to answer, 'if only out of gratitude'!

So much the more we have to undertake this task, as Penfield shows us where the physicists are supposed to enter the stage, and even specifies in what problems:

> So many questions still confront us! But to ask them is the first step toward solution. I am confident that they will be answered in time. After adopting the dualist hypothesis one can quite logically call upon the physicists for help. *Can electrical energy take two forms*? *What is the nature of mind*? *Has it a structure*? *What is electricity*? Whatever the answers to these questions may be, *the mind is present*. [(Penfield, 1978), Chapter 20, § d; *emphasis added*, *a*/*n*]

As we said, one cannot speak of the energy in the case mind, for we cannot speak of a physical structure of the mind: *the energy is always connected with a physical structure*. Fortunately, though, for us, the physicists who are willing to undertake the Penfield program from the point where the great sage brought it, *one cannot* speak of a *physical structure* of the charge too, which, incidentally, justifies the question: what is electricity? And so we have at our disposal a precedent from which we can learn. And, from this precedent, *one can learn* to speak of a *structure of charge*, and *this is dualistic* indeed, as we shall show here in due time. So, it may be here the place to look for *the nature of mind*, because the charge, which is known to support the mind, *has a structure*. Not a physical structure, by any means, but a structure nonetheless, mathematically describable!

So, we shall dwell for a while around the question of Penfield about electricity. And, for the physical definition of electricity we adopt here the views of Poincaré, as the wisest ones [see (Poincaré, 1901), Chapter II: *The Hypotheses of Maxwell*]: they show that it is connected with some *state of deformation* of matter, and our mathematical problem would be to describe such a state from different angles. On the other hand, the status of the findings of Galvani and Humboldt by the midddle of the 20th century are well and meaningfully described, in many specialty works [we recommend, for instance, (Gesell, Brassfield, & Lillie, 1954), which is also advisable in matters of details]. However, in order to see what distinguishes the nervous matter from other categories of matter that fell under the attention of physicists and neurophysicists, we find it better to give an excerpt in which Sir Edgar Douglas Adrian summarizes its distinctive properties, and, in our opinion, the best way:

> When a nerve *is removed* from a frog and connected with an amplifier and recording system, *occasional impulses* appear if the fibres are allowed to dry and a *short discharge* can be produced *by pinching and cutting*, but if drying is prevented there is *no sign of activity except during the actual infliction of an injury*. Mammalian nerves give a very different picture. In a medium-sized *nerve trunk* from the cat or rabbit, set up in moist, warm atmosphere, *large and rapid fluctuations of potential are nearly always present and the disturbance may last for an hour or more*, in spite of repeated irrigation. Very small nerves *may give a steady base line* and in nerves prepared for recording motor or sensory discharges *a disturbance, initially present*, *may subside during the exposure and manipulation*, but in all mammalian experiments *the danger of an unsteady background* is increasingly present as the condition of the nerve is a closer and closer approach to the normal. [(Adrian, 1930), *emphasis added*, *a*/*n*]

As we said, in passing from optics to physiology the habit of constructing rays has not changed. Indeed, the above characterization, can be taken as essential for the creation of a matter artifact which can be aptly called *a ray of nervous system*: a piece cut and removed from a whole nerve. What is, however, interesting for us, is the essential characterization of the matter of nervous system, which can be read in the phrases ‘removed’, ‘cut’ or ‘nerve trunk’: *even removed from its natural place*, *the nervous matter retains the property of exerting electric impulses*. If in its natural place one can suspect that an electric excitation within nervous matter might come out along the mechanical activity, of the muscles for instance, when isolated, that matter can by no means avail of this condition: it is ‘drying’, and its electrical condition cannot be influenced but by a ‘moist and warm atmosphere’. The muscles, then, cannot but only perturb this state, if they are generating charge at all. In a word: an *isolated nervous ray either retain the charge* from the whole to which it has been a part of, or *it has the capability of creating charge from the background of the universe*. Thinking of the possibility of memory storage of the nervous system, it appears as quite attractive the idea that both these possibilities are characteristics of the nervous matter: it is able to retain charge, just to the extent that it is able to create charge out of the de Sitter background of the universe that we inhabit.

Based on the previous phenomenology, then, Lord Adrian of Cambridge raised the big question that concerns the physics directly:

> A large number of units discharging independently yet *all of the same frequency* would give a composite effect *in which this frequency could be detected*, but *the amplitude of the oscillations* would be small unless there were some tendency for many of the units *to discharge at the same moment*. Thus the units *must be synchronised* with one another, and *we have to explain how this can take place*. [(Adrian, 1930), *emphasis added*, *a*/*n*]

Synchronization: this is the big problem of the physics of waves in general, even without having the same frequency. It is only using signals of given frequencies that Wilder Penfield was able to elicit recorded facts of experience hidden in the brain. And, of course, we have to agree with the conclusion of Lord Adrian, namely that:

> The *most reasonable explanation* seems to be that *an active fibre* can cause a *slight momentary increase in the stimulus to other fibres* and that it can do so *owing to the action current* which it produces. (*emphasis added*, a/*n*)

Many a time can one see this conclusion repeated ever since in different forms, and each time more emphatically, with the prominence given to it by the obvious feature of an intuitive truth. However, the explanation it deserves remained unsatisfactory, to say the least. We must, therefore, address this problem, 'if only out of gratitude', as Penfield would say, but for this we need further insights of neurophysiological nature, in order to dispose of a phenomenology.

The basic question to be answered here is: *what are those units*, *which Lord Adrian also call* '*fibres*' *in his observations*? We, the physicists, can learn what they are from a remarkable synthesis due to Valentino Braitenberg, who helps us in imagining a real ray within the nervous system, and completing the concept of ray over that brought in physics by Louis de Broglie. Quoting:

> In view of the uncertainty of the translation of the neurological reality into logical diagrams the basic element should be carefully defined and should not be burdened with too many unproven assumptions.
>
> a) The *basic element* of nerve nets is the *insulated nerve fiber*. It varies in length between 10 μm and 1 m, and in thickness between about 1/10 μm and over 20 μm.

The operation performed by such a fiber is *the transmission of an event* from one end to the other *without interference with the other events in other fibers*.

b) The *direction of transmission is fixed for each fiber*.

c) The *velocity* of transmission varies between a few cm/s and over 100 m/s. *It is fixed for each fiber* and *varies with the thickness of the fiber* (the thicker the faster).

d) The transmission of an event is a consequence of the occurrence of « excitation » within a region, called the *pickup-field*, associated with each fiber.

e) The transmission of an event produces « excitation » within a region, called *field of excitation*, associated with each fiber. Field of excitation and pickup field of the same or different fibers may be overlapping.

The phenomena called « event » and « excitation » are complex physico-chemical changes. It is convenient to treat the event which is transmitted in nerve fibers as a binary signal and to relegate to the intervenient excitation those effects which appear as monotonic (« facilitation ») or non monotonic (« inhibition ») functions of the number of active fibers.

The places where the quantity excitation determines the transmission or not transmission of an event in fibers are *knots* of a net. Note that according to the definitions given the knots include, besides the « synaptic junctions » between neurons, also *branching points of axons* (which are generally uninsulated) and, generally, *uninsulated segments*. In praxis we shall have to revert to the conventional identification of fibers with « neurons » due to the lack of precise physiological data on the interaction outside of the synaptic regions. [(Braitenberg, 1959); *mixed emphasis*; *the underlined words are emphasized in original*, *a*/*n*]

This is, indeed, the best synthesis of the neurophysiological work a physicist can ever dream! In order to honor it, we cannot but follow it faithfully. Thus, in the following development we shall try to use extensively the terms defined by Braitenberg, in order to solve some of the issues triggered by the phenomenology behind these definitions.

A general conclusion for us, the physicists, would be about the structure of a 'nervous ray', ultimately a physical structure, capable of conducting the signals generated by electricity in the animal bodies, especially in the brain. In the above excerpt, Braitenberg implicitly defines a physical ray, and in the very sense of Louis de Broglie at that: *a pencil of nervous fibers*. Each one of these fibers is characterized by the *direction of propagation* of the nervous signals and the *velocity of propagation* of these signals. Moreover, it seems that we have here a natural delimitation of a 'least part of a ray', as Newton would express it, by the two precise differentiae of the concept of field here: the *pickup field* and the *excitation field*, defining the 'synaptic junctions'. As it turns out, this phenomenology is quite enough for helping us in constructing the mathematics of the interpretation of the ray inside of the nervous matter. But, before anything else, let us see where physics stands in the problem of the obvious physical background of the nerve impulses: *the electricity*.

2.7. A General View on Electricity for Physics

When we said above that a muscle can produce electricity by mechanical activity, we were thinking implicitly of the deformation of the muscle. The deformation of matter is a general phenomenon occasionally connected to the presence of electricity. A well-known occasion is the mechanical constraint of a piece of matter (more specifically, the quartz, to name the first well-known case) offering the phenomenon of *piezoelectricity*, by electric polarization. The organic matter, however, is prone to a reverse characterization: an electric polarization leads to a deformation, of the muscles for instance. As a matter of fact, this is what physicists made out of the old discoveries of Luigi Galvani and Alexander von Humboldt, starting by the beginning of the 20th century, and we have strong reasons to believe that the situation has not changed at all ever since, in spite of the fact that there is a strong current of opinion to the contrary. Our argument: there is not a physical clue as to the *general relation between charge and deformation*, in spite of the fact that, classically speaking, there was – and still is, as a matter of fact – a concept relating the charge with a state of deformation of matter, the so-called *constrained deformation* [(Poincaré, 1901), Chapter II: *The Hypotheses of Maxwell*]. The reason seems to reside in the manner of *interpretation of the charges*, that is, by *fluids*. Quoting:

> One might still suppose the existence of *two inducing fluids* intertwining each other, *the molecules of one of them acting on the molecules of the other*, *whose molecules would be disturbed from their equilibrium positions*. However, if this hypothesis has the advantage of reducing the *special elasticity of the inducing fluid to the elasticity as we usually consider it*, it has the inconvenience of being more complicated than that of the existence of a *single fluid*. Thus, we think that the Maxwell's hypothesis of the inducing fluid is only transitory, and it will *be replaced by another more logical one*, *as the progress of Science will allow it*. One can object to us that Maxwell *has not introduced* this inducing fluid hypothesis; but, as we said in the beginning of this chapter, if the word does not exist in the work of this physicist, *the thing is there*; only, what we have called *inducing fluid* is designated by the word *electricity*; in the language of Maxwell the electricity of the dielectrics *is assumed elastic*, while the electricity of the conductors *is assumed inertial*. These *different properties* attributed to two fluids designated by the same name are the source of the lack of clarity presented by certain passages from the Maxwell's work. *It is just in order to avoid this obscurity that we have introduced the expression inducing fluid* in expounding the Maxwell's ideas. [(Poincaré, 1901), § 36; *our translation*, *emphasis added*, *a*/*n*]

More than a century went by since the great natural philosopher wrote these words. The 'progress of Science' is with us by now, but it still has not allowed to 'replace the hypothesis': it is one of those hypotheses necessary for interpretation, as this concept has come to us from the necessities of the wave mechanics. There is an interpretative fluid serving in the description of the deformation, that can be designated as *inducing fluid*, and an *electric fluid* serving for conduction, *connected to the charge*. The way these two fluids intertwine with each other shall be described by us in this work. However, the principle of this description was also given by Poincaré in the work just cited above, and remains the same as in those old times. Quoting:

> Let us finally show that the Maxwell theory leads to the same expression for the thickness of an electric layer located at the surface of a conductor, as the usual theory does.
>
> Let S be the surface that *separates the electricity of the inducing fluid in the state of normal equilibrium*, and S′ *the surface of separation in the state of a constrained equilibrium*. The free electricity being the excess of the quantity of electric fluid contained in the conductor in the state of constrained equilibrium, over the same quantity existing normally, the charge of the conductor is the quantity of fluid comprised between the two surfaces S and S′. *This fluid being incompressible*, the *charge in every point is* therefore, *proportional to the normal distance separating the two surfaces*. Let us consider a molecule of the inducing fluid located, in the state of normal equilibrium, en un point *m* of the surface S; in a state of constrained equilibrium, this molecule will come in the point *m′* of the surface S′. The triangle *mnm′*, whose side *mn* is the normal distance separating the two surfaces, can be considered as a triangle right in *n*. The thickness of the electric layer is therefore equal to the projection of the displacement along the normal to surface (actually, the displacement is normal to surface, but we do not need here to consider this property of the inducing fluid). [(Poincaré, 1901), § 40; *our translation*; *emphasis added*, *a*/*n*]

This idea too, will be followed by us in connection with the concept of physiological ray: as we shall see, *the charge can be relegated to a deformation*, and *vice versa*. It will help us in explaining the essential characteristics of the *fibers*, *defined by a velocity flux*, as in the Valentino Braitenberg's summary of nervous matter physiology. However, the general idea of physics here still remains that of understanding and, therefore, describing a *universal connection* between charge and deformation. In achieving this, the neurophysiology offers to physics the best working ambit.

3. Incentives from Physics for the Choice of Mathematics

Expressed in broad strokes, the previous chapter suggests some general requirements for the mathematical instruments in treating the brain universe. The presence of rays, as well as the definition of electricity makes the *theory of surfaces*, both analytical and differential, a central piece of this mathematics. On the other hand, the basic geometry of such a universe must be a Riemannian geometry, just like the geometry of the universe we inhabit, more specifically the conformal Euclidean geometry of the so-called Maxwell fish-eye optical medium [(Mazilu, 2024), § 1.2]. What makes this geometry especially attractive is the fact that it is prone to a general physical interpretation in terms of a concept of universe, inasmuch as it can be always obtained in connection with an absolute geometry (*loc. cit. ante*, § 3.4). But then, the theory of surfaces, while it should play a central role in the mathematical equipment of this physics, is by no means sufficient by itself. More physical requirements are in need special mathematical tools in that equipment, and we shall suggest them in this chapter, in connection with the physical incentives pointing toward them.

3.1. A Mathematical Perspective on Planck's Quantization: Planck's Differential Equation

To start with, an important *mathematical message* can be extracted, based on the special considerations of physics, coming from the Planck's procedure of quantization. Namely, as we said before, Max Planck may very well be acknowledged as having found that the considerations of thermodynamical equilibrium *cannot* be applied at all to the contents of Wien-Lummer enclosures, which, in our opinion at least, are the experimental counterparts of universes. After all, the temperature of that kind of equilibrium can only be *understood* as a statistic – *viz*., *the variance* parameter of the Maxwellian distribution of velocities – just for one of the physical components of such a content, that is, *the matter*, and, even that, taken in the special form of an ideal gas. Certainly, then, one can say that Planck has found a *fundamental property of statistics* to be applied in physics: *it should be described in terms of their variances*.

It is, indeed, in terms of the variances that Planck has guided his statistical reasoning on the two laws of radiation he had at his disposal, as special cases of the spectral density of radiation, in order to conclude to his general law of radiation. Those special cases have been assigned to two ranges of absolute temperature, associated with the two ranges of corresponding wavelengths: the *theoretical* Rayleigh-Jeans case, and the *heuristic* Wien's case of the radiation laws. Thus, Planck was able to assign variances for these two cases, and then, considering them as two limit cases which would represent two *statistically independent* processes, he assigned a variance to the *whole process* of radiation which was the sum of the variances in the two known cases [see (Mazilu, 2022), §2.1 for details]. A specific nonlinear ordinary differential equation thus has emerged, that entered the stage of theoretical physics ever since, describing the radiation as a stochastic process characterized by a family of exponential distributions having *quadratic variance distribution function*. One can say that Max Planck has just described a first *ontologically fundamental* case of this type of statistics, a type of which, epistemologically speaking, the statisticians themselves became fully aware only much later, apparently by the middle of the second half of the 20^{th} century (Morris, 1982).

Continuing this view on the achievement of great old sage of modern physics, we can completely *give up* the idea of thermodynamical equilibrium, apparently ineffectual for the case in point, and talk directly of a *process of fluctuations* of the spectral density of radiation, as of a *stochastic process* characterized by *a family of quadratic variance distribution functions*. Then, the physical system itself, whatever this may be, *taken as a universe*, can assume a definite geometrical description as *a moment in the sense of Stroud*, starting from the observations of a purely mathematical nature that will follow here, to which we shall try to attach a physical meaning. Remarkably enough, this physical meaning preserves the very classical considerations of statistical physics, – that is, that a physically meaningful statistic cannot involve but *conserved quantities*, as in the case of the absolute temperature of the ideal gas – however, with the benefits of participation of a purely Riemannian geometry, which, in this case, is far from being a hypothesis.

Obviously, in matters mathematical, the physiology does not have equations in this case, as Wilder Penfield would demand [see (Braitenberg & Lauria, 1960), for a discussion of the matters of modeling in the nervous system]. If the physiology ever comes *by itself* to equations once in a while, these are basically 'qualitative equations', if we may say so, of a specially conceived symbolic logic, intended to cope with the brain functionality. However, in our opinion, the symbolic logic's rules *must be a consequence of the physical laws* of behavior of the matter in general, of the brain matter in particular, and this fact has not been but partially fulfilled at best, in the theoretical physiology of cells. Our idea, that we think represents best the line of thinking of the Wilder Penfield, is that the physiological facts, too, should stay at the basis of physics in a fundamental way: being observational facts, they should be taken in the way the observational facts were taken in the classical natural philsophy, namely to stay at the basis of that philosophy, in the manner conceived by Newton. The archetype data of this construction of Newton was embodied in the group of three Kepler laws for the planets of solar system, and this is the part to be played, for us anyway, by the neurophysiological facts.

Assume, therefore, in the spirit of physics that came to us starting ever since Newton built its base, that we describe *the fluctuations* of the energy density of light in a Wien-Lummer enclosure with an equation that explicitly transits any scale by its very form:

$$u_0 = \frac{\alpha u + \beta}{\gamma u + \delta} \tag{3.1.1}$$

It is in keeping up with Planck's original case for the moment, and just for educating the guesses of our intuition, that the symbol u designates the spectral density *of radiation*. For, in this case, we read the equation (3.1.1) as follows: the spectral density of radiation has the instantaneous value u, that can be any value whatsoever, in a general environment described by the parameters (α, β, γ, δ), in which it assumes a *stationary value* u_0. The parameters (α, β, γ, δ) are four coefficients representing all the properties regarding *some other quantities that describe the physical content of enclosure*, *but related to the matter contained in it and to the enclosure itself*, *not to the light*, which is the object of study for which the enclosure is used. It is only a particular incident, if we may say so, that this value can be deemed as an *equilibrium value*, like, for instance, in the case of equilibrium thermodynamics. This incident is only due to the very manner of study of the radiation in a Wien-Lummer enclosure, that is, just like any system in thermodynamical equilibrium. With an old philosophical expression, one might say that 'the truth of the Wien-Lummer enclosure stays in the fact that *it is a universe*', which is exactly why we take it as such. This is a universal property, while the thermodynamical equilibrium cannot be taken as such, for it is quite a particular occurrence, subjective to a great extent, we might say.

As to the truth of this very enterprise of our intellect, it depends exclusively on the fact that such enclosure has the characteristics of the universe of our experience, as proved in later times by the NASA experiments regarding the background radiation, which show that it is a kind of blackbody radiation at the special temperature of 3K. We are, therefore, entitled to take this universe as a prototype of the concept, of which the brain should be, again, just another particular case. So, in general, a concept of universe is characterized by fluctuations of some physical category looking like the light from our prototype universe. These fluctuations cannot be completely arbitrary: they are connected to the physical properties of the system which, naturally, are accessible to our wits only by experiment. They are determined through a model-enclosure, whose state is characterized by a stationary value of the physical property of our *object of study* within that enclosure, *i.e.*, in the general case, of *an analog of the radiation* from the prototype case.

According to this philosophy, one rational way to read into Planck's procedure of quantization *per se*, in order to make a general law out of it, is the following, amounting, as we shall see right away, to the characterization of a physical system as being of the nature of Stroud's moment. Namely, we have many different possible stationary values of the spectral density of the physical process of light, and these are happening within the same Wien-Lummer enclosure. Then, if a stationary value is indeed a constant, the u-process can be described, in general, by an equation like (3.1.1), assigning the stationary value to the system by the very properties of the enclosure containing our system. Hence, if we express the steadiness of the stationary value by a differential equation that shows this very fact, we have:

$$du_0 = 0 \quad \therefore \quad du = \omega^1 u^2 + \omega^2 u + \omega^3 \tag{3.1.2}$$

Here, the differential forms $\omega^{1,2,3}$ represent a coframe for an algebraic space known to be Riemannian: the $\mathfrak{sl}(2,R)$ *Lie algebra*. Obviously, this is a manifold that can represent geometrically – and physically we claim, bearing in mind our physical problem – a universe in a particular situation, that is, *a state of that universe*. The algebraical expressions of components of such a coframe can be given in terms of the four parameters and their variations [see (Mazilu, 2022), §5.4], and they are:

$$\omega^1 = \frac{\alpha d\gamma - \gamma d\alpha}{\alpha\delta - \beta\gamma}, \quad \omega^2 = \frac{\alpha d\delta - \delta d\alpha + \beta d\gamma - \gamma d\beta}{\alpha\delta - \beta\gamma}, \quad \omega^3 = \frac{\beta d\delta - \delta d\beta}{\alpha\delta - \beta\gamma} \tag{3.1.3}$$

The equation (3.1.2) is thus an equation *in pure differentials*. This means that, taken as such, it can be used, let us say, in a fractal calculus, definitely in a stochastic calculus. However, maintaining the differential point of view, if by some chance we can refer that equation to a parameter – like the inverse temperature from the classical case of Planck – thus establishing a measure for the (continuous) variations of the parameters, we can transform it into an ordinary differential equation of the Riccati type. And that chance always exists by the very geometrical theory of the problem: the parameter in question is simply the arclength of the geodesics of any metric of the parameter space locally described by the coframe from the equation (3.1.3) above. Of course, the geometry is here just a 'category' as it were, for that parameter: it may carry many physical meanings.

Indeed, as we have already mentioned, we are dealing here with a metric geometry, more to the point with a family of ∞^6 Riemannian geometries, having as possible metrics quadratic forms in the differentials (3.1.3), with constant coefficients in the simplest of cases. The discriminant of quadratic polynomial from equation (3.1.2) is just one of those metrics for a special choice of the six constants entries for the metric tensor: it is *the Killing-Cartan metric*. It dictates the nature of solutions of the differential equation (3.1.2). As it can frequently happen with the $\mathfrak{sl}(2, R)$ kind of algebra, the differentials (3.1.3) are constant rates with respect to a parameter. To wit, along the geodesics of the metric just mentioned, the differential equation (3.1.2) becomes an ordinary differential equation, because the differential forms (3.1.3) are *conserved quantities*, with conservation laws to be written in the general form of some rates with respect to a single independent variable:

$$\frac{\omega^1}{d\phi} = a^1, \quad \frac{\omega^2}{d\phi} = 2a^2, \quad \frac{\omega^3}{d\phi} = a^3 \tag{3.1.4}$$

Here, the independent variable ϕ is the arclength of the geodesics of the Riemannian geometry of the Killing-Cartan metric, and a^k are constants, so that we can write:

$$\frac{du}{d\phi} = a^1u^2 + 2a^2u + a^3 \tag{3.1.5}$$

This is the most general form of *Planck's equation*: it leads to classical quantization in the particular take $a^3 = 0$, and with ϕ representing the inverse of absolute temperature [see (Mazilu, 2022), §2.1]. Extending the name to the original differential equation (3.1.2) that generates in a rational way the equation (3.1.5), one can say that the Planck's equation is universally valid, but only on special geodesics of the Killing-Cartan metric of the parameter space it becomes an ordinary differential equation. The Riemannian geometry comes, therefore, just naturally into this physics, by the way of algebra.

The most general case, including the Bose-Einstein and Fermi-Dirac statistics, the right hand side of the Planck equation (3.1.5) has a positive discriminant. In the opposite cases, where the discriminant of the right hand side of equation (3.1.5) is negative, the parameter ϕ has another exquisite *statistical interpretation*, needed by us, this time, in the description of a Lorentz quantum, as announced in the § 2.5. This interpretation involves the one-dimensional Cauchy distribution correlated with the elementary probability represented by the differential $d\phi$, which becomes quite obvious if we write the equation (3.1.5) in the form:

$$d\phi = \frac{du}{a^1u^2 + 2a^2u + a^3} \tag{3.1.6}$$

This transcription shows, indeed, that $d\phi$ can be taken as an *elementary probability of a distribution of Cauchy type* for whatever the symbol u may represent. Indeed, these cases cannot be those of Planck's blackbody radiation, obviously. In order to get a grip on them, let us treat them separately.

So, it is at this point that we should like a mathematical detour on the concept of charge, that can be of help later on in our developments. In order to understand the necessity of this detour, let us recall that the Planck's radiation law, which is our prototype in this physics, comprises in itself two different statistics: one is the genuine Planck's statistics, involving quadratic variance distribution functions, while the other is the Jeans' statistics on the frequencies. The Planck's law of radiation is the product of the variance of the first kind of distribution, with the density of the distribution on frequencies: one can say that this radiation law represents a *density of variance*, in general. Fact is, that if the charge enters the stage, we cannot exclude the possibility of a third kind of statistics entering the stage, and this in quite a straight way: *via* the Planck's concept of resonator, originally defined by Planck as a dipole. Let us, therefore, get a little deeper into this case.

3.2. The Quantal Nature of the Charge: Statistical Estimation Turned Cosmological

The way *we perceive* the charge was always a mystery fostering issues of the natural-philosophical nature, the most important of which was embodied in the question*: why is the charge always accessible to us only as an integer number of the electron or proton charge*? A first modern quantum-mechanical breakthrough towards the solution of this problem has been made by Paul Adrien Maurice Dirac, who introduced the concept of *magnetic monopole* (Dirac, 1931). This kind of pole is nowhere to be found *in our experience*: the perceived matter always comes to our senses in matter pieces characterized by pairs of magnetic poles. Nevertheless, the 6th decade of the 20th century brought a great elucidation in what became the critical problem of supraconductibility, and this elucidation was based on a quantum philosophy, whereby the magnetic flux plays a central part according to Dirac's theory of magnetic isolated poles. This elucidation reverberated, as it were, by a clever physical analogy due to Yoichiro Nambu, into an idea of solution for the problem of structure of the elementary particles (Nambu, 2008). The reader interested in general theoretical details of this analogy, can consult a beautiful article of Steven

Weinberg, which voices, as a matter of fact, a quite remarkable point of view in theoretical physics in general (Weinberg, 1986).

The key concept of physics, at that epoch in the solution of problem of supraconductibility, was the so-called *Cooper pair* (Cooper, 1956). Roughly speaking, but with a significant benefit for our understanding, this pair represents the fundamental unit of 'electric moving charge', if we may, which can explain this kind of conductibility (Bardeen, Cooper, & Schrieffer, 1957). Incidentally: we find the fact quite significant, that Leon Neil Cooper was intensely involved, as a physicist of course, in the problems of *structure of the brain* [see, for instance, the biographical notice and bibliography in (Apartsin, Cooper, & Intrator, 2014)], but this may be subject of another story. For now, though, the name of Cooper will be taken exclusively in connection with a natural philosophy brought by his discovery from the point of view of the present work, about *the brain as a universe*. We present here this natural philosophy along the following line of facts.

First of all, we take notice that Cooper's discovery is referring to a solution of *the Planck's equation* (3.1.5), but along the lines initiated by Planck himself: as mentioned above, the Fermi-Dirac distribution, which is mentioned by Cooper as a target in his 1956 work, can certainly be assumed as generated by a solution of the original Planck's equation. We see here the presence of a universe of the Maxwell fish-eye type, in the form of a Lorentz quantum, in the solution of this problem: a mixture of (plane) waves – the phonon background in a solid – creates the opportunity of a different 'zero' of the charge in a universe where the Kepler planetary unit has as components only particles of the same kind, except for their spin, in order to be in accordance with Pauli's exclusion principle. This general natural-philosophical idea may come easily to our mind, indeed, in view of the logic presented by us above, but the fact is that the analogy proposed by Yoichiro Nambu in 1961 (*loc. cit. ante*), has brought especially the charge to attention of particle and field theorists. And they attached to this solution of supraconductibility a fundamental meaning, involving the Cooper pairs along with an idea *quantization of charge* (Yang, 1970).

As we see it, the problem here is one of *interpretation* (Yang, 1962): find those 'material points' (taken in the sense of Hertz, of course), the ensemble of which can explain the supraconductibility. And sure enough, in the intuitive presentation of Richard Feynman (see the ever-celebrated *Lectures*, Volume 3, Chapter 21), the Cooper pair is taken as a boson, which is why the superconductivity phenomenon occurs: below the *critical temperature* – a parameter specific to every solid – there do not exist but such free bosons *in the very same state*, like the bosons always do. And they move unperturbed forever, if the ambient temperature does not change, which explains the phenomenon. One can say that *there are not collision forces* to liberate the electrons of the pairs to their own isolate states. Better expressed yet, there may not be *collision events* in a conductor of this type, at least not as many as to make a strong case for the existence of free electrons. Consequently, the whole theory is based on the prototype bosonic field, that is, the one satisfying the Schrödinger equation: the classical wave function is here turned into a field representing the so-called *order parameter* (Weinberg, 1986).

It is at this point that Chen-Ning Yang intervenes (Yang, 1970), with considerations on the quantization of charges. He noticed that in the case of an ensemble of charged fields, ψ_j say, each having the charge e_j, there is a *gauge transformation* involving just the charge (Yang & Mills, 1954), which can be written as:

$$\psi_k \rightarrow \psi'_k = \psi_k \cdot \exp(i \cdot \alpha \cdot e_k) \tag{3.2.1}$$

where we preserved the original symbolistics of Yang. There are many definitions of the concept of gauge transformation, most of them being quite hard to understand in simple terms. Here, however, in equation (3.2.1), the meaning of such a gauge is quite obvious: it says that the general facts of our experience, theoretically explained by the existence of the fields ψ_k, are explained in the very same way by any possible copy ψ_k' of that field, obtained from a given one by the rule (3.2.1). With equation (3.2.1), we just 'fixed the gauge' of the theoretical description, as they say on occasions: we use *specifically the copy* (3.2.1) *of the field* in illustrating those general facts of experience.

On this occasion, therefore, we do not have so much to do with a transformation *per se*, as much as with an 'acknowledgment', as it were: practically, it would mean that we just need to be aware, and acknowledge this awareness, of course, that the theory giving the 'explanation' of behavior of the fields in the 'construction' of the universe we inhabit, is done based on a 'copy' of the field, taken in any place at any time, but with the charge entering the explanation *via* the 'phase' ($\alpha \cdot e_k$). The charge e_k is *spacetime independent*, but the parameter α may depend on the location in space and on the time. These properties, suggested by facts of our experience, are reproduced by the rules of handling the fields ψ and the charges e, as we know them according to our experience, for any field and charge from an incidental list of that experience.

The trouble starts brewing when considering these 'charged' fields together. Notice that, in order to make a proper phase out of the monomial ($\alpha \cdot e_k$), the parameter α needs to have the physical dimensions of an inverse charge. And this is the observation that directed the attention of theorists toward the charge quantization in connection with the fundamental physical structures of the matter: the inverse of a charge is the physical characteristic of a magnetic flux (Dirac, 1931). In its turn, the magnetic flux is an essential 'observable' in the supraconductibility phenomena: it can be handled by physical procedures involving supraconductors, as proved, for instance, by the existence of the Meissner effect (see Feynman Lectures, *loc. cit. ante*).

The case considered by Yang in his little article from the year 1970 just cited above, is that of a world of protons and electrons: according to our experience, these are fundamental physical constituents of matter in the universe we inhabit, and their charges are equal in magnitude, with a high degree of certainty, but opposed in sign. Notice that the two fundamental constituents of matter differ, according to classical dynamics, only by their inertial properties. However, from a cosmological point of view – which is the natural point of view to be adopted when one discusses a universe, *as we do* – there is a possibility that *the magnitude* of their charges may differ from each other beyond our capability of deciding quantitatively (Lyttleton & Bondi, 1959, 1960). Even such a level the discrepancy in the fundamental charges of our experience, practically unnoticeable, can, however, make a significant difference in the appearance and theoretical description of *the universe we inhabit* (Hoyle, 1960), which is what interests us here in the first place. According to the ideas promoted by Yang's work, which implicitly involves a cosmological point of view in his consideration of elementary particles, in this case the quantization of charge cannot be a fact, as first assumed. In order to see this, while introducing our view alongside, we need to consider the Cooper pair as a fundamental particle – just like Yang did (Yang, 1962) – but keeping in mind that it is such a particle *only in a world where the supraconductibility is a phenomenon of current experience*, as C. N. Yang recommended himself.

Indeed, we have to notice that the ensemble of all charged fields described by (3.2.1) corresponding to the same charge *is a group*, more precisely, a *commutative group*, due to the known properties of the exponential:

$$\exp(i \cdot \alpha \cdot e) \cdot \exp(i \cdot \beta \cdot e) = \exp\ \{\ i \cdot (\alpha + \beta) \cdot e\} \qquad (3.2.2)$$

One can say that the commutativity is a consequence of the corresponding property of the addition of numbers: for the field of a certain charge, the gauging can be done with no restriction, since the multiplication of the exponentials produces an exponential of the same charge. This property of the exponentials is the one that lets the idea of quantization of charge slip in, to start with. The situation changes, and radically at that, when one is bound to consider more fields, however, with more than one charges of different magnitudes, for in that case the property of commutativity is jeopardized. In order to see this, it is sufficient to consider just two such charges, for which (3.2.2) reads:

$$\exp(i \cdot \alpha \cdot e_1) \cdot \exp(i \cdot \beta \cdot e_2) = \exp\ \{\ i \cdot (\alpha \cdot e_1 + \beta \cdot e_2)\} \qquad (3.2.3)$$

As one can see, this compound exponential does not represent a charged field, unless the condition:

$$\alpha \cdot e_1 + \beta \cdot e_2 + \gamma \cdot e_3 = 0 \qquad (3.2.4)$$

is satisfied, for some incidental e_3. If the three unit charges exist, such that (α, β, γ) are arbitrary, and the condition (3.2.4) is still satisfied, these units cannot be scalars, as the usual charges: they must possess the characteristics of the base vectors in linear spaces.

According to Chen-Ning Yang, the current situation of the charges in physics, is the following: if the two charges from the equation (3.2.3) are not a multiple of the same unit charge, the group property just mentioned above is broken. It is maintained only if the two charges are, to use the expression of Yang's, 'commensurate', *i.e.*, speaking for the general case of many charged fields, only if their charges are integer multiples of the same *universal unit*, *e* say:

$$e_1 = n_1 \cdot e,\ e_2 = n_2 \cdot e,\ e_3 = n_3 \cdot e\ \ldots \qquad (3.2.5)$$

Then, the equation (3.2.3) becomes

$$\exp(i \cdot \alpha \cdot e_k) \cdot \exp(i \cdot \beta \cdot e_l) = \exp\{i \cdot e \cdot (\alpha \cdot n_k + \beta \cdot n_l)\} \qquad (3.2.6)$$

no matter of the pair (*k*, *l*), and the exponent from the right-hand side of this equation becomes, indeed, a *bona fide* phase, as in equation (3.2.1). This appears to be the only situation that can save the condition of the known charges of our experience – of being expressed by real numbers – along with the universality of an equation of the type (3.2.4).

This reasoning can be extended, indeed, to as many charged fields of this type as we happen to have in reality: the end result is the same, but a price must be payed. According to Yang this is a special property of the group characterized by the composition rule from equation (3.2.2): it describes a *compact group*. By this it is always meant, in physics at least, a certain generalization of the finite group of exponentials to a continuous parameter group with the parameter (or parameters, if the case may occur) covering a finite-volume space. Quoting Yang's own words:

> If the different e_j's (= e_1, e_2, …) of different fields *are not commensurate* with each other, the transformation (3.2.1) is different for *all real values* of α, and *the gauge group must be defined* so as *to include all real values* of α. *Hence the group is not compact*.
>
> If, on the other hand, all different e_j's *are integral multiples* of *e*, a *universal unit of charge*, then for *two values* (*original emphasis*, *a*/*n*) of α different by an integral multiple of $2\pi/e$, the transformation (3.2.1) for any fields ψ_j are the *same* (*original emphasis*, *a*/*n*). In other words, *two transformations* (3.2.1) *are indistinguishable* if their α's are the same modulo $2\pi/e$. *Hence the gauge group as defined by* (3.2.1) *is compact*. [(Yang, 1970); *emphasis added*, *except as mentioned*, *a*/*n*]

According to Steven Weinberg, the symmetry is 'Z_2-broken'(Weinberg, 1986): the world has just two representative charged gauge fields, α and $\alpha + 2\pi\hbar/e$, for any α. However, the possibility of the introducing the noncompact groups needs to be taken in consideration too, and this is the price to be payed, of which we are talking about, for it raises an important problem: *to define a priori such a group*. Considering the charge quantization exclusively through the pair solution of the problem of supraconductibility may be quite deceitful from a cosmological perspective. For, quoting again from Yang:

> In the experiment of flux quantization, one finds that magnetic flux trapped in superconducting rings are in whole units of $2\pi c\hbar/2e$. What should *e* be *if the electron and the proton do not have the same charge*? The answer is that *e should be the electron charge*, *since electron pairs*, not the protons, *are the* "*basic group*" that possess off-diagonal long-range order *in a superconductor*. To illustrate this point further, let us assume that spin-up electrons and spin-down electrons have charges $-e$ and $-e'$, respectively, and that the basic group is a pair of electrons with opposite spins. The flux unit would then be $2\pi c\hbar/(e + e')$. If, on the other hand, there are two kinds of spin-up

electron with charges *e* and *e′* which are incommensurate, and two kinds of spin-down electrons with similar charges, and, furthermore, if pairs of electrons *e–e*, *e–e′*, *e′–e*, and *e′–e′* of opposite spins all have off-diagonal long-range order in a superconductor, then the flux unit is an integral multiple of $2\pi c\hbar/2e$, $2\pi c\hbar/(e + e')$, and $2\pi c\hbar/2e'$. Hence it is ∞. To summarize, the existence of a finite flux quantization unit merely reflects on the quantized *nature of the charge of the basic groups in a superconductor*, and *does not* necessarily imply that electric charge is *always* quantized. But if the flux quantization unit were found to be ∞, one would have concluded that the *electric charge is not quantized* [(Yang, 1970); *emphasis added*, *a*/*n*].

The quantum effects connected with that 'off-diagonal long-range order' are explained by Yang in his work dedicated to the problem of interpretation (Yang, 1962). The overall conclusion of this theory is that one cannot declare universality of charge quantization based on just considerations of supraconductibility, for this is quite a particular state of matter: this specific phenomenon is not referring, in fact, to an elementary particle, but to a compound, that is, to that 'basic group' which, in Yang's theoretical understanding, is what we have called here a 'Cooper pair'. This fact led us to the conclusion that Yang's approach to the concept of interpretation is actually the right one, only has a wrong address, and this can be 'felt', so to speak, even from his discussion in the last excerpt above: if the charges in the universe are 'incommensurate' the group describing the gauge *must be noncompact*. Consequently, the exponential is not a proper functionality for representing the field or, better yet perhaps: the exponential *alone* cannot represent the gauge. It takes a certain *physical cycle in matter* in order to do that. In any case, from a groupal point of view we need to involve more functional forms, in order to cope with the condition of more than one group variables and/or parameters.

Had the quantization been pursued *ad litteram* along the original Planck's procedure, this ambiguous situation would never occur. First of all, in the spirit of Planck's quantization we would have to admit along with the charges of proton and electron – which, from the cosmological point of view, are supposed to stay at the basis of this world as it appears to our senses – the charge of the *neutron*. While being a short-lived compound, this fundamental particle can still count as a material particle, more precisely, as a *material point* in the sense of Hertz. So, instead of one universal unit referring to the magnitude of charge, we should have three such units, referring to the components of the universal charge, a fact that, from the perspective of the physics of the last century, would appear as quite natural. So we would have to assume a different approach to the Yang's theory of 'basic groups', and the following approach which appears to have much of a statistical nature imbedded in it when connected with the theory of charge, seems to be most recommended in view of the Planck's quantization.

First, we have to ask ourselves: what happens if the considerations of discrepancy invoked by C. N. Yang in his little article, are referring not to the magnitude of the universal unit charges, but to the magnitudes of their *universal units* separately? In fact, what would happen if the situation in the universe we inhabit would be exactly that invoked by Yang?! To wit: *pairs of electrons of different charges*, both negative and positive and, let us not forget, *protons of positive and negative charges* that can be paired with the electrons in planetary physical structures, as well as in the structure of ephemeral neutron. And, along this line of recalling physics' facts, perhaps is best not forgetting to add to the list the *pairs created from light in vacuum*. The answer is that the measurement of charge itself would be a 'Planck problem', even *independently of any considerations of magnetic flux*. And, if the universe is taken as a Wien-Lummer enclosure containing matter and charge, the quantum of charge would appear as a consequence of Planck equation (3.1.6): the charge would be characterized by a Cauchy distribution. This observation turns our attention to the practical determination of such a kind of distribution, since this determination involves a kind of 'sample estimation'.

Professor Peter McCullagh from Chicago University, has once noticed a fact that picked up the interest of statisticians of many persuasions along the time. Namely, there is an important property of the one-dimensional Cauchy distribution, whose elementary probability is given by us in equation (3.1.6) above, mainly due to its important consequence in estimation. This property is manifest as a benefit of a *complex* parameterization of this distribution (McCullagh, 1996); [see also (Mazilu, Agop, & Mercheș, 2021), especially Chapter 4 of that work]. The parameters of families of statistical distributions are usually taken as real, especially when it comes to their evaluation by some statistics on samples. However, McCullagh has disclosed a clear advantage of representing these parameters in a complex form, which, at least when it comes to the case of the Cauchy distributions, specifically facilitates their sample estimation. He started with the observation that a *normalized density* for this distribution of a single statistical variate, Y say, can be written in the form

$$f_Y(y\,|\,\theta) = \frac{|\theta_2|}{\pi|y-\theta|^2}, \quad \theta \equiv \theta_1 + i\theta_2 \tag{3.2.7}$$

where θ is the 'complex parameter' of the distribution, and y is the value associated to Y in a real range. For the sake of understanding what the estimation may mean here, we mention that the real part of the parameter θ gives the *location* of the data described by this distribution, while the imaginary part roughly characterizes the *dispersion* of the data, as in any regular measurement. One can write the equation (3.1.6) formally, as the elementary probability $f_Y(y|\theta)\cdot dy$, taking a general variate value y instead of Planck's u, with the complex parameter θ calculated in terms of the three parameters (a^1, a^2, a^3), assumed real. This involves standard algebraical procedures of calculations, that we may skip here, momentarily just for the sake of a continuity of our story.

And that story goes on saying that this complex representation of the parameter of Cauchy distribution, brings to the fore one of the most important consequences of the existence of this type of distributions. Namely, if Y is a statistical variate that belongs to the Cauchy family of distributions with the complex parameter θ, *i.e.*, expressing it symbolically, if the variate $Y \approx C(\theta)$ – in words, meaning that 'Y is a statistical variate that belongs to a univariate Cauchy distribution from a class having the McCullagh parameter θ' − then we must have the consequence:

$$Y \approx C(\theta) \quad \Leftrightarrow \quad \frac{aY+b}{cY+d} \approx C\left(\frac{a\theta+b}{c\theta+d}\right) \tag{3.2.8}$$

That is: the statistical variate given by a homographic transform of Y belongs to the Cauchy class of the homographic transform of the complex parameter. This property allows us to give efficient estimators for the complex parameter θ, based on *the principle of maximum likelihood*. The statistical theory then reveals what one already knows about the Cauchy distribution, from the statistical practice. First, with the information of only two measured values we cannot have a reliable estimation for the mean: it can be any value between the two measured ones. As to the *variance estimator*, which would indicate the type of 'spreading' of the Cauchy variate values over the population described by the Cauchy distribution, it is also indeterminate, but this is quite a natural characteristic, as it were, of this type of distribution: it has no finite moments of higher order. This time, however, a certain kind of 'definiteness' is present, that may mean everything from physics' point of view.

First, at this point we can easily see the advantage of equation (3.2.8): it shows that the best assessment of the Cauchy distribution by sampling data, involves just as many measured values of Y as *a reliable assessment of a real homography*, or Möbius transformation, in the phrasing of Peter McCullagh himself. That is, we need to have *three* measurements in a sample of the statistical variable Y, in order to assess a Cauchy distribution the best possible way. The general estimator will then be calculated from a particularly convenient Cauchy distribution through a well-defined transformation based on sampling. Let us do some calculations here. The equations of the log-likelihood procedure with three measured values are given by:

$$\sum_k \frac{1}{y_k-\theta} + \sum_k \frac{1}{y_k-\theta^*} = 0, \quad \frac{3}{i\theta_2} + \sum_k \frac{1}{y_k-\theta} - \sum_k \frac{1}{y_k-\theta^*} = 0 \tag{3.2.9}$$

where the lower index k runs, in the sum, through values *1*, *2*, *3*. The meaning of the two equations is that *the likelihood of the sample needs to be stationary for a safe evaluation of the probability density*, which is exactly the property required by Planck's equation. The first one of equalities (3.2.9) shows that the sum involved in it is purely imaginary, inasmuch as its real part vanishes. In this case the second equality comes down directly to the imaginary value in question, *viz*.,

$$\sum_k \frac{1}{y_k-\theta} + \frac{3}{2i\theta_2} = 0 \tag{3.2.10}$$

or its complex conjugate. Now, the calculation of the real estimators for θ_1 and θ_2 is straightforward, but it may appear a little bit tedious. We can simplify it on both accounts, but with a substantial profit for theoretical physics too, as we shall see, by using the property (3.2.8) of the Cauchy distribution.

To sumarize: the procedure of estimation here amounts to choosing three *convenient* values for Y – the original choice of McCullagh's being (-1, 0, 1) – and calculate the complex estimator of θ for them. Then, take the homographic transformation of this estimator through the homography that carries the values -1, 0, 1 into some generic sample values y_1, y_2, y_3 of Y, and that is our estimation of the parameter θ, given the sample of 'volume' three (y_1, y_2, y_3), if it is to use the statistics' practice terms. Indeed, such a *real* homography is always well defined, inasmuch as only three entries, represented by some adequate choice from among the different ratios of the parameters a, b, c, d – the entries of the matrix realizing the homography in (3.2.8) – are sufficient in establishing the correspondence. In general algebraical terms, this is a reflection of the fact that the *homographic action* of a 2×2 matrix is well determined by three particular values of the variable and of their transformed values. To show this, let us consider that the values (y_1, y_2, y_3) correspond to the values (-1, 0, 1) in this order. If the matrix of this homography has the entries a, b, c, d, then we can find the values of these parameters from the system of equations:

$$y_1 = \frac{-a+b}{-c+d}, \quad y_2 = \frac{b}{d}, \quad y_3 = \frac{a+b}{c+d} \tag{3.2.11}$$

Solving this system in terms of a, b, c, d, [see (McCullagh, 1996), p.794], gives for these values a simple infinity of possibilities, for they are established only up to a common factor, by the system of equations:

$$\frac{a}{y_2y_3 + y_1y_2 - 2y_3y_1} = \frac{b}{y_2(y_3 - y_1)} = \frac{c}{2y_2 - y_3 - y_1} = \frac{d}{y_3 - y_1} \tag{3.2.12}$$

Nevertheless, the homographic action of the matrix is determined *in a unique way*, indeed, provided we disregard the permutations within samples in consideration.

This last issue is, however, irrelevant to some extent, for, when it comes to the estimators themselves, their expression in terms of sample values are invariant with respect to any permutations of the three values. Indeed, the problem is now to find the complex estimator θ for the particular values (-1, 0, 1), taken in this order. This can be easily done from equation (3.2.10) and its complex conjugate, which, in this case provides the system:

$$\theta_1 = 0, \quad 3\theta_2^2 = 1 \tag{3.2.13}$$

no matter in what order we take the three values used to calculate the estimators. The numerical values are, obviously, provided here by the McCullagh's choice of the three particular values.Therefore, in this particular case we have simply $\pm i/\sqrt{(3)}$ as an estimation for the very parameter θ, which turns out to be purely imaginary indeed. On the other hand, the estimator according to arbitrary data (y_1, y_2, y_3) will then be obtained through the homography with entries given by equations (3.2.12). According to the theorem expressed in equation (3.2.8), this shows that between the general value of the parameter θ, and the particular value $\pm i/\sqrt{(3)}$ there should be a homographic relation given by equation

$$\theta = \frac{(y_2 y_3 + y_1 y_2 - 2y_3 y_1)\frac{i}{\sqrt{3}} + y_2(y_3 - y_1)}{(2y_2 - y_3 - y_1)\frac{i}{\sqrt{3}} + (y_3 - y_1)} \tag{3.2.14}$$

which, obviously, depends on the particular correspondence between the triplets (*–1*, *0*, *1*) and (y_1, y_2, y_3), for there are three such possibilities of (3.2.14) given by the permutations of the three values of (y_1, y_2, y_3) with respect to the fixed triplet (*–1*, *0*, *1*). However, the real estimators calculated from this formula by taking the real and imaginary parts of θ, are:

$$\theta_1 = \frac{\sum y_1(y_2 - y_3)^2}{\sum(y_2 - y_3)^2}, \quad \theta_2 = \sqrt{3}\frac{(y_2 - y_3)(y_3 - y_1)(y_1 - y_2)}{\sum(y_2 - y_3)^2} \tag{3.2.15}$$

with the summations extended over all three positive permutations of the indices of the sample. Therefore, the conclusion follows, that the real estimators are the same no matter what is the order of correspondence in the samples of size three. Thus, any one of the three possible correspondences (3.2.14) will do, giving for the estimators of the real and imaginary parts of the McCullagh complex variable – *i.e.* for the *estimators of location and dispersion* of the Cauchy population, using samples of size three, in statistician's language – the same real values (3.2.15), as already stated before.

The McCullagh's complex estimator of the Cauchy distribution has an important algebraical, and even geometrical connotation, as we shall see, worth considering from a physical point of view. If the three generic values (y_1, y_2, y_3) are the eigenvalues of an Euclidean tensor having a physical meaning, they are simply the (real) roots of a cubic equation. Then, the complex estimator θ is the root of the corresponding *Hessian equation*, playing the part of a general resolvent of that cubic equation [see (Burnside & Panton, 1960), Chapter VI]. Indeed, in terms of the roots of a cubic equation its Hessian equation is:

$$\left(\sum(y_2 - y_3)^2\right)y^2 - 2\left(\sum y_1(y_2 - y_3)^2\right)y + \left(\sum y_1^2(y_2 - y_3)^2\right) = 0 \tag{3.2.16}$$

and its roots are θ above and its complex conjugate: the algebraic expressions from equation (3.2.15) are the real and imaginary parts of these roots. The sum and the product of the two complex estimators are given by the mean and the standard deviation of the three values of the samples of Y, with respect to the system of three probabilities invariant to positive permutations of the components of Y:

$$p_1 = \frac{(y_2 - y_3)^2}{\sum(y_2 - y_3)^2}, \quad p_2 = \frac{(y_3 - y_1)^2}{\sum(y_2 - y_3)^2}, \quad p_3 = \frac{(y_1 - y_2)^2}{\sum(y_2 - y_3)^2} \tag{3.2.17}$$

which they determine quite naturally.

Now, in matters of physical interpretation of this 'sample estimation', a problem still remains: namely that of the 'seed' numerical values (*–1*, *0*, *1*) of Peter McCullagh. In a practical situations regarding the charges of our experience, we should have to replace the McCullagh's triplet (y_1, y_2, y_3) with ($-e$, 0, e). Therefore, we are compelled to assume that the three values (-1, 0, 1), carry the meaning of an *essential information* for the classical definition of charges: they represent the sign of the charges. Then, the Planck's quantization approach to the problem of charge would mean, first of all, that between such an information and the value of charge, there is a homographic relationship. This relation can be taken as essential in 'calculating' the *information content of the charge*, so necessary in the problem of the memory of a universe.

When limiting to a purely statistical point of view, the 'seed' (-1, 0, 1) appears as just a choice among many others of the kind, having, therefore, mostly the *subjective* attribute of a convenience: one chooses these values to have an *easy direct estimation* for the two real parameters of the Cauchy probability density (3.1.6). However, *if* this estimation is based on samples referring to the whole universe, as in our case, it can count only in connection with the Planckian homographic action (3.1.1), and we are tempted to ask: what if one *has no possibility of other choice ever*? All of a sudden the sample above, along with its estimations, *become objective*: the equation (3.2.11), supporting the estimation process, now starts to represent a correlation of the physical magnitude represented by y in an equation like (3.1.6), in a universe with the three *a priori* stationary universal values of this quantity, taken just as the triplet ($-e$, 0, e), only according to Planck's quantization procedure. And, from a physical point of view, when taking these values as stationary values in the sense of Planck (see §3.1 above), we need to add that they are referring to an *instantaneous universe* – an instanton, *ad litteram*, in our acceptance – where there is no possibility of some other sample choices, insofar as the very equation (3.1.6) is referring to any stationary universe, just like the Planck's procedure of quantization generating it.

This situation can be seen as a fundamental property of the family of universes having the same de Sitter background, like any universe we can imagine. Namely, there is no other possibility: there should be three unit charges in every existent universe, and the transition from a universe to another is given, according to Yang's observation, by a noncompact group of homographies. The procedure of constructing of such a group involves the equation (3.2.11), which lead to a firm conclusion: in any universe there is a cubic equation having the three specific unit charges (y_1, y_2, y_3) as roots. Then the general transition between universes of this family is effected by a transformation involving the Hessians of the two corresponding cubic equations. This view will be developed along the present work, in spite of the fact that statistics of the case is a little more involved. We can get a complex quantity playing the part of the root of the Hessian of a cubic equation with real roots *via* the two *Novozilov statistics* (Novozhilov, 1952), which require integration on continua of 'hidden variables', as it were, over the space of a Lorentz quantum [see §2.5; for details on the geometry of families of cubic equations one can consult (Mazilu & Agop, 2012), Chapter 7; the § 5 of that chapter has the definitions of Novozhilov statistics characteristic to an Euclidean tensor].

Regardless of any statistics, fact is that here, from a historical point of view, the electrodynamics learned an old lesson, manifest explicitly in Yang's approach: *it needs a statics* necessary for interpretation, which in view of the history of electricity cannot be described but by *static charges*. So, Wu and Yang (1969) constructed a first static solution of the Yang-Mills field equation, that met the requirements of a field theory erected on the idea of so-called *pseudoparticles*, instantons in particular. This way, the noncompact groups made their entrance in the mind of theorists. However, if we follow the above suggested natural philosophy in solving the Yang and Mills problem, we see that such groups stay at the basis of the theory from start.

For once, the theorem (3.2.8) changes the view, not only by giving a new perspective on quantization, through the very fact that the structures of the SL(2, R) type are compact only in special conditions: the main representatives of this algebraic structure are conspicuously non-compact. So they should be, if it is to apply them to some other worlds. And when we say ‘other worlds’, we do not have in mind just the blatant example of the brain, where the creation of charge may be supposed as being connected with the homographic action of the neurons, but even the world of our experience: the Félix Cernuschi’s idea of *excited particles* in the relativistic case, for instance, or even the Fennelly’s ‘Gödelised hadron’ for that matter, can thus be mathematically shaped, and therefore physically understood. Let us insist on this point for a while, for it may be of essence latter.

All these arguments plead for thoughtful concern, if we may say so, of the role played by the three numbers (*–1*, *0*, *1*) of our experience, or of their equivalents in any other experience, when attached to charges. Fact is that the charges have never been defined in our experience, but only by interpretation, in connection with some other quantities (lengths, distances, durations, etc) or physically known processes (deformation of matter, thermodynamical processes, and such), so that in case we attach numbers to charges, we need to recall these issues. In what follows from this instalment of our work, this is our main concern.

3.3. A General Group Related to Charges and Its Target

One can say that, the dynamics in the realm of electricity, has effectively started with the works of André-Marie Ampère. What Ampère added to Newtonian theory of forces in order to make it usable in the dynamics connected with the charges, was apparently intended to bring their concept back to the natural-philosophical Newtonian definition. Maintaining Newton’s conception in the representation we make of the force, the previous section suggests rephrasing the Corollary III of the Proposition VII from the First Book of *Principia*, for instance, in order to make it compatible with the post-Newtonian experience. Tentatively, this reformulation should start somehow in a form leaving the possibility of introducing the considerations of scale transition as, in fact, was the case historically. To wit, in such a rephrasing, we may have for the Newton’s Corollary III, for instance:

> The force by which the body P in any orbit revolves about the center of force S, is to the *electric* force by which the same body *revolves in the same orbit*, about *the same center of force* S, as the solid etc etc …

Such a reformulation of the definition of Newton would make the defined forces capable of generating a dynamics at the microscopic scale of the world. However, we do not have too much of a possibility of filling in for suspension points in the new Newtonian definition above. In order to do this, we have to appeal to the forces between currents, as defined by André-Marie Ampère. And this is possible, indeed, but only realizable within the quantization procedure of Planck.

Perhaps, it is best if we present our incentives for choosing this path of our reasoning, so that our reader may judge by himself as to the virtue of our option. It all started from the Newtonian definition of the static forces. Recall, again, the definition of these forces as given in that celebrated Corollary III of Newton:

> The force by which the body P, in any orbit revolves about the center of force S, is to the force by which the same body *revolves in the same orbit*, about *any other center of force* R, as the solid etc etc …

Notice that in the previous definition here, it is tacitly implied that the center of force of the *gravitational forces*, which generates the orbit dynamically, let us say *R*, is the same as the center of *electric forces* generating the same orbit. However, the original Newton's definition involves *two* different centers of force from the plane of the orbit: *S* and *R*. His definition is referring to the ratio of the forces acting on the body *P* towards these points, and, if we know the force towards one of the points, we can have the force toward the other one. This was the original case of Newton, and he considered the forces toward the center of the Kepler ellipse – the universal planetary orbit in the sky – known *a priori*: they are, indeed, elastic forces. And it is only by using this *a priori* knowledge, that the Newtonian definition produces the static forces usually considered today as 'Newtonian': acting toward the focus of the Kepler ellipse, and having the magnitude inversely proportional to the square of the distance from that focus.

Two things struck our mind when reaching these conclusions. The first of them is regarding the 'corporality of forces', so to speak, in the sense of a Mach's principle: the forces must act toward bodies, not just in directions. In this understanding, the force toward the focus of the Kepler ellipse makes sense, physically speaking: there, in the focus, the Sun is placed, in the case of planets, and thus one can say, with Mach, that this force is physical. And, assuming that the force toward Sun is a also a dynamic force, the classical dynamics produces, indeed, a Kepler ellipse as trajectory of the planet. However, such a force is simply a figment of our imagination according to the very Newtonian definition of the forces, since in the center of ellipse there is no body to ascertain its 'corporality' in the Mach's sense. This, in particular, may be taken as the reason of the first re-formulation given by us above of the Corollary III of Newton: the natural-philosophical idea is that the forces are due not to the bodies *per se*, but to their physical properties (mass, charge, and so on), and if we have two such properties cumulated into the same body, the two centers of force occupy, actually, the same position. The problem, then, would be transferred into one of the geometry of the very physical properties generating the force, and the bases of this geometry were set by Newton himself, through his theory of forces.

In this connection, we must mention the second one of the two things that struck our mind, as mentioned in the previous paragraph: it is connected to *orbit*. This one was understood by Newton as a *static object*, however, provided by the principles of dynamics. The geometry, in its analytic form, discovered that the orbit can be *located as any position*, but in an abstract space. Then the problem is transferred on the geometry of these objects, for the forces generating them are not necessarily connected with the focus. This fact is plainly proved by the case of *binary stars*: each one of these systems exhibits an elliptic orbit, which has a particular point *different from the focus* that plays the part of the center of force, whose position in the plane of the system is established by the Kepler's second law.

These considerations regarding the Newtonian theory of forces where utterly disregarded due to the analytic theory of the forces enabled by the Poisson equation toward the end of the 19th century. To wit: the fundamental property of Newtonian forces that allowed Ampère's equation of forces between *elementary currents*, thus starting the theoretical electrodynamics in the first half of 19th century, would have a precise analytic representation. Namely, if the forces were conservative and central, as in the case of Newtonian forces, *their magnitude did not necessarily depend exclusively on the distance* between the two places of the action at distance. This property is mathematically transparent when we assume a typical central conservative force with the magnitude depending, possibly, also on coordinates separately, *i.e.* a force that, viewed as a vector in an arbitrary Cartesian reference frame with origin in the center of force, assumes the form: $\boldsymbol{f}(\boldsymbol{r}) \equiv f(x, y, z)\cdot\boldsymbol{r}$, in view of the centrality property. This field must fulfil the *Helmholtz conditions*: a *scalar condition* amounting to the equation $\nabla\cdot\boldsymbol{f}(\boldsymbol{r}) = 0$ everywhere in space, except the origin, and a *vectorial condition*, in the form $\nabla\times\boldsymbol{f}(\boldsymbol{r}) = \boldsymbol{0}$; these are automatically satisfied by the original Newtonian forces. From the second of these conditions we have, in detail:

$$\nabla f(x,y,z)\times\boldsymbol{r} = \boldsymbol{0} \quad \therefore \quad \nabla f(x,y,z) \propto \boldsymbol{r} \tag{3.3.1}$$

if the force is not a constant in the chosen reference frame. The first Helmholtz condition then becomes:

$$\boldsymbol{r}\cdot\nabla f(x,y,z) + 3f(x,y,z) = 0 \tag{3.3.2}$$

and shows that *the magnitude of force* must be a homogeneous function of degree -3 in the coordinates. Thus, only in the particular case when this function is r^{-3}, are we getting the Newtonian forces going inversely with the distance squared. Otherwise, the magnitude $f(x, y, z)$ can very well be the reciprocal of a general third-degree homogeneous polynomial in the three coordinates separately, or a $-3/2$ power of a homogeneous quadratic form, as actually happened in the original Newton's case that led to the definition of the Newtonian forces. In general, therefore, the mathematics allows for the magnitude of forces any other form leading to a homogeneous function having the degree -3, not even involving the dependence of the magnitude of forces exclusively on the distance between the bodies they connect. Combining (3.3.1) and (3.3.2) we find that the most general force satisfying both Helmholtz conditions:

$$\boldsymbol{f}(\boldsymbol{r}) = -(\frac{r^2}{3}) \cdot h_{-5}(x,y,z) \cdot \boldsymbol{r} \tag{3.3.3}$$

where h_{-5} must be a homogeneous function of degree -5, as indicated by its lower index. This expression of the vector force can be rearranged to appear *as proportional to a Newtonian force*:

$$\boldsymbol{f}(\boldsymbol{r}) = h_0(x,y,z) \cdot (\frac{\boldsymbol{r}}{r^3}), \quad h_0(x,y,z) \equiv -(\frac{r^5}{3}) \cdot h_{-5}(x,y,z) \tag{3.3.4}$$

with the coefficient h_0 – a function *homogeneous of degree zero in coordinates*.

In other words, the most general force field satisfying the two Helmholtz conditions *concurrently* – taken as essential properties of Newtonian field of forces, and extended as fundamental properties of any conceivable central force – must be proportional to a genuine Newtonian force field, with the proportionality described, in a system of Cartesian coordinates, by a factor which can be either *a function of the ratios of coordinates*, or *a constant*, as in the genuine Newton's case. We can maintain the Newtonian spirit of the definition of forces in a static ensemble of identical *dyons*. Invented in order to describe the structure of the nuclear matter (Schwinger, 1969), this kind of particles possessing electric charge along with a magnetic one, can be taken as a typical classical charged particle along the lines of a natural philosophy of charges (Katz, 1965). It can also be taken as the typical element of the Planckian interpretative ensemble – by assuming that the physical properties generating the forces that maintain the equilibrium on ensemble, are simply spread in space in a manner that can be represented by functions of the ratios of coordinates. That is, they are general functions, homogeneous of order zero. The Ampère forces between the elementary currents are such functions, expressed trigonometrically [see (Mazilu, 2024), §1.5].

The Ampère generalization, however, is not quite complete, for it is referring only to a *space geometry*, with no involvement of physics whatsoever. In the classical case of genuinely Newtonian forces, one would have the *masses*, or the *charges*, or even both in fact, located at any two positions involved in the interaction at distance: the forces generated by them are *bilinear* in those physical quantities. In keeping with the idea of continuity and space extension of the matter, we may think of some *mass elements*, or *charge elements* – Hertzian 'higher order infinitesimals' – located at the two positions involved in the action at distance, and entering the equation of force through bilinear expressions, *i.e*. by their product. This was the original Newton's case, and so the equation of force may appear in the case of currents, considered as such infinitesimals. The problem remains, though, concerning the *measure* of the current elements: while in the case of genuinely Newtonian forces one can think of the differentials of mass or charge, in the case of currents issues of relative directions of their orientations occur. The common view arising just about the opening of the Ampère's epoch was that a current element should be represented by the time rate of variation of the charge, $d\boldsymbol{q}$ in our notation here – known as the *intensity* of the current – multiplied by the element of 'wire', thought to be a *line* in space: $Id\boldsymbol{l}$.

It is, in this respect, the case to draw attention on a circumstance that connects the *gauging length* of Berry and Klein (1984) with the inertial and gravitational masses of the attractive body of the classical Kepler problem, and implicitly with the charges. The following considerations, destined to support this statement in all its details, are associated with the name of Ivan Vsevolodovich Meshcherskii, the founder of the modern dynamics of the *variable mass body* [(Mestschersky, 1902); see also (Мещерский, 1952), pp. 199 – 213; we use a Latin transcription as above for this name, because, in pronunciation at least, it is identical to the Russian name written in Cyrillics in the second citation here]. These considerations are referring to the Keplerian motion, which is the key motion in defining the fundamental properties of the system of classical dynamics. It is important to keep in mind, therefore, that this is a problem of dynamics, and involves the inertial masses of bodies among other quantities generating forces according to classical natural philosophy. Being the motion that inspired the definition of *Newtonian static forces*, one can think that the following considerations must also enter the very concept of forces. We cannot decide on the solution of this fundamental problem of natural philosophy just yet: all we can do is a certain clarification of the Newtonian point of view, serving the concept of interpretation. It is worth noticing, in this respect, that the modern wave mechanics was forced to admit a consequence of this point of view (Niederer, 1972), a fact which we found quite significant, from gnoseological point of view, and even for ontological conceptions of the world.

To start with, let us notice that in the plane of motion, referred to the center of force, the *Kepler motion* of a particle is described by a system of differential equations:

$$\ddot{x}+\kappa\frac{x}{r^3}=0, \quad \ddot{y}+\kappa\frac{y}{r^3}=0, \quad r^2\equiv x^2+y^2 \tag{3.3.5}$$

This system is a consequence of the second principle of classical dynamics. Assuming only gravitation for now, the constant κ is, according to Newtonian view, *a monomial* that includes a characteristic of the universe – the gravitational constant, – the gravitational mass of the source of gravitational field, which implicitly contains the hypothesis that only the mass is the source of Newtonian force of the problem, and the ratio between the gravitational mass of the moving particle and its inertial mass. It is this monomial that, in order to satisfy the Ampère general conditions on forces, must be a homogeneous function of degree zero, according to Helmholtz's conditions, shown above. If this is the case, then our contention is that we are here in the realm of action of a noncompact gauge group belonging to the SL(2, R) algebraical structure.

In order to show this, and properly characterize that realm along, we assume that it is sufficient to reproduce the Meshcherskii's theory up to the point where it meets the essential mathematical requirement of the modern Berry-Klein gauging theory, and then draw the appropriate conclusions . To wit, Meshcherskii took notice of the fact that if, for a plane dynamics, we are free to gauge the coordinates by a function of time, and the time by another function of time, according to the recipe:

$$\xi = f(t)\cdot x, \quad \eta = f(t)\cdot y, \quad d\tau = \varphi(t)\cdot dt \tag{3.3.6}$$

Then, under the special condition

$$\varphi(t) = k\cdot f^2(t) \tag{3.3.7}$$

where k is a constant, the system (3.3.5) reduces to

$$k^2\xi'' + \frac{k}{f}\frac{\xi}{\rho^3} + \frac{2\dot{f}^2 - f\cdot\ddot{f}}{f^6}\xi = 0, \quad k^2\eta'' + \frac{k}{f}\frac{\eta}{\rho^3} + \frac{2\dot{f}^2 - f\cdot\ddot{f}}{f^6}\eta = 0 \tag{3.3.8}$$

Here $\rho^2 = \xi^2 + \eta^2$ is the square of the gauged position vector, a dot over represents derivative with respect to original time, and the accent denotes derivative with respect to the new time. The last terms of these two equations appear as the components of an elastic force directed toward the center of the force of magnitude inversely proportional with the square of distance if:

$$\varphi(t) = (\alpha t^2 + 2\beta t + \gamma)^{-1} \tag{3.3.9}$$

where α, β and γ are three arbitrary constants. With an appropriate choice of these constants, in order to include the arbitrary constant k, the system (3.3.8) can be written as:

$$\xi'' + \frac{\kappa}{f}\frac{\xi}{\rho^3} + (\alpha\gamma - \beta^2)\xi = 0, \quad \eta'' + \frac{\kappa}{f}\frac{\eta}{\rho^3} + (\alpha\gamma - \beta^2)\eta = 0 \tag{3.3.10}$$

In summary, the equation (3.3.6) means that by the gauging:

$$\boldsymbol{\rho} = \frac{\boldsymbol{r}}{\sqrt{\alpha t^2 + 2\beta t + \gamma}}, \quad d\tau = \frac{dt}{\alpha t^2 + 2\beta t + \gamma} \tag{3.3.11}$$

the equations of motion (3.3.10) can be written as

$$\xi'' + \kappa_0\frac{\xi}{\rho^3} + (\alpha\gamma - \beta^2)\xi = 0, \quad \eta'' + \kappa_0\frac{\eta}{\rho^3} + (\alpha\gamma - \beta^2)\eta = 0 \tag{3.3.12}$$

provided that the coefficient κ varies in time as

$$\kappa(t) = \frac{\kappa_0}{\sqrt{\alpha t^2 + 2\beta t + \gamma}} \tag{3.3.13}$$

where κ_0 is a constant. In our case, this condition targets the whole monomial mentioned above.

First, let us summarize the mathematical facts: the second equality in equation (3.3.11) gives an elementary probability of Cauchy type in the time of this dynamics. For once, according to our discussion from the previous section, one can say that a Planck's procedure of quantization is in force for the physical magnitudes generating the central force that activates this dynamics on the moving body. The Cauchy variate is here the time of dynamics. On the other hand, the vector from the first equality (3.3.11) has the magnitude given by:

$$\rho^2 = \frac{r^2}{\alpha t^2 + 2\beta t + \gamma} \tag{3.3.14}$$

and this magnitude is an *invariant function* over a group of transformations in two variables with three parameters, having the structure of SL(2, R) (see I, § 2.4):

$$t \to \frac{\alpha t + \beta}{\gamma t + \delta}, \quad r \to \frac{r}{\gamma t + \delta} \tag{3.3.15}$$

representing a 'radial motion', as it were. Let us prove this statement, for it is of importance in what follows.

The infinitesimal generators of the transformation (3.3.15) are:

$$X_1 = \frac{\partial}{\partial t}, \; X_2 = t\frac{\partial}{\partial t} + \frac{r}{2}\frac{\partial}{\partial r}, \; X_3 = t^2\frac{\partial}{\partial t} + tr\frac{\partial}{\partial r} \tag{3.3.16}$$

and they satisfy the structural equations specific to an $\mathfrak{sl}(2, R)$ Lie algebra, which we take usually in the form:

$$[X_1, X_2] = X_1, \; [X_2, X_3] = X_3, \; [X_3, X_1] = -2X_2 \tag{3.3.17}$$

The functions (3.3.14) are the basic solutions of the equation:

$$(\alpha X_1 + 2\beta X_2 + \gamma X_3) \cdot f(t, r; \alpha, \beta, \gamma) = 0 \tag{3.3.18}$$

as one can easily verify using the equation (3.3.16). Any function of ρ^2 is a function invariant by the group (3.3.15), in the sense of equation (3.3.18). The group in question is given by the action (3.3.15), characterised by the infinitesimal generators (3.3.16). Notice that the function (3.3.13) is a 'ρ-function', if we may be allowed to give a name, provided the constant κ_0 is taken as a *gauge length*. This is the point where Meshcherskii theory meets the modern theory of gauging of the forces (Berry & Klein, 1984).

The bottom line of these considerations: a group like (3.3.15), that targets the *radial kinematics* and the time in a certain universe appears to have precise meanings. For once, the invariant functions (3.3.14) on this group, targets the physical constant entering the Newtonian gravitation, as suggested by equation (3.3.13). In matters of interpretation, *the particles of the ensemble serving for interpretation are identical*, so the equation (3.3.13) is referring to their masses and charges, in an equilibrium interpretation of the forces. And, inasmuch as the denominator of (3.3.14) is, according to a 'Planckian philosophy' *the variance of a time ensemble*, if ρ is 'cosmological', then r has, indeed, the meaning of a standard deviation on such an ensemble, as stipulated before. This is what happens in reality: in a Newtonian cosmological problem, when starting to speak of the dominant force in the universe, r^2 is taken as a variance on the interpretative ensemble rather than a dynamical variable. This gives a kind of 'Jeans interpretation', but referring to the Newtonian forces.

A known case is offered by Fritz Zwicky who, based on a genuine interpretation involving stationary ensembles of nebulae (the Coma cluster), proved that the universe is still Newtonian pending a physically sound explanation of the frequency displacement in the observation of the nebulae (Zwicky, 1942). Obviously, an interpretative ensemble needs to be at least stationary – if not straightaway static – according to a Planckian philosophy, in order to describe a statistics of the distances between interpretative particles of the ensembles. In general, for the definition of such an ensemble, we need to consider closely the matter constituents, and these considerations bring forth the old Schwinger's idea of dyons: such an ensemble cannot logically exist but under the concurrence of three static Newtonian forces. Namely, we need to have on each and every interpretative particle the action of three such forces: one atractive gravitational force and two repulsive electric and magnetic forces. So every interpretative particle carries a gravitational mass, and two charges, electric and magnetic. This particle can then be deemed as a *heavy dyon*, and it can be conceived as such at any scale of a universe.

3.4. A Maxwellian Incentive: the Action *of* Charges

While it is fresh in our mind – and, of course, will be described in more detail as we go along with this work – let us insist a little further on the *concept of dyon*, in order to see how it can be used for interpretation in the case of the concept of charge in general. The classical theory of charges [(Mazilu, 2024), §2.2] assumes that between the static force – of the Newtonian kind, for instance – and the Lorentz force – incorporating a certain idea of motion – there is a rotation determined by the charges (Harrison, Krall, Eldridge, Fehsenfeld, Fite, & Teutsch, 1963). Let us expound this point, which, apparently, generated the idea of Julian Schwinger leading to the concept of *dyon* (Schwinger, 1969), but which turns out to be essential for electrodynamics in many other ways.

For once, from the point of view of Planck's quantization procedure, Schwinger's idea is providing a reliable theory of active charges, that can liberate, *e.g*., the Newton's 'intelligent Agent' of his tiresome 'duty' of providing the 'just impulse' in a well-known cosmological problem of creation, if this expression of ours may be permitted. Which duty, as a matter of fact, cannot be but a figment of our imagination: after all, such an 'Agent' only *prescribes laws*, according to which everything and everyone proceeds. That is all: 'It' does not carry out the procedure, like people always claimed in history, just "by the tail of implication", as Stroud would say! Let us recall that we adopt the Katz's natural philosophy (Katz, 1965), which implies, among other things, that the charge is always 'dual': it exists as electric, as well as magnetic charge, amassed into the same interpretative particle. As a matter of fact, this is Schwinger's starting point in his theory of the dual-charged particle for whose name he recommended 'that dyon is the best choice'. And the starting point of such a theory is always the static, or permanent force, if it is to speak in terms of the time extension.

The *permanent force* of Coulombian type is assumed to be a linear combination of the classical field intensities. Then by a Lorentz transformation this force field can be transformed into a field characteristic to a uniform motion (Harrison, Krall, Eldridge, Fehsenfeld, Fite, & Teutsch, 1963):

$$\boldsymbol{f} = q_e\boldsymbol{e} + q_m\boldsymbol{b} + \frac{1}{c}\boldsymbol{v}\times(-q_m\boldsymbol{e} + q_e\boldsymbol{b}) \tag{3.4.1}$$

Here q_e is the *electric charge*, while $\boldsymbol{e}$ is the *electric field intensity*, giving the electric part of the force $q_e\boldsymbol{e}$. Likewise, q_m is the *magnetic charge* and $\boldsymbol{b}$ is the *magnetic field intensity*, giving the magnetic part of the force, denoted by $q_m\boldsymbol{b}$ in equation (3.4.1). Then, in a world where the gravitation provides only a background, without dominating the events by its Newtonian force, one can assume that the Maxwell electrodynamics is still valid, as Schwinger himself did (*loc. cit. ante*). Assume, therefore, that the field intensities $\boldsymbol{e}$ and $\boldsymbol{b}$ are Maxwellian indeed, but in a 'broader' sense, namely that they satisfy the 'symmetric' Maxwell equations, destined to cope with the two types of charges, and with the forces generated by these:

$$\begin{aligned}\nabla\cdot\boldsymbol{e} = 4\pi q_e\rho, \quad \nabla\times\boldsymbol{e} = -\frac{1}{c}\frac{\partial\boldsymbol{b}}{\partial t} - \frac{4\pi}{c}q_m\boldsymbol{j}\\ \nabla\cdot\boldsymbol{b} = 4\pi q_m\rho, \quad \nabla\times\boldsymbol{b} = \frac{1}{c}\frac{\partial\boldsymbol{e}}{\partial t} + \frac{4\pi}{c}q_e\boldsymbol{j}\end{aligned} \tag{3.4.2}$$

Here ρ is the numerical density of the particles on which the force is supposed to act, while $\boldsymbol{j}$ is their current. One can see right away that assuming no magnetic charge – $q_m \equiv 0$ – the equations (3.4.2) are the classical Maxwell equations:

$$\nabla\cdot\boldsymbol{E} = 4\pi q_e\rho, \quad \nabla\cdot\boldsymbol{B} = 0, \quad \nabla\times\boldsymbol{E} = -\frac{1}{c}\frac{\partial\boldsymbol{B}}{\partial t}, \quad \nabla\times\boldsymbol{B} = \frac{1}{c}\frac{\partial\boldsymbol{E}}{\partial t} + \frac{4\pi}{c}q_e\boldsymbol{j} \tag{3.4.3}$$

where the capitals denote the field intensities in this situation. The symmetric case, namely that with zero electric charge – $q_e \equiv 0$ – describing a Maxwellian magnetic universe, is perfectly feasible. This time the corresponding Maxwell equations are:

$$\nabla\cdot\boldsymbol{E} = 0, \quad \nabla\cdot\boldsymbol{B} = 4\pi q_m\rho, \quad \nabla\times\boldsymbol{E} = -\frac{1}{c}\frac{\partial\boldsymbol{B}}{\partial t} - \frac{4\pi}{c}q_m\boldsymbol{j}, \quad \nabla\times\boldsymbol{B} = \frac{1}{c}\frac{\partial\boldsymbol{E}}{\partial t} \tag{3.4.4}$$

It would appear thus that the Maxwellian electrodynamics is just a matter of choice, therefore subjective, even though firmly based on our experience. Indeed, *this experience* decides our notions of finite, and being aware that these notions decide, in turn, the concepts of infrafinite and transfinite, where there cannot be but a *fictitious experience*, the Maxwellian electrodynamics can be deemed as 'subjective'.

However, still fact of experience is the so called *Lorentz force* acting upon a charge. Referring it to the equations (3.4.3), this force can be calculated with the Lorentz's formula that, by now, has become classical through years of use, and we transcribe it in the form:

$$\boldsymbol{F} = q_e\left(\boldsymbol{E} + \frac{1}{c}\boldsymbol{v}\times\boldsymbol{B}\right) \tag{3.4.5}$$

This formula displays the fact that Lorentz force involves the motion of particles used for the interpretation of an electric fluid. Since such a force can be considered a fact of experience too, we might as well take notice of the fact that the motion is involved in the force, and the charge includes in its value dynamic units too. As known, these units are in the ratio c^2 with the static units, where c was *assigned* by Maxwell as the speed of electromagnetic light.

The situation of force can still be described in a symmetric Maxwellian electrodynamic, and thus helps maintaining the magnetic part of the static force in the picture. Assume, indeed, instead of a static electric charge, a 'dynamic', if we may say so – understanding by this term 'different from static' – electric charge e in the picture of force given by equation (3.4.5) *i.e.*,

$$\boldsymbol{F} = e\left(\boldsymbol{E} + \frac{1}{c}\boldsymbol{v} \times \boldsymbol{B}\right) \tag{3.4.6}$$

which reduces to (3.4.5) for $q_m = 0$. Then the dual symmetric force can be maintained in the picture, according to the equation (3.4.1), by the following transformation of the fields:

$$e\boldsymbol{E} = q_e\boldsymbol{e} + q_m\boldsymbol{b}, \quad e\boldsymbol{B} = -q_m\boldsymbol{e} + q_e\boldsymbol{b}, \quad e^2 = q_e^2 + q_m^2 \tag{3.4.7}$$

which also gives the Maxwellian electrodynamics represented by the differential equations (3.4.4). This means that the force must be written expressly as carrying the condition of those equations:

$$\boldsymbol{F}_{q_e=0} = q_m\left(\boldsymbol{B} - \frac{1}{c}\boldsymbol{v} \times \boldsymbol{E}\right) \tag{3.4.8}$$

So should be the Lorentz force must be understood in the general case of equation (3.4.6), under conditions (3.4.7). It contains the static electrical force at $q_m = 0$, and the static magnettical force at $q_e = 0$.

Notice the fact that the existence of duality transformation (3.4.7) of the Maxwellian fields amounts to the existence of a matrix, $\boldsymbol{D}$ say, such that:

$$\boldsymbol{D} \cdot |\boldsymbol{e}\rangle = |\boldsymbol{E}\rangle, \; e\boldsymbol{D} \stackrel{\text{def}}{=} \begin{pmatrix} q_e & q_m \\ -q_m & q_e \end{pmatrix} \tag{3.4.9}$$

where the two kets have as components the corresponding vectorial fields:

$$|\boldsymbol{e}\rangle \stackrel{\text{def}}{=} \begin{pmatrix} \boldsymbol{e} \\ \boldsymbol{b} \end{pmatrix}, \; |\boldsymbol{E}\rangle \stackrel{\text{def}}{=} \begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{B} \end{pmatrix} \tag{3.4.10}$$

Assuming, therefore, that there are entities *acting on fields* in order to *make them forces*, or *vice versa*, according to equation (3.4.7), these entities must be endowed with pure charges of two types – electric and magnetic – and represented by a matrix as the one above. One thus can say that a *dyon* – thus calling the physical entity acting upon fields in order to get the forces – is represented here by a matrix as in equation (3.4.9).

Let us conclude here with a rule of representation of the dyons: starting from a static case, *the charges act on fields by dyons*, geometrically representing rotations for once. This action *turns the fields into forces*, which are then capable to generating a dynamics, for instance in the Newtonian sense. This seems to be the whole idea of the classical natural philosophy: namely, in passing from statics to a certain dynamics, it is the forces that must undergo a transformation. Apparently, the notorious case of *inertia as a force* falls under this general concept of transformation of forces.

3.5. The Action *of* Charges: Applying the Lesson

The framing of a dyon as a means of transformation between fields and forces, may not be feasible from physical point of view, but it is certainly convenient from the perspective of concept of interpretation. Speaking of transformation from its physical point of view, the Planckian approach of the charges, directs us to think that a dyon can be taken as the physical entity representing a resonator, inasmuch as it possesses two charges, and we may assume that a dipole can be conceptually assimilated as a dyon. The matrix ***D*** above, can then be taken as just a particular case of such resonator, effecting only a rotation of the fields. Along this line of reasoning, a general case of dipole can be expected to do more than that. So we are set here to characterize a dipole from this point of view.

Our experience opens a different outlook on the physics of charges, if we are to account for the modern physical-theoretical facts related to the concept of quantization, particularly for the dyons. Namely, the transformation (3.4.7) should be just a particular one, among many transformations transcending the scales of space and time. We need to find this class of transformations *independently of the Maxwell equations*: for, obviously, it is only by these equations that one can make a case against the isolated magnetic poles. However, the main reason for such an option appears to us way deeper, mostly from a gnoseologic point of view. Namely, these equations assume a particular space-time behavior of the field variables and charges, which is, apparently, destined to *preserve the scales of space and time*, as Einstein has once shown by his very construction of the special relativity (Einstein, 1905). Explicitly, this idea was expressed by Vladimir Fock, in his theory of gravitation [see (Fock, 1959), Appendix A, for a detailed explanation of this statement]. The preservation of scales is in obvious contrast with the definition of Newtonian forces, which assumes that the concept of force transits the scales of space and time. It compels us to special considerations when defining a static reference state of the fields based on Maxwellian electrodynamics (Wu & Yang, 1969). Therefore, we cannot leave the realm of the Newtonian definition of forces just yet, without showing what the dynamics of electromagnetism had to add to the concept of Newtonian forces.

In the case of classical electrodynamics, we can simply *avoid the Maxwell equations* by a condition analogous to that describing the propagation of light as dependent of the constitutive description of the medium sustaining this propagation [(Mazilu, 2024), §§6.3–4]. This condition fits perfectly in the physical description of the light phenomenon in its relation to vacuum (*loc. cit.* §6.3). We postulate that it can be expected to have also the capability to describe physically any medium that sustains a propagation of any physical quantity in a universe, under the general condition of the Planck's quantization procedure. However, the propagation of the electromagnetic light in matter only ‘arouses forces’ in the medium in which it takes place: this is the essential characteristics of the Fresnel physical optics, to start with, and we can expect it to be, from an operational point of view, universal, given, for instance, a certain kind of transport theory. However, such a transport theory requires the possibility of interpretation and, consequently, some kinds of special fields in matter, generated by the charges themselves. Let us put these ideas in a mathematical form.

Assume, therefore, that those forces ‘aroused’ in pairs, are described mathematically by a *general* linear homogeneous action of the 2×2 matrices: this action *creates* pairs of forces, like in the electromagnetic prototype, *via* the ‘quantal recipe’ represented by the eigenvalues of a matrix like (2.5.2). In other words, the case of equation (3.4.7) is just a particular, one as we said, referring *only* to the rotation of the existing fields: the *duality rotation* as they usually call it in Maxwellian electrodynamics. It concerns only a *particular* linear action generated by the charges. A homographic action, as required by the Planck’s quantization, can be connected with this linear action simply by considering it a transformation of the ratios of forces, as in the Newtonian definition: *it can represent a connection between the ratios of charges and the ratios of forces*. Leaving aside the mathematical details on this subject for now, let us proceed with the general linear transformation, as announced. The very same principle as that of propagation for light, can then be invoked for the *general transformation of fields* ‘*propagated*’ *into forces* ‘*aroused*’, whose components give the eigenvalues of the force aroused:

$$\left|e'\right\rangle = \alpha\left|e\right\rangle + \beta\left|b\right\rangle, \quad \left|b'\right\rangle = \gamma\left|e\right\rangle + \delta\left|b\right\rangle \tag{3.5.1}$$

Here (α, β, γ, δ) are some ‘generalized charges’, by analogy with the classical case (3.4.7) of a pure duality rotation. Taking the lesson offered by duality rotation, presented for guidance in the previous section, we can say that the coefficients of this linear transformation must be somehow connected to charges in general. So, *we claim* that this transformation must leave the *tensor of Maxwell stresses*, *constructed this time with the fields* ($|e\rangle$, $|b\rangle$) say, *invariant*, just like its prototype dual transformation does for the classical Maxwell tensor. That is, *the field aroused* must contain *the same information as the field propagated*, and this information is contained in the Maxwell tensor of stresses. Then, a classical duality transformation is just a particular case of the transformation (3.5.1), which should be a definition of charges in general, valid for any universe, so we might not need an explanation for it after all. Fact is, though, that the classical duality transformation needs such an explanation, and it goes along the following lines.

If there is a de Sitter background of charge in the universe we inhabit, as the Einsteinian cosmology compels us to believe [see (Mazilu, 2024), Chapter 3, especially § 3.2], then this background certainly transcends the scales of space and time. And the only way of being able to explain the facts of experience related to the space and time scales transcendence for the charge, must be exactly in the manner of the transcendence of the light from our prototype physical universe. For, it is, certainly, a fact that *the light we perceive transcends the scales of space and time*, carrying a fundamental information with it, and this fact can be allotted to the property of invariance of the Maxwell tensor of the electromagnetic light. More to the point, the light comes to us from that hazy spherical zone of the Stroud sphere of anyone of us, is captured into some finite Wien-Lummer enclosures – an everyday example of which can very well be our eye, at least for the limited case of some frequency ranges – and, then, is analyzed *via* some microscopic processes, involving *grosso modo* the physical transfomation of light into charge, all of these processes bearing the mark of infrafinite.

And so it should happen with the charges too, for, inasmuch as they are in the 'background' of any universe, they plainly transcend the space and time scales: charges exist in the universe at large, as well as in the finite world of our experience, and also in the microscopic world. The only difference between them stays in the fact that, the forces they generate at different scales of space and time, behave differently *with respect to gravitation and inertia*, which appear to be universal. The most obvious fact here is that, with respect to gravitation, the forces generated by the electric and magnetic charges are to be neglected at the transfinite scale, but they prevail over gravitation at finite and infrafinite scales. Let us proceed to a mathematical description of the most general invariance of the Maxwell tensor under the transformation (3.5.1).

For the moment, the kets $|e\rangle$ and $|b\rangle$ are just triplets of numbers arranged into column matrices, destined to represent the Fresnel aroused *vector* forces whose magnitudes are given by the two eigenvalues of the 'quantum matrix' from equation (2.5.2). There are an infinity of such triplets, according to the position we assume for those eigenvalues: if these are magnitudes of forces, then they are, most probably quadratic forms in the components of the kets $|e\rangle$ and $|b\rangle$. The very same may happen if the eigenvalues in question are some components of the forces, like in the case of spin forces. The tensor of 'Maxwell stresses' is then expected to represent best such a situation, in any case. It has the components constructed from the components of the field kets $|e\rangle$ and $|b\rangle$ according to the rules:

$$t_{ij} = \lambda e_{ij} + \mu b_{ij}, \quad e_{ij} \overset{def}{=} e_i e_j - \frac{1}{2} e^2 \delta_{ij}, \quad b_{ij} \overset{def}{=} b_i b_j - \frac{1}{2} b^2 \delta_{ij} \tag{3.5.2}$$

Here λ and μ are two parameters depending on the properties of the medium where the transformation (3.5.1) takes place, and $\boldsymbol{\delta}$ is the Kronecker unit tensor, as usual. The special conditions of invariance on the Maxwell tensor thus constructed, are expressed by its *constancy* just like in the classical case from § 3.4 above. Therefore, we shall have the equations $t'_{ij} \equiv t_{ij}$ satisfied in any position of the medium, where the prime is referring to the transformed tensor, corresponding to the primed constants that characterize the medium. When written in detail, based on the definitions from equation (3.5.2), these invariance identities for the Maxwell tensor will lead to a connection between the 'general charges' (α, β, γ, δ) of the transformation (3.5.1) and the parameters λ and μ characterizing the medium in its two instances, or we should say, *states*: 'unprimed' and 'primed'. These conditions of invariance result in the following system of equations, connecting two possible states:

$$(\alpha\delta - \beta\gamma)^2 \lambda\mu = \lambda'\mu', \qquad \alpha\beta\lambda + \gamma\delta\mu = 0$$
$$\alpha^2\lambda + \gamma^2\mu = \lambda', \qquad \beta^2\lambda + \delta^2\mu = \mu' \tag{3.5.3}$$

Let us proceed to a solution of this system for (α, β, γ, δ), in order to see in what conditions these parameters are meaningfully obtained from the parameters λ and μ characterizing the medium at the location where the transformation (3.5.1) takes place. For once, from the first and the last two equalities of the equation (3.5.3), we must have

$$(\alpha^2\lambda + \gamma^2\mu)(\beta^2\lambda + \delta^2\mu) = (\alpha\delta - \beta\gamma)^2 \lambda\mu \quad \therefore \quad (\alpha\beta\lambda + \gamma\delta\mu)^2 = 0 \tag{3.5.4}$$

showing that the system is compatible for *any* values (α, β, γ, δ), in view of the second equation from the first row of (3.5.3), which is the only constraint these parameters must have: that is, only three of these parameters can be independent.

One can say that the problem raised by the system (3.5.3) *has always a solution*. Using a concept largely vehiculated only in geometry, the equation (3.5.4) defines *a porism* – there is an infinity of transformations a medium can produce for the action on fields – just like the Planck's equation in fact [see § 3.1, equation (3.1.2)], even though we have not treated it as such yet. Mention should be made, on this occasion, that the only physical model based on the concept of porism – at least the only one we are aware of – is that of Harry Bateman for the atom, based on the concept of *Steiner*'*s porism* in the three-dimensional case (Bateman, 1908). The atom is imagined as an ensemble of spheres representing the electrons, all of them tangent to two fixed spheres. The Steiner's porism then says that if such an ensemble exists for a given number of electrons, then an infinity of ensembles exist in the same *conditions*: that is, the same number of electrons and the two spheres not intersecting each other. We mention this model, because we consider it useful, if not even indispensable from a gnoseological point of view, in understanding the mechanisms of transition between the static function of the Newtonian forces and their dynamical function, which allow us to settle for the deepest meaning of the dynamical description of the Kepler problem. As we see it, this is one of the most important problems of the Newtonian mathematical philosophy.

Coming back to our discussion, for now, the equation (3.5.4) will tell us that the transformation makes sense anywhere in a medium characterized by the parameters λ and μ. So, in general terms, the transformation (3.5.1) can be generated by *any* matrix:

$$\boldsymbol{A} \equiv \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \quad \textit{provided} \quad \alpha\beta \cdot \lambda + \gamma\delta \cdot \mu = 0 \tag{3.5.5}$$

The transformation is completely defined, and preserves the Maxwell tensor. Some constraints, according to this description of the transformation, can only arise for transformations in a particular background medium at the location where they are acting. Let us describe *two* such cases, each one of them having an intuitive meaning, especially for the case of the brain universe.

The first case is that of a medium *homogeneous at any scale*, just like the ether has been once considered in physics: $\lambda = \lambda'$ and $\mu = \mu'$. In the brain universe, this means pure propagation of some fields within a *given glial structure*. For once, this condition cannot be satisfied but only by a matrix $\boldsymbol{A}$ of determinant ± 1, as can be seen right away from the first equation of the first row of (3.5.3). Indeed, on this condition the three remaining equations from (3.5.3):

$$\alpha\beta\lambda + \gamma\delta\mu = 0, \quad (\alpha^2 - 1)\lambda + \gamma^2\mu = 0, \quad \beta^2\lambda + (\delta^2 - 1)\mu = 0 \tag{3.5.6}$$

are compatible only within the following suplimentary constraints:

$$\begin{matrix} \alpha\delta - \beta\gamma = 1 & & \alpha\delta - \beta\gamma = -1 \\ and & or & and \\ \alpha - \delta = 0 & & \alpha + \delta = 0 \end{matrix} \tag{3.5.7}$$

Now, the matrix in question can be characterized by the parameters of medium, just using one of the equations (3.5.6) in the specific conditions from (3.5.7): the relation between λ and μ is always linear and can be taken directly from the first equation from (3.5.6). In conclusion, for a homogeneous medium, there are two types of matrices. The first one has determinant unity and is 'almost' a rotation, either Euclidean or hyperbolic:

$$\boldsymbol{A} = \begin{pmatrix} \cos\theta & m^{-1}\sin\theta \\ -m\sin\theta & \cos\theta \end{pmatrix} \quad or \quad \boldsymbol{A} = \begin{pmatrix} \cos(i\theta) & m^{-1}\sin(i\theta) \\ -m\sin(i\theta) & \cos(i\theta) \end{pmatrix} \tag{3.5.8}$$

where θ is an arbitrary real parameter and i is the imaginary unit of complex numbers. In other words, the *imaginary rotations*, usually expressed by hyperbolic trigonometrical functions *are also allowed* here, due to the trigonometrical identities

$$\cos(i\theta) \equiv \cosh\theta \quad and \quad \sin(i\theta) \equiv i \cdot \sinh\theta \tag{3.5.9}$$

allowed by the fact that the product $(\beta \cdot \gamma)$ can have either sign for arbitrary real factors. On the other hand, the second condition from equation (3.5.7) can only be a null-trace matrix – an involution – having the determinant negative unity:

$$\boldsymbol{B} = \begin{pmatrix} \cos\tau & m^{-1}\sin\tau \\ m\sin\tau & -\cos\tau \end{pmatrix} \quad or \quad \boldsymbol{B} = \begin{pmatrix} \cos(i\tau) & m^{-1}\sin(i\tau) \\ m\sin(i\tau) & -\cos(i\tau) \end{pmatrix} \tag{3.5.10}$$

where τ is the arbitrary parameter this time and we have used the identity equation (3.5.9) for trigonometrical functions. The parameter m is defined by the identity $m^2 \equiv \mu/\lambda$ in both situations.

Notice that *the product* of two matrices (3.5.10) having *the same* parameter m, with either regular trigonometrical entries, or hyperbolic trigonometrical entries, but of different parameters τ, is always a matrix (3.5.8) of parameter m, having as parameter θ the difference of the two parameters τ. For once, one can say right away that these last two parameters carry the very same basic meaning, whatever this may be. But, more importantly, we conclude that *a matrix* (3.5.8) *cannot be*, *in any one of its actions*, *but a composition of two matrices* (3.5.10), for the same *m* and different parameters τ. This fact makes out of the matrices (3.5.10) a fundamental instrument of the present theory, and becomes of essence when constructing a stochastic process of the action of the medium of propagation. When undertaking such a job, it is also important to keep the mind open on the fact that a product of two involutive matrices – that is, 2×2 matrices having zero traces – cannot be an involutive matrix in general: it requires special conditions for the relation of the two factors.

The matrices of the type (3.5.10) are the general expression of the possibility of a quantum measurement, as defined by a matrix like the one from the equation (2.5.1). This last one can be obtained from (3.5.10) for *m* complex and of magnitude unity. Thus, let us stress once again: we can come to the natural conclusion that the measurement in a universe is realized with a matrix of the form (2.5.2) and provides two eigenvalues completely independent of the matrix of transformation. It is important to notice further that these two eigenvalues can fit into Fresnel's fundamental principles of optics as expressed by Poincaré and reproduced by us in the § 2.5, if the direction (θ, φ) in equation (2.5.2) is a direction of propagation of a plane wave within a Lorentz quantum: they can be taken as the forces ($|e\rangle$, $|b\rangle$) 'aroused' in the plane of the wave propagating in that direction; these values are independent of the direction of propagation of wave within the Lorentz quantum. However, they can depend on the direction of 'vibration of the molecule', and this fact makes some statistics out of those eigenvalues, as we shall show here in due time.

A special problem connected to this aspect of the theory, arises for the case of composition of two matrices (3.5.10) having the same *m*, one with real angular parameter, the other with imaginary parameter: the result is a matrix of the kind (3.5.8) with the same *m*, but constructed with trigonometrical functions of a complex variable. We know but of one case in the history of theoretical physics that suggests the feasibility of such a case from a physical point of view (Husimi, 1953). The work just cited shows that the extension in complex of the real independent variable of the Bloch equation transforms it into a Schrödinger equation. This means that the complex argument is susceptible to represent a transition from statistics to dynamics and *vice versa*. As we said, we shall return to this issue in due time.

The previous theory is referring to a homogeneous medium: the same parameters λ and μ all over the places in the medium. From this point of view, a *second case* may be considered now, which is inspired by the presence of the ratio of the two parameters λ and μ of a medium: what, may we ask, if the transformation is realized between two different states of a medium, however under the condition that *the ratio* of the two parameters λ and μ of the two states is the same, instead of each one of its factors? In the physics of our experience this phenomenon can be labeled as *vacuum tunneling*: a transition between two kinds of vacua. In the brain universe, on the other hand, it can describe either the transition between two glial structures, or a transition from cancel waves to message waves. Going back to (3.5.3), and choosing the working ratio as λ/μ, the first row of the system gives:

$$(\alpha\delta-\beta\gamma)^2=(\mu'/\mu)^2, \quad \alpha\beta\lambda+\gamma\delta\mu=0 \tag{3.5.11}$$

while the second row provides the equality:

$$\frac{\alpha^2\lambda+\gamma^2\mu}{\beta^2\lambda+\delta^2\mu}=\frac{\lambda}{\mu} \tag{3.5.12}$$

These conditions are independent of each other. Combining them in order to eliminate the ratio λ/μ results in the equation:

$$\gamma(\alpha\delta-\beta\gamma)(\alpha^2-\delta^2)=0 \tag{3.5.13}$$

which basically means: $\delta^2=\alpha^2$, provided, of course, the matrix of transformation *is not singular*. Therefore, in appropriate conditions of reality we obtain again the matrices (3.5.8) or (3.5.10) up to a numerical factor.

That is, first of all, we have in detail:

$$\boldsymbol{A}=\begin{pmatrix}\alpha & \beta\\ \gamma & \alpha\end{pmatrix} \quad and \quad \beta\cdot\lambda+\gamma\cdot\mu=0 \tag{3.5.14}$$

but, this time, with no 'trigonometrical restriction' on the parameters α, β, and γ. This matrix belongs to the class (3.5.8), which means that it belongs to a *group of matrices*. For the sake of future developments, we shall write the equation (3.5.14) in the form of a general definition of these matrices:

$$\boldsymbol{A}=\begin{pmatrix}\alpha & \beta\\ -\lambda\cdot\mu^{-1}\cdot\beta & \alpha\end{pmatrix} \tag{3.5.15}$$

with no condition on its determinant, that is, other than it is never 0.

On the other hand, we can have matrices that belong to the class (3.5.10), *i.e.*, they represent involutions. In detail we have:

$$\boldsymbol{B}=\begin{pmatrix}\alpha & \beta\\ \gamma & -\alpha\end{pmatrix} \quad and \quad \beta\cdot\lambda-\gamma\cdot\mu=0 \tag{3.5.16}$$

which can be written in the convenient form for future use:

$$\boldsymbol{B}=\begin{pmatrix}\alpha & \beta\\ \lambda\cdot\mu^{-1}\cdot\beta & -\alpha\end{pmatrix} \tag{3.5.17}$$

again, under condition that it is never singular.

Summarizing the results, the most general matrices accomplishing the transformation (3.5.1) under the condition of preservation the Maxwell tensor, are always of the form (3.5.15) or (3.5.17), provided the first condition from equation (3.5.11) is satisfied, which means:

$$\alpha^2+\beta^2\cdot\frac{\lambda}{\mu}=\left(\frac{\mu'}{\mu}\right)^2 \quad or \quad \alpha^2+\beta^2\cdot\frac{\lambda}{\mu}=-\left(\frac{\mu'}{\mu}\right)^2 \tag{3.5.18}$$

This means that, if all symbols are referring to real values, in the case (3.5.17), the determinant of the matrices must be always negative.

The reader may have had already sensed where we are heading with this theory: notice that the equations (3.5.8), (3.5.10) or, in general, the classes (3.5.15) and (3.5.17), are obtained from some conditions of homogeneity of what seems to be a medium supporting the 'propagation', like the vacuum in the case of light in the physical universe of our experience. In the brain universe, it may be the case of 'charge propagation' through the glia, for the glia plays the part of such a medium: it sustains the existence of the communication networks in the brain universe, and also grows appropriate sub-structures.

3.6. Involving Charges in Transformations: the Iwasawa Decomposition

A theory of forces within a Lorentz quantum, can be conceived as a quantal measurement of the kind described by us in § 2.5 above: the forces 'aroused' by a plane wave of arbitrary direction of propagation occur in the plane of wave, in pairs of eigenvalues, independently of that direction. They are the magnitudes of two vectors in arbitrary position with respect to each other. Respecting the independence, these forces should not depend but on the matter and, according to the classical theory of electromagnetic fields the general vectors raised by propagation are not orthogonal to each other, but in an arbitrary position. In other words, a Lorentz quantum represents, for a given universe, *an ensemble of classical nets*.

Speaking of the nets, let us consider the classical *Chebyshev representation* of a certain surface in terms of a couple of two non-normalized eigenvectors of a general matrix. These vectors can be considered simply as two column matrices having three entries as components, just like the two 'electromagnetic fields' above:

$$d\boldsymbol{r}=\boldsymbol{e}_1du+\boldsymbol{e}_2dv \tag{3.6.1}$$

Here u and v are the coordinates on the surface. Then, the first fundamental form of this surface is

$$d\boldsymbol{r}\cdot d\boldsymbol{r}=\boldsymbol{e}_1^2(du)^2+2(\boldsymbol{e}_1\cdot\boldsymbol{e}_2)dudv+\boldsymbol{e}_2^2(dv)^2 \tag{3.6.2}$$

which is, obviously, a metric form describing the surface on which the plane is applied. Thus we have at our disposal a *metric tensor*:

$$\boldsymbol{h}\overset{def}{=}\begin{pmatrix}\boldsymbol{e}_1^2 & \boldsymbol{e}_1\cdot\boldsymbol{e}_2\\ \boldsymbol{e}_1\cdot\boldsymbol{e}_2 & \boldsymbol{e}_2^2\end{pmatrix}=\begin{pmatrix}\lambda^2 & \lambda\mu\cos\theta\\ \lambda\mu\cos\theta & \mu^2\end{pmatrix} \tag{3.6.3}$$

This represents *a gauging* of the local geometry of the surface, induced by the two vectors in arbitrary position with respect to each other. The two gauging vectors have the magnitudes λ, μ obviously.

In a usual plane Euclidean geometry, and in the case of a symmetric matrix, there is no problem in making the two directions orthogonal: there is always a rotation with the help of which a general *symmetric* matrix can be reduced to the diagonal form. The matrix from equation (3.6.3) is ideal for an application of this theorem: we can always find a rotation matrix, $\boldsymbol{K}(\varphi)$ say, where φ is the rotation angle, that reduces the matrix $\boldsymbol{h}$ to its diagonal form. However, *in the process of interaction of a plane wave with a message wave*, the form of the matrix is by no means 'out for debate', so to speak, for nothing is at our disposal. Like in the original case of cutting the garments (Chebyshev, 1946, 1970), one has to account for a special deformation in the application of a plane on a geometrical form, or *vice versa*. Unlike that classical case, however, neither the plane of 'cutting' – *i.e.*, the plane of the wave from a Lorentz quantum – nor the geometrical form of the 'body' to be covered by garments – *i.e.*, a network ray carrying the message wave – is at our disposal in a finite geometry. That is, a matrix 'applying a ray to a plane wave', or *vice versa*, cannot be a matter of choice, for we cannot control anything of the kind. So much the more, this should be the case if the interaction involves a *Lorentz quantum*: a 'chaos', if we may be allowed, of plane waves of all orientations and frequencies, which would require some kind of statistics in order to calculate the necessary projections.

Nevertheless, we can recognize that a message wave has encountered a cancel wave when the two eigenvectors of the matrix are transformed into an orthonormal basis. In fact, this operation may count as a 'geometrization of the kets' for these matrices are to be defined independently of any reference frame: they are simply triplets of numbers physically defined, in our case through a scenario involving the Planck's procedure of quantization [see (Dirac, 1966), § 2]. This is, for the moment, just our geometrical 'intuition', so to speak, but we can easily assign a physical reason to it. Think, indeed, of a radio set in a sea of plane radio waves in vacuum: in order to serve its purpose, this radio set must be tuned to a certain frequency. This means that it must be capable to 'choose' a certain plane wave from the 'chaos' of the vacuum waves in which it is engulfed. In terms of electromagnetic description of the vacuum waves, we can say that the radio set *operates a transition* between the vacuum electromagnetic waves – which, as known, consist of *perpendicular* electric and magnetic fields – and its own electromagnetic fields propagating through the wires – the genuine classical *rays of electricity* – in the form of Ampère currents.

And while we are on the subject, let us mention that a radio set can also be required to act as a *transmitter of electromagnetic waves*: in this case, according to the general philosophy delineated right above, it realises an application of an arbitrary electromagnetic fields from the matter, into perpendicular electromagnetic fields characteristic to vacuum. It is worth our while, remembering that, according to the Lorentz characterization of the physical object we call a Lorentz quantum (see § 2.3 above), this will explain the emission of a quantum from the atom, if this will be conceived as a fundamental 'radio set'. For, according to Gilbert Newton Lewis, considering the Bohr description of the encounters between light and the planetary atoms, it would be 'inappropriate' to speak of the particles during such an encounter. Therefore, it would fall to physics to explain *how* can it happen that

…. we are to asssume that it (*a Lorentz quantum*, *a*/*n*) *spends only a minute fraction of its existence as a carrier of radiant energy*, *while the rest of the time it remains as an important structural element within the atom* … (*original emphasis here*, *a*/*n*) [(Lewis, 1926c); *emphasis added*, *a*/*n*]

and what are the physical processes and laws describing this situation *conceptually*.

From a Planckian perspective, the above situation is the feature of a universe, in general – and therefore 'becomes prone to be incorporated', as it were, into the concept of universe – with just a simple change in terms: a Planck resonator – *a general radio set* – produces a transition between the waves in 'vacuum' and the waves in 'matter', taken as categories, with specific realizations. For instance, in the case of brain, a vacuum is accomplished by the 'glia' while the matter is the 'nervous matter'. Then, if we assume that the 'meeting' of the message waves with the cancel waves is replicated by the transition between two pairs of vectors, we may be in position to describe this encounter mathematically. Indeed, according to a transformation theory, this operation can be achieved by a matrix representing more than a pure rotation, for one of the two pairs of vectors cannot be orthogonal. In order to see how such a matrix can be calculated, let us write the transformation it generates in the form:

$$\left|\boldsymbol{e}\right\rangle = \boldsymbol{G}\cdot\left|\hat{\boldsymbol{e}}\right\rangle,\quad \hat{\boldsymbol{e}}_1^2 = \hat{\boldsymbol{e}}_2^2 = 1,\quad \hat{\boldsymbol{e}}_1\cdot\hat{\boldsymbol{e}}_2 = 0 \tag{3.6.4}$$

where the components ($\boldsymbol{e}_1$, $\boldsymbol{e}_2$) of the arbitrary ket $|\boldsymbol{e}\rangle$ are themselves kets, non-normalized and non-orthogonal triplets of numbers. We say that the equation (3.6.4) determines the matrix $\boldsymbol{G}$ up to an arbitrary phase, and this phase can acquire a precise physical meaning. Indeed, if α, β, γ, δ, are the entries of this matrix, and it is of the form (3.6.3), then by (3.6.4) we must have:

$$\boldsymbol{G} \equiv \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},\quad \alpha^2+\beta^2=\lambda^2,\quad \gamma^2+\delta^2=\mu^2,\quad \alpha\gamma+\beta\delta=\lambda\mu\cos\theta \tag{3.6.5}$$

Taking in consideration the representation suggested by the second and the third equalities here, we can write:

$$\alpha=\lambda\cos\varphi_1,\quad \beta=\lambda\sin\varphi_1,\quad \gamma=\mu\cos\varphi_2,\quad \delta=\mu\sin\varphi_2 \tag{3.6.6}$$

where between the two phases φ there should be a relation. This is given either by using the fourth equality from (3.6.5), or by calculating the area of the parallelogram supported by the two vectors, components of $|\boldsymbol{e}\rangle$, assumed to have a common origin:

$$\left|\boldsymbol{e}_1\times\boldsymbol{e}_2\right| = \lambda\mu\sin\theta \quad\leftrightarrow\quad \alpha\delta-\beta\gamma=\lambda\mu\sin(\varphi_2-\varphi_1) \tag{3.6.7}$$

We can choose $\varphi_2 = \phi_1 + \theta$, in which case the matrix $\boldsymbol{G}$ is

$$G \equiv \begin{pmatrix} \lambda\cos\phi & \lambda\sin\phi \\ \mu\cos(\phi+\theta) & \mu\sin(\phi+\theta) \end{pmatrix} \tag{3.6.8}$$

In a way, the form of this matrix is a pure convention, in the sense that the trigonometric functions of the sum of angles could very well be placed on the first row of the matrix and nothing would have changed. However, the convention can go only up to a point, inasmuch as it is connected to the concept of interpretation, as we shall see shortly here: this aspect of the form of matrix is simply connected to the difference between wave and particle.

In order to get a glimpse into the physical meaning of this construction, we assume, as we said above, that it is valid in the brain universe, in connection with the *local interaction* of the cancel waves in the Planck medium of this universe, *i.e*. in *the glia*, with the matter existing in this medium in the form of *neurons*, as well as some other kinds of cells. The glia is, therefore, to be treated as an optical medium in which the brain matter exists, and from an optical point of view, such a local interaction can be described as the physical action of the optical medium on the rays existing within this medium. On the other hand, from a physical point of view, the action of the medium described by a matrix (3.6.8) can be represented by an *Iwasawa decomposition* of the matrix [see (Simon & Mukunda, 1993, 1998), for the general idea and the literature to be consulted; see also (Mazilu, 2024) §6.5, for a similar case in the physical universe].

An Iwasawa decomposition of a matrix involves, in general, three matrix factors, each one of unit determinant, representing three basic physical operations, 'implemented' physically, as it were, by the optical device upon the optical phenomena taking place within it, such as those involved in the construction of a ray: reflection, refraction, focusing, propagation, etc. The Iwasawa decomposition, therefore, works for matrices like those from equations (3.5.8) and (3.5.10). In such a decomposition, we are not interested in the physical structure of the optical medium *per se*: this medium is only identified through its actions, as in the physical optics, represented by the phenomena of the kind just listed above. Let us assume that this decomposition involves the following factors, depending each one on one parameter only:

$$N(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \quad A(y) = \begin{pmatrix} y^{-1} & 0 \\ 0 & y \end{pmatrix}, \quad K(\phi) = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \tag{3.6.9}$$

and take the general form of the matrix G in the following factorization:

$$G = N \cdot A \cdot K^{-1} = \begin{pmatrix} y^{-1}\cos\phi + xy\sin\phi & xy\cos\phi - y^{-1}\sin\phi \\ y\sin\phi & y\cos\phi \end{pmatrix} \tag{3.6.10}$$

Now it is clear that the form of this matrix depends exclusively on N in the following sense: notice that the case in (3.6.10) is for N upper triangular. Should N be lower triangular, the lines in (3.6.10) would switch places in the matrix, and the matrix G would appear as in equation (3.6.8). Whence the conclusion: the form of G, having the sum of angles in the first line or in the second line, is dictated by the matrix factor N: if this matrix is upper triangular, the first line of the resulting matrix ***G*** apears with a sum of angles; on the other hand, if ***N*** is lower triangular, the resulting matrix ***G*** would appear with the sum of angles in the lower line. The bottom line is that the form of the matrix ***G*** depends on the property of 'triangularity' of the component factor ***N*** of the matrix in its Iwasawa decomposition, and this property is pending on the meaning of the Iwasawa parameter x: if x ia zero, all the differences disappear. On the other hand, one can see that the Iwasawa factorization is pending on the determinant of the matrix ***G***: the three factors (3.6.9) are of unit determinant, and if the determinant of ***G*** is not a constant, we cannot write an Iwasawa factorization for it. In terms of (3.6.5), the area determined by the two vectors in equation (3.6.7) needs to be constant, *and this means a quantization*.

A physical interpretation of parameters of the Iwasawa decomposition is now in order, which indicates a generic physical structure. From the physical optics' point of view, the three parameters of the Iwasawa decomposition have the following meannings: ϕ represents a *rotation angle*, obviously, for the rotation 'implemented', as it were, by the matrix $\boldsymbol{K}(\phi)$. y represents a general *magnification* – occasionally, it may be a 'negative' magnification, as it were, *i.e.*, a *reduction* – 'implemented' by the matrix $\boldsymbol{A}(y)$. As for x, on which we are mostly interested, it is a simple geometrical *displacement*, that, nevertheless, can be taken under a twofold meaning: either as a *displacement proper of a certain particle*, or as a *measure of propagation of a wave*. This is reflected in the kind of 'triangularity' of the matrix $\boldsymbol{N}(x)$, and it may be a matter of convention too. However a rule needs to be observed: if $\boldsymbol{N}(x)$ is taken as upper triangular for the *displacement of a particle*, then the *propagation of a wave* is to be described by a lower triangular $\boldsymbol{N}(x)$, and vice versa. This is the only *physical difference* – and, therefore, an essential difference at that – between the two forms of ***G*** from the equations (3.6.8) and (3.6.10), for the formal geometry of these matrices is the same. From this, formal, mathematical point of view, one could say that 'the waves and the particles are equivalent', as often uttered ever since the times of Louis de Broglie.

Then, the Iwasawa decomposition (3.6.10) says that the 'optical device model' of the medium in question acts in the plane of the wave, in the manner envisioned by Fresnel: first, by a rotation of the two orthonormal vectors from equation (3.6.4). A magnification is applied after this, on one of the resultant rotated vectors, while, concurrently, a reduction is applied on the other one. Finally, a 'translation' is applied on one of the vectors resulting from the previous two operations: it can be either a *displacement proper* of a particle, or in fact a *propagation* of a wave, depending on the property of 'triangularity' of $\boldsymbol{N}(x)$ as we have shown above. The action on the vectors $|\boldsymbol{e}\rangle$, on the other hand, which involves the realm of a ray in order to get the orthonormal basis $|\hat{\boldsymbol{e}}\rangle$, is accomplished by a matrix which is the inverse of the previous one, so we must have the decomposition:

$$\boldsymbol{G}^{-1} = \boldsymbol{K} \cdot \boldsymbol{A}^{-1} \cdot \boldsymbol{N}^{-1} \equiv \boldsymbol{K}(\phi) \cdot \boldsymbol{A}(y^{-1}) \cdot \boldsymbol{N}(-x) \qquad (3.6.11)$$

Therefore, now we have the following sequence of operations: first a displacement executed upon one of the vectors of non-orthonormal basis, or a propagation on the other, depending on the form of the matrix ***N*** as discussed above. This is followed by a magnification on one vector and a concurrent diminution on the other, and then the result is rotated: the rotation is the last operation performed by the medium in this decomposition.

As to the values of the three parameters x, y and ϕ, they can be found in terms of the four entries of the matrix $\boldsymbol{G}$, an identification which allows an exquisite physical interpretation for these very entries, and therefore of the matrix itself. Notice, first, that these entries are charges, to start with. The mathematical idea to be followed further, goes on using the equations (3.6.5) and (3.6.6) in order to identify the entries of $\boldsymbol{G}$ with those of its Iwasawa decomposition. This operation provides three equations under the assumption of a constant determinant, which allows us to normalize the matrix, and so to assume that *it has a unit determinant*. This system of equations gives the following values of the decomposition parameters:

$$x=\frac{\alpha\gamma+\beta\delta}{\gamma^2+\delta^2}=\frac{\lambda}{\mu}\cos\theta,\quad y^2=\gamma^2+\delta^2=\mu^2,\quad \tan\phi=\frac{\gamma}{\delta}=\tan\varphi_2 \tag{3.6.12}$$

It is these equalities that suggest an the interpretation of the entries of the matrix $\boldsymbol{G}$, which, at least in what we are concerned, results from the physical meaning of the decomposition parameters. But, before anything, first comes an important observation, purely geometrical in character.

Professor Marian Ioan Munteanu of University Alexandru Ioan Cuza from Iasi, Romania, uses the '*nak* Iwasawa decomposition', as he calls it, depicted in equation (3.6.10) above, in order to characterize some *Weingarten surfaces* (Munteanu, 2023). Inasmuch as, among other things, we have determined that this decomposition should be associated with a general description the Yang-Mills fields [(Mazilu, 2024), § 4.4], having as an important consequence the fact that the gravitational field can be described as a 'protective field' in the sense of wave mechanics (*loc. cit. ante*, §§ 4.3 and 6.5), it seems worth our while in making here a connection with a hot subject of geometry, which turns out to be a hot subject of physics too. The point of connection is the matrix (3.6.10): we just concluded that it is *physically necessary* for the description of fundamental interaction between the message waves and the cancel waves. Here, we further intend to deepen the association of the Iwasawa decomposition with the facts of physics, and the work of Professor Munteanu draws our attention a bit further, to the fact that it must be connected to the *theory of cycles*, eagerly promoted by Professor Vladimir Vladimirovich Kisil as a natural extension of the great Felix Klein's concepts of a unitary view of the geometry, with the help of continuous group theory (Kisil, 2012, 2023). Thus, the work of Munteanu suggests that the *nak* decomposition should be the key in the connection between the mathematical and physical facts. For once, it produces, indeed, conservation laws in a physical description of the Planck's procedure of quantization, as we shall show here right away.

Meanwhile, though, let us explain the gist of *our* approach, which is based on a general theory of functions (Barbilian, 1974), of the kind that started being revived only lately based on some general algebraical principles (Kisil, 1997, 2012). Notice the fact of fundamental importance in the 'physics of matrix $\boldsymbol{G}$', as it were: its entries *can be taken as charges* from the background of the universe, which can be physically considered a four-dimensional *de Sitter continuum of charges* [(Mazilu, 2024), Chapter 3, §5.2]. To wit: if we take the two lines of the matrix ***G*** as two-dimensional charge vectors, as one usually proceeds in handling the algebra of Iwasawa decomposition – where, nevertheless, one works as a rule with the vectors given by the columns of the matrix, instead of lines, though the theory is just the same – the displacement parameter *x* is the 'ratio of the two charge vectors', as it were, *i.e.*, the projection of the first vector along the inverse of the second. The magnification parameter, on the other hand, is given by the magnitude of second charge vector, whose (relative) orientation also decides the rotation angle. One can say that, in this case, the component charges of second line of the matrix 'dominate' this Iwasawa decomposition. Analogous results can be obtained when the first line is taken to dominate the decomposition. So, this time we can decide on the kind of matrix ***G*** – that is either (3.6.8) or (3.6.10) – by the line which dominates the Iwasawa decomposition.

The equation (3.2.16) for instance, represents a cycle. According to Dan Barbilian a general cycle can be obtained by taking the trace of the product of matrices:

$$\begin{pmatrix} \frac{z+z^*}{2} & -z\cdot z^* \\ 1 & -\frac{z+z^*}{2} \end{pmatrix}\cdot\begin{pmatrix} y & -y^2 \\ 1 & -y \end{pmatrix}=\begin{pmatrix} \frac{z+z^*}{2}y-z\cdot z^* & -\frac{z+z^*}{2}y^2+z\cdot z^* y \\ y-\frac{z+z^*}{2} & -y^2+\frac{z+z^*}{2}y \end{pmatrix} \tag{3.6.13}$$

and putting it equal to zero, which coincides indeed with the equation (3.2.16), up to sign, for that trace is:

$$-y^2+(z+z^*)y-z\cdot z^* \tag{3.6.14}$$

as one can easily see right away. Within this mathematical philosophy, a certain cycle can be obtained from a zero radius cycle, by acting on it with the Hessian matrix The trace of the cycle from the right hand side is the Hessian of the cubic having the three roots that generate its coefficients.

4. A Suggested Way of Communication in the Brain

Let us reiterate to our reader that, according to Hillis' scenario above (see § 2.1), there are two kinds of waves within the brain matter: the *message waves*, *circulating through the network of neurons* – which we take as *natural rays* in the brain universe – and the *cancel waves*, circulating *by propagation* in the manner described in physical optics, in the background of this universe, *i.e. the glia*. In such a scenario, the cancel waves must prevent, on occasions, the spreading of information initially carried by the message waves, and this 'operation' must be done, in general, by a process that needs to be physically understood and, therefore, mathematically *described*. In order to discern the necessary steps of a physical explanation towards a proper understanding of this situation, we appeal, again, to some concepts created by the necessities of the modern computer era, namely the *secrecy of* *communications* between individuals, *via* computers. In brain, however, the 'social secrecy' does not mean too much, but we choose the positive side out of this 'concept', if we may be allowed to use the word. Thus, we argue that, the general philosophy leading to 'secrecy' in society may be able to lead us, in turn, to a physical understanding of the functionality of neurons during communication. In view of our developments thus far, parts, at least, of what we can learn appear the way we describe in this section of our present work.

4.1. Messages and the Information They Carry

In the modern-day experience of humanity any message carrying some information, like, for instance, a file, an article, or even a book, *can be represented as a real number*: this is the basis of the modern technology of communications. The novelty brought in the physiological researches by the ideas of John Joseph Hopfield (Jr) must, in our opinion, be connected, *before anything else*, with this specific circumstance of the communication technology (Hopfield, 1982). More to the point, in matters of an analogy targeting the operational brain, our experience tells us that the 'files' of messages, representing images, written messages, audio messages, etc – *i.e.*, the kinds of messages which are addressed to our senses – can be represented as real numbers. Now, one cannot expect the neuron to write, or paint, or anything of the kind, but, in view of the previously described Planck procedure of quantization (§§ 3.1. and 3.2.), as it is *applied to the charges*, one is entitled to assume that within the universe of brain *charges are produced*, *of any values*. The meaning of this statement is taken 'in the sense of Planck' (§ 3.2.): not just integer multiples, of a fundamental charge of our experience, but, for instance, charges in the take of Edward Witten, who assumed that for dyons an excess charge might exist as real-number 'multiples', due to the presence of the magnetic charge in the universe (Witten, 1979) – and, moreover, corresponding to *three* basic charges of the universe we inhabit, not just one. It is these 'real-number charges' that, we think, carry somehow the messages. They are 'transporter charges', which may turn out to provide the the 'ink of the brain', or its 'paint brush', as it were.

A primeval process of producing such charges is by *sensory organs*: under the stimulus of external factors, these organs issue charges destined to special regions in the brain, which we can aptly call *sensory patches* (auditory, visual, etc). According to a Planckian natural philosophy, such regions can be characterized by three different charges out of a physical background, as presented in the § 3.2 above. In this view, a sensory patch is described by a class of Cauchy probabilities having the McCullagh complex parameter 'estimated' from 'samples of volume three' of physical charges, as the *roots of the Hessian equation* (3.2.16), corresponding to the cubic having the three charges as roots. Then, denoting z such a complex root, and z^* its complex conjugate, a patch is aptly represented by an involutive matrix whose fixed points are z and z^*:

$$I \stackrel{\text{def}}{=} \begin{pmatrix} z+z^* & -2z\cdot z^* \\ 2 & -(z+z^*) \end{pmatrix} \tag{4.1.1}$$

One can say that the general trend in theoretical physics today, entitles us to the conclusion that the brain is not an isolated case in this respect: if, for instance, Fennelly's idea of a 'Gödelised hadron' holds true, then every *fundamental physical particle is a universe* of such a possibility.

But, regardless of any purely physical speculations on this issue, so far as the quantization process gives us the rights to *represent such a charge by a real number*, we conclude that it can carry a 'written message', so to speak, in the brain, and the idea can be then advanced that *the messages in brain are*, in fact, connected to the *moving charges*, more to the point, to *physiological charges*, if we may be allowed to use a specific name. By analogy with the real numbers representing recorded information in social life, these charges too, can carry information to be stored or transmitted within the brain, and, for once, this circumstance entitles us to call them, simply *messages*. Then, according to the previous sections of this work, three charges of the world of brain generate three real numbers – let us say, *fundamental messages* – that can be arranged in a 2×2 matrix as shown in the equations (3.2.11) and (3.2.12). These matrices act homographically on the real or complex numbers, and linearly on pairs of such numbers. Therefore, we can conclude to their action *on messages*, and if a message is addressed to some place in brain, as requested in the Hillis' scenario, *the homographic action of the matrices must be the main mathematical tool for the description of physics of such communication in the brain*, in view of the brain condition as a universe (see § 3.2).

Here, in making up our mind on the problem and the way of its solution, we shall take advantage of yet another scenario of the modern social technology of communications, related, this time, to the secrecy of communications between machines, as we said. This scenario is related to the communications *via* a *public key cryptosystem*, with the help of which we can hope to learn how the brain neuronal units, and its waves connect amongst them. The secrecy, which is the main concern of every cryptosystem known thus far, is just a social aspect of the communications. As we said, one expects that for the communications in the brain the secrecy should not be as important as in the human society, but some other characteristics of a kind of social links – like *privacy*, for instance – would certainly prevail, and these are, indeed, of interest for us. Let us repeat it once more: there should be no secrecy in the universe of brain, or in any universe for that matter, at least not of the kind involved in the social universe. On the contrary, there should be, naturally, *addressed* messages, and even these must 'allow' to be 'corrupted', so to speak, in order to be rendered ineffective once a private address has been reached; this is an issue quite contrary to the secrecy.

We chose a *public key cryptosystem* based on Chebyshev's polynomials (Bergamo, D'Arco, De Santis, & Kocarev, 2005) as a guidance in our constructions. The reason of this choice is that the system of Chebyshev's polynomials has a property that can be suitably replicated by the 2×2 matrices: the commutativity. To wit: the set of Chebyshev polynomials is, algebraically speaking, a *commutative semi-group* with respect to the composition of polynomials. Of course, one cannot claim the same property of commutativity for the whole algebra of the 2×2 matrices. However, we can claim such a property for significant subgroups of matrices, that can have an interpretation in which the theoretical physics may be meaningfully introduced for the description of the functionality of the brain. On the other hand, in the brain universe there are 'social' groups of neurons known to exist, acting in 'synchrony', as it were, in order to accomplish the basic functions of the brain. Let us, therefore, first describe such subgroups of matrices, and then extend the physical image of the brain based on these, by showing how are they to be used for communication in the brain universe.

Consider a family of 2×2 matrices generated by adding an involutive matrix – algebraically represented, in general, as a matrix of null trace, of the type (3.5.16) or (4.1.1) – to the identity matrix or, more generally to a multiple of the identity matrix. For a general discussion of using such matrices in the physical case of special relativity, in terms of involutions and their algebraical properties, one can consult our recent work [see (Mazilu, 2024), especially § 2.3.] Thus, by this procedure of 'updating the identity matrix', as it were, we generally get a linear family of matrices depending on *two parameters*, say (p, q) – which can be taken as *coordinates of the matrix over the family* – according to the equation:

$$\boldsymbol{A}_{pq} \stackrel{def}{=} p\cdot\boldsymbol{1}+q\cdot\boldsymbol{I}, \quad \boldsymbol{I}\equiv\begin{pmatrix} a & b \\ c & -a \end{pmatrix} \tag{4.1.2}$$

Here the parameters a, b, c are fixed, and we expect them to completely characterize the family. For the necessities of physiology, if such a matrix *represents a neuron*, then we just can baptize its family, bearing in mind the physiological topography of the cortex: *e. g.*, a sensory patch, a memory location, and such like.

There are, indeed, on the brain cortex surface and within its volume, concentrated regions receiving and/or storing information that comes from different sensory organs [see, for instance, (Penfield, 1978); virtually any figure of the book will do; however, there are great many other, more comprehensive works to be consulted in this respect, easy to be found by everyone interested; the brain is, understandably, overly represented in social media!]. Such regions need to be coordinated with each other, and there are some other concentrated regions within cortex, doing this specific job. All this organization is controlled, according to Wilder Penfield, from the upper stem of the brain, which is surrounded by the cortex (*loc. cit. ante*, Figures 8 and 9). Let us call these co-ordinated regions *joint action regions* (*JAR* for short). The action in question may be of different kinds: storing sensorial information, coordinating two or more sensorial regions, learning, motor, etc.

So, assuming the representation of a neuron by a matrix, we may say that any time when a neuron is targeted, a JAR is targeted first. For the neuron is found at one of the addresses from the region 'located' by the numbers (a, b, c) playing the part of an 'area code', if we may be allowed to push a little the analogy with the communications in society. One can say that (p, q) are the coordinates of a matrix in the JAR represented by the family of matrices generated by the two matrices (***1***, ***I***). The multiplication of two matrices of a given JAR is commutative, and this is one of the basic algebraic properties to be exploited in a public key cryptosystem of the type mentioned above. Indeed, multiplying two such matrices of the same family, one of them having the coordinates (p_1, q_1) while the other has, respectively, the coordinates (p_2, q_2) in the family, and represented within the given JAR by the matrices:

$$\boldsymbol{A}_{p_1q_1} = p_1 \cdot \boldsymbol{1} + q_1 \cdot \boldsymbol{I}, \quad \boldsymbol{A}_{p_2q_2} = p_2 \cdot \boldsymbol{1} + q_2 \cdot \boldsymbol{I} \tag{4.1.3}$$

results in a matrix (4.1.2) of the same family, having the 'coordinates' in the JAR:

$$p = p_1 \cdot p_2 + q_1 \cdot q_2 \cdot (a^2 + bc), \quad q = p_1 \cdot q_2 + q_1 \cdot p_2 \tag{4.1.4}$$

Of these coordinates, only the first one depends on the 'area code', as it were. The group property of such a two-parameter family of matrices now becomes pretty obvious. For once, the family is closed to multiplication: the product of two matrices of the family, belongs to the family. Further, we have an identity of the family for $p = 1$ and $q = 0$, and there is an inverse for each of its members: if (p, q) are the coordinates of the matrix in the family, its inverse has the coordinates

$$\frac{p}{p^2 - q^2(a^2+bc)}, \quad \frac{-q}{p^2 - q^2(a^2+bc)} \tag{4.1.5}$$

Moreover, in view of the symmetry of the bilinear forms from equation (4.1.4) with respect to the indices of coordinates, one can say right away, that such a two-parameter family (4.1.2) is a *commutative subgroup* of the multiplicative group of the 2×2 matrices, and this proves our contention.

These algebraic properties are then reproduced in any action of the matrices belonging to the JAR family, in particular by the *homographic action* on the real or complex numbers. Thus, denoting the homographic action on a number x by $\boldsymbol{A}(x)$, we have:

$$\boldsymbol{A}_s\left(\boldsymbol{A}_r(x)\right) = \boldsymbol{A}_r\left(\boldsymbol{A}_s(x)\right), \quad r \equiv (p_1, q_1), \quad s \equiv (p_2, q_2) \tag{4.1.6}$$

In view of this property of commutativity, let us replicate the public key cryptosystem based on the Chebyshev polynomials [see (Bergamo, D'Arco, De Santis, & Kocarev, 2005), §III A], by using the homographic action of the matrices instead of Chebyshev polynomials. In this case, the homography, like a neuron, would act by that 'incrementing or decrementing' of the charge serving as a 'backpointer' in the Daniel Hillis' scenario.

So, like in the case before, in the 'quotation' concerning Hillis' scenario, here too, we only have to replace conveniently some of the terms from the original just cited (*loc. cit. ante*). Assume, therefore, two neurons, called *Alice* and *Bob* – apparently, these are the universal names of partners in communication in the science of cryptography when it comes to scenarios like this one, chosen apparently just in order to justify the necessity of secrecy of communication – 'living' in a JAR from the brain matter, somehow described by the 'coordinates' (a, b, c), *via* the matrix ***I***. Assume, further, that Alice wants to connect with Bob – it is, as always the female who makes a first move, 'she chooses the one that will choose her', in spite of the fact that it may appear otherwise – as requested in the Hillis' scenario: she first needs to 'give him a pointer'. In order to do this, according to the cryptosystem in question, *Alice* must follow a *key generation algorithm*, with the following steps:

> 1) *Generates* a pair of real numbers, $s \equiv (p, q)$ say, representing her coordinates in the JAR; 2) *Selects* a random real number, x say, – not to be confused with the parameter of the Iwasawa decomposition, at least not always – and then 'computes' $\boldsymbol{A}_s(x)$ by the homographic action of the matrix $\boldsymbol{A}_s$ belonging to the family of commutative matrices of parameters (a, b, c), like those from equation (4.1.2); 3) Then she *sends* a *public key* message *in the network*, as a pair of numbers $(x, \boldsymbol{A}_s(x))$, and *keeps* the pair of coordinates s as a *private key* for herself.

In order to communicate with Alice, *Bob* collects her public key $(x, \boldsymbol{A}_s(x))$ which is at anybody's disposal from the network. Then he proceeds with an *encryption algorithm* for his message, in four steps, as follows:

> 1) *Represents* his message as a real number, m say; 2) *Generates* a pair of real numbers, r say, representing his address in the specific network, analogous to s, the address of Alice; 3) *Computes* the three numbers $\boldsymbol{A}_r(x)$, $\boldsymbol{A}_r(\boldsymbol{A}_s(x))$, and $X = m \cdot \boldsymbol{A}_r(\boldsymbol{A}_s(x))$; 4) Then he *sends* to Alice the *ciphered* message given by the pair $C \equiv (X, \boldsymbol{A}_r(x))$.

Now, Alice can recover the message m addressed specifically to her by Bob, from the encrypted one she receives*,* by a *decryption algorithm* following two steps:

> 1) She *computes* $\boldsymbol{A}_s(\boldsymbol{A}_r(x))$, using her private key s and the component $\boldsymbol{A}_r(x)$ of Bob's ciphered message; 2) she *recovers* the message m of Bob, simply by *computing* $m = X/\boldsymbol{A}_s(\boldsymbol{A}_r(x))$, from the first component of the Bob's ciphered message, which is actually a *bona fide* deciphering.

It is easy to see that this cryptosystem is correct: all the operations listed above make sense exactly as in the original Chebyshev cryptosystem [see (Bergamo, D'Arco, De Santis, & Kocarev, 2005), §III B].

As we said, this scenario is taken here only for guidance, in view of the fact that, as the physiology seems to imply, the physics here is quite complicate, and there is but only a scarce possibility of guidance that it can offer by itself. The reason for this state of the fact is, in our opinion, that the physics asks deliberately for considering the brain a physical system, without even a hint that it might be even 'willing to change its own mind', so to speak [see, for instance, Gérard Toulouse's exquisite work (Toulouse, 1992), especially the section of that work entitled "Historical steps"]. The previous scenario, reveals instead a few issues that must be in our attention here, all of them springing from the fundamental one: the commutativity, which is the fundamental algebraical property that makes the previous scenario effectual, is valid *within the same social group* (a, b, c) – a JAR. The essential trend of these issues is not just physiological, but mainly mathematical, and therefore quite general, if not straightforwardly universal: how can one calculate a specific address (p, q) in a JAR (a, b, c)? But, let us put this issue aside for the moment, assuming that it can be done mathematically, and just analyze the consequences, hoping that the solution will come to us by itself, along the lines of physiology.

An indication of the place of solution is handed to us right away. Namely, from the point of view of the modern technology of social communication of the mankind, such a cryptosystem seems quite unsafe at the first sight: it can be easily attacked, and the secrecy is gone. However, for once, we can say, and not just for the sake of those strictly interested in the technology of communications in society, that such cryptosystem can be made unbeatable with a careful choice of the parameters a, b and c, *in connection* with p and q. The mathematical point of this whole theory is that the neurons, just like matrices, must be *associated in cycles* [for the physiological necessity see (Kubie, 1930); for the associated mathematics see (Kisil, 2012) especially Chapter 5 of that work]. The prototypical example of such a cycle in physics is the classical Kepler's ellipse, that served, *in this capacity*, the theoretical physics close to four centuries since its Newtonian beginnings. This association points out, among others, to the study of the *Weingarten surfaces* (Munteanu, 2023). What we are interested in, though, is the logical reason for such an approach. Just anticipating, we shall settle here for reasons closer to the inception point of this theory [(Fillmore & Springer, 1990); see also (Cnops, 2002), especially Chapter 4].

It is this connection that, in our opinion, one can deliver the detailed solution to our *problem of address*. As it is now, though, that is, quite 'undisciplined' from a theoretical point of view that will refer to the concept of a universe, the procedure described above makes the communication reasonably safe, indeed, for the transmission of fairly large texts, for instance. But this is not our main point here: as we said, the nature *does not function by secrets* of the kind that make the human society and this can raise difficult issues for the mathematical formalism, due to its physical implications. In the brain, for instance, the signals *must interact* in order to be 'stopped', to say nothing of some other physical processes we may have working in the brain in order to create a mind, or interact with a mind, as the case may occur. So, in the brain universe, the secrecy of the social kind should not be an issue: it only comes down to *privacy*, as we said, *i.e.*, to the assurance that the message delivered is received *at the place indicated by its address*. The scenario right above, like the one of Hillis before it, can only be used by us just to guide, and 'discipline' as it were, our work. This means that every operation of this scenario *must have a physical correspondent in the brain process* of handling the charge. And this is, in fact, our overall task here, to which we shall concentrate in the remaining part of the present instalment of our work. For now, though, we only notice that the analogy of the brain with a computer [(McCulloch, 1949); see also (McCulloch, 1965), *passim*], resorts on the possibility of the nowadays much-discussed *association* in the brain, originally advanced by Donald Hebb as a 'social theory', so to speak (Hebb, 1949).

In summary, the whole moral of the present section amounts to the conclusion that the involution ***I*** is the mark of such an association, primarily represented by a JAR: the mathematics of a family of involutions offers a manner of description of the common actions of a 'society of neurons'. Such a society exists in the background of the brain universe, and therefore should be dictated by the structure of this background, and we need to formalize its ‘joint action’, in the first place. It is time, therefore, to consider this fundamental issue right here, in order to help developing the physical theory of the brain functions.

4.2. The Neurophysiological Problem of Individual Addresses

In the above cryptographic scenario the essential problem to be solved is that of the address of an *individual neuron*. An essential step toward the solution of this problem is the discernment of the physiological possibilities of communication of the biological cells in general. Thinking of a neuron *as of a biological cell*, the usual direction of any messaging in which the whole neuron may be involved, is apparently fixed from a physiological point of view by Ramón y Cajal’s principles of neurophysiology [see (Kandel, 2006), Chapter 4; also (Kandel, 2018), Chapter 1]: through its soma dendrites a message goes towards the *cell soma*, and then into the *cell nucleus*, where, after some processing, the message is directed along *the axon*, to be again collected and transmitted either directly by some *axon synapses*, or through the terminal *dendrites* of the neuron towards the terminal synapses. It is, in such a scenario, within *the nucleus* of the cell, that we can speak of a genuine address of the neuron. One can say that, what the neurophysiology brings into the analogy regarding the rays of different universes, is the fact that a physical concept of ray must include *the source of ray* into its very concept. This is that part of the ray that ‘accrues’ whatever is necessary from the universe, and creates, out of that accrual, a ‘cut’ which is propagated along the ray. In the case of the optical rays that part was a slit or a hole in a screen. Louis de Broglie described it through a function *independent of time*, which dictates the initial amplitude and the phase that allow for the interpretation of the wave propagating along the ray [(Mazilu, 2020); see especially § 2.1]. In the brain, though, one can hardly say that the source of the wave propagating along the ray can be a slit or a hole: all we need to consider for the concept is the essential physical aspect that these devices can generate. That is, we need to consider *the static conditions defining an amplitude and the phase of the interpretative ensemble*, which can be *a priori* assured by the holographic property of a universe as a concept [(Mazilu, 2024); see § 4.5].

Fact is that, in the case of a cell, the nucleic acids have been found of essence in the *transmission* of the genes along a line of living individuals. And, in the problems related to the structure of these nucleic acids, the biochemistry was well ahead of us in using the cryptographic approach, by some three quarters of a century. Everything started with the realization of the *double helix structure* of the nucleic acids (Watson & Crick, 1953). The theoretical physicist George Gamow then entered the stage immediately, with the observation that *the handling of proteins in the nucleus has regular geometrical traits* (Gamov, 1954), which point directly to a *cryptographic scheme* of the genetic material (Gamow & Yčas, 1956). And so, the modern *genetical coding* was born. Our idea is that the charges involved in the biochemistry of the nucleic matter can be taken as those charges involved in the structure of the brain universe, just like the charges of the electron, proton and vacuum in the chemistry of the physical universe. So, it happened that we started to think of *the addresses of neurons as supported by the genes themselves* along these very lines.

This idea had, nevertheless, some fundamental shortcomings. First of all, the 'genetic code' in every cell, contains the characteristics of the *whole* individual whose body contains that cell: every cell in a body has that same genetic code, so one cannot think of 'locating' a single cell by such a code. We have to settle for some other possibility of locating a cell in the brain. Luckily, towards the end of the 20th century, and under the pressure of experimental facts, the views changed on the structure of nucleic acids: they were not only transmitters of the physical properties of individuals, but – and this is our own twist on a common conclusion – *can also serve for transmitting messages within an individual*. We quote here from a work concluding the specific achievements of the previous millenium:

> *Celular functions*, *such as signal transmission*, are carried out by '*modules*' made up of many species of interacting molecules. Understanding how modules work, has depended on combining phenomenological analysis with molecular studies. General principles that govern the structure and behaviour of modules may be discovered from synthetic sciences such as engineering and computer science, from stronger interactions between experiment and theory in cell biology, and from *an appreciation of evolutionary constraints*. [(Hartwell, Hopfield, Leibler, & Murray, 1999); *emphasis added*, *a*/*n*]

Thus, the great idea has been passed into the third millenium, in connection with the protein matter of the cells, namely that this one was entirely analogous to overall architecture of nervous matter of the human brain, which has the prevalent part of it – that is, the free cortex, *dedicated to learning* – largest in the whole animal regnum (see § 1.3 above). It appears, indeed, that this property of the human brain as a whole, replicates, in fact, a corresponding property of the genetic material in the cells' nuclei, *namely of the existence of a large quantity of non-coding genes*, previously deemed as 'junk', because, apparently, it did not serve to 'coding' *per se*. This quantity is, for the human genetic material, the largest in the whole world of living beings [see (Shabalina & Spiridonov, 2004), especially the Figure I of the work, the only figure, in fact]. The amount of proteine-coding material in humans is about 2%, and this is less than in any being on Earth. *Transcribed* non-coding material in humans amounts to 43% of the proteines, while 55% are *untranscribed* (for the meaning of these terms see *loc. cit. ante*, and the specialty literature given there). Just like the free part of the cortex, serving for learning purposes, these 'non-coding parts' of the nuclear matter can also serve for learning, which from the perspective of the rays in the brain universe should mean the transmission of information *within individuals*. Notice, in passing, that this similarity between the whole and its fundamental physical component, may be a solid reason for taking *the man* as a universe, not just *the brain*. That is, the hypothesis can be forwarded, that *the non-coding material from the* *cells*' *nuclei*, *contains in a certain way not only the address of the individual as a whole*, *but even the address of the cell itself*. That 'certain way' must be uncovered as we go along with the theory. The 'coding' *per se* is pretty much referring to the properties of the double helix, as once explained by George Gamow (Gamov & Yčas, 1956). For understanding the other terms of this discourse, we recommend the short but comprehensive work of Namrata Iyer on this issue (Iyer, 2011). A critical analysis of the idea of coding with its *pros* and *cons*, based on the experience from the second half of the 20th century, is offered in (Stegmann, 2016).

The problem is thus reduced, at least in what we are concerned, to the connection of the *double helix*, through analogy of course, with the structure of the fundamental complex of electricity: *the planetary atom*. The next section is dedicated to the physics related to planetary atom, more to the point, to its quantal inception point and the problem of its forces. In this physics we shall be able – again, hopefully, let us not forget the hope! – to read some way to conceive the physics of electricity in the brain universe.

4.3. The Edifying Case of Zenaida Uy's Gauge

Having in mind this last idea of analogy, we shall get into an interesting historical roundup of some notions apparently unrelated to each other, in order to describe the planetary structure of the matter, essential in making the BKS case (see § 2.2. above), of an 'atom' as a 'double helix'. As a matter of fact, we can make up our mind on the position of this case in our knowledge, just by turning back to Joseph James Thomson, at the beginning of the 20th century, well before the BKS fundamental work (Bohr, Kramers & Slater, 1924). It is hoped that this excursion in time will help us in making up our mind in *all* issues connected to this concept, not just for the case of brain. So, let us read again the fundamental assumption of the Thomson's natural philosophy regarding *the dynamics of negative electricity* that he discovered by the end of the 19th century. Quoting:

> In the Philosophical Magazine for August 1907, I discussed a theory of radiation from hot bodies which regarded the radiation as arising from the impact of *negatively charged corpuscles* with the molecules of the body; the impact starting electric pulses which collectively constitute radiation from the body. When we resolve, by Fourier's theorem, this radiation into its constituent harmonic vibrations, we find that the amount of light of any given period depends upon *the ratio of that period to the time occupied by a collision*. It was shown, moreover, that this radiation would *not conform* to the Second Law of Thermodynamics, unless the *time occupied by a collision* varied *inversely as the kinetic energy of the corpuscle* before it came into collision, and in addition, that the time of collision of a corpuscle moving with a given speed *must be constant and independent of the nature of the molecule against which the corpuscle collides*. I showed that *the first* of these conditions would be satisfied if *the forces exerted during the collision* between a corpuscle and a

molecule *varied inversely as the cube of the distance* between them; *the second* condition will be satisfied if *the collision is regarded as taking place* not between the corpuscle and the molecule as a whole, but as *between the corpuscle and systems dispersed through the molecules*, these systems being of the same character in whatever molecules they may be found, and repelling the corpuscle with forces varying inversely as the cube of the distance between them. *Forces of this type would be exerted by electric-doublets of constant moment* with their negative ends pointing to the corpuscles. [(Thomson, 1910); *our emphasis*]

First of all, it is not too much to say in order to make here a connection with the Maxwellian theory of electricity, bearing in mind its characterization by Henri Poincaré (see § 2.7 above). Indeed, assuming that the collision of 'negatively charged corpuscles' must be allocated for a physical explanation to the *inertial properties* of matter, one can see in those 'systems dispersed through the molecules' mentioned by Thomson, the interpretative ensembles for a Maxwellian 'inducing fluid', as described by Poincaré. Then, a host of facts, apparently not having at all connection among them – at least according to an orthodox natural philosophy, obviously – come into a common place.

To wit, notice the closing phrase of the excerpt above: the 'electric-doublets of constant moment' are essential in the description of the forces. Keeping in mind the statistical nature of these forces, the *dipoles* enter just naturally in the physical theory here, giving a natural-philosophical reason to Planck's approach of quantization [see (Mazilu, 2024), § 1.2]. And, if this statistical theory of forces is described by a static Yang-Mills electrodynamics, then the *atomic nucleus* has, indeed, its equivalent in the *cellular nucleus*. In view of this observation, we can further venture to say that the 'double-helix ladder' of the genetic matter in the world should be described as a 'Planckian property' of the matter in general. Assume, indeed, that the atom itself represents a universe. Then its physical structure is a 'double helix', just like the DNA; the only difference is in the structure of the very dipoles. This is the main theme of the present section of this part of our work.

Now, we can say that the Newtonian forces are the essential properties of a ray, for, according to J. J. Thomson, an atomic structure, in general – not just a planetary structure – should be based on a corresponding structure of the forces acting within the atom. And here we have the novelty destined, in our opinion, to 'complete' the Newtonian analysis of the forces [(Newton, 1974); Book I, especially Sections II and III]. To Thomson, these forces are given by:

(1) A radial repulsive force, varying *inversely as the cube of distance* from the centre, diffused throughout the whole of the atom, combined with

(2) A radial attractive force, varying inversely as the square of the distance from the centre, *confined to a limited number of radial tubes in the atom* [(Thomson, 1913); *original emphasis*]

The *radial repulsive force* is universal in space, and Thomson seems to imply that it is connected to the definition of that part of the energy that we call *potential*. On the other hand, the *radial attractive force* acts only within selective *solid angles*. In what follows right away, we will concentrate on a theoretical description of these two forces in a relation that can be rightfully called a 'Planckian property' of the matter.

Let us insist a here on a forgone expression, if we may, of the conclusion of the Thomson theory, written in the modern language of the Yang-Mills fields. In the previous instalment of the present work, we presented a case illustrating the static Yang-Mills fields of Zenaida Uy (see I, §4.6). We re-present here the logical reason of this case again, this time from the point of view of the modern concept of interpretation, along the idea of an optical field as advanced in § 3.5 above. This repeat is intended to serve our present purpose of delineating the analogy between *the nucleus of the planetary atom* and *the nucleus of a living cell*, by clarifying the angle of approach. After all, we can declare that, if a neuron is a ray, just like the planetary atom in a kind of BKS theory (see § 2.2 above), such an analogy becomes a necessity. It is inspired by the exquisite work of Nicholas Manton presenting the nuclear matter of the atom from the point of view of the *deformation of matter in general*, just like Henri Poincaré did before, when he presented the concept of charge (see § 2.7 above), only adopting the modern point of view of *skyrmions* (Manton, 1987).

Assume a fictitious static world of particles in equilibrium under the Newtonian forces between them, in an ensemble to be used for the interpretation. When we say 'Newtonian forces', we understand *not* the general forces defined by Newton, but those central forces with magnitude depending exclusively on the distance between material particles in a specific way: *inversely proportional to the square of this distance*, and *directly proportional to the square of the physical quantity generating the force*. In this case, though, the *distance* cannot be a dynamical variable, but rather a statistical one, serving for interpretation in the manner described by Fritz Zwicky (see § 3.3). The square of this statistical variate becomes *the variance of distance between the particles* over the interpretative ensemble. Then, it is the inverse of this variance to be taken as the Newtonian force realizing the equilibrium over this interpretative ensemble. One can say that the Newtonian forces are, *in this interpretative instance*, those forces once invoked by James Hopwood Jeans on the occasion of the second edition of his celebrated *Report on Radiation* (Jeans, 1924).

This case, to be briefly described now, is referring exactly to such an approach of the problem of radiation, for it considers the forces exactly from the perspective we just presented above. This is one of those moments of history, when the man realizes that he needs to change his mind on the case of microcosmos, and it was, indeed, forced upon our intellect by the concept of quantum. Remarkably enough, it is referring to those concepts that were extended with no restriction, as we said before, at another scale than the one where they have been discovered: the Newtonian forces. To wit, Jeans had the following observation regarding the *electromagnetic forces* connected to the static field intensities, *according to the Maxwellian theory of light*, which seems to be in blatant contradiction with *the idea of quantum*, but appears implicitly as an incentive for the concept of interpretation, starting directly from the classical mechanics:

> We can avoid all contradiction with the conceptions of the quantum theory, and at the same time retain equations (*of Maxwell electrodynamics*, *a*/*n*), if we suppose that *X*, *Y*, *Z*, (*the components of the electric field*, *a*/*n*), *or some functions of these quantities*, *measure*, *in some way at present unknown*, *the probabilities of jumps in the velocity and perhaps also in position* of an electron which forms part of an atomic system.
>
> In the limiting case in which *X*, *Y*, *Z* change *only very slowly with time*, the whole field of radiation may be analyzed, by Fourier analysis, into trains of waves of very low frequency. For *radiation of low frequency the quantum is very small*, and the jumps required by quantum-theory are so slight that the motion of the material electrons approximates closely to continuous motion. When the jumps in the electron velocities are very slight, they must be very frequent if finite change is to be produced, and instead of considering the probability of a jump occuring in one second, it is more natural to discuss the "average number"of jumps per second. But the "average number" of jumps per second in the velocity of an electron is proportional to the increase per second of the velocity, and so again is proportional to electron's acceleration. On the classical mechanics, *X*, *Y*, *Z* are proportional to the acceleration of the electron; on the quantum-mechanics we have seen that they are proportional to this acceleration *in the special case of radiation of very long wave-length*, but that they assume some other meaning in the general case of radiation for which the quantum is not of inappreciable amount. Thus, the quantum-mechanics must include the classical electrodynamical theory as a special case *appropriate to radiation of great wave-length*; this consideration explains why *the true black-body radiation formula* (Planck's formula) *reduces to the equipartition formula* when the wave-length of the radiation is very great. [(Jeans, 1924), §88; *our emphasis*, *a*/*n*]

One can notice right away the fact that the role of forces, as it is assigned by Jeans, is that of fundamental *adiabatic variables* that can serve in the construction of a quantum mechanics, as this has been done initially (Heisenberg, 1925). For now though, our observation, in view of the discussion preceding this excerpt, is that those functions are indeed related to probabilities, as Jeans implies, however, not 'in a manner at present unknown', but quite on the contrary: as forces, they are *inverses of the variance of distance*, taken as a statistical variate over the interpretative ensemble. Apparently, these forces represent a kind of 'likelihood functions' over the ensembles used for the concept of interpretation, keeping in mind the statistical practice of describing a statistics by the 'maximum likelihood' (see § 3.2 above, for a significant example regarding the Cauchy density for the one-dimensional case). This conclusion explains quite a few facts related to the concept of interpretation, to be briefly reviewed in this section.

Theoretical physics allows us to think *logically* of the interpretative ensembles as characterizing a stationary state of the physical universe, but the common present-day scientific opinion is that these ensembles must be *associated with a reality*. Moreover, not the classical physics, but the general relativity even allows us to describe a like ensemble in the language of mathematics (Israel & Wilson, 1972). Such an ensemble, and along with it the corresponding state of a universe is, nevertheless, fictitious from physics' point of view, which, obviously, involves the experience of humanity. The essential reason for this appears to be the non-existence of some conceivable *static* Maxwellian electromagnetic fields. However, such a state *can be imagined* for Yang-Mills fields (Wu & Yang, 1969), and a few cases of significant solutions have even been described in physics [see (Marciano & Pagels, 1975); also (Uy, 1976), and I, §4.6]

What is the *gnoseological* basis of the assumed existence of such an interpretative ensemble? According to our experience, there are *three* known physical quantities that generate forces the way Newton characterized them for cosmological explanation needs: these are the gravitational mass and the two charges – magnetic and electric. In the spirit of interpretation of all times, one can *imagine* a world with matter composed of identical particles where these forces act simultaneously, and therefore can make an ensemble of particles in equilibrium [(Mazilu, 2020), Chapter 3, especially §5.1]. We emphasize once again: *imagine*, since, according to our experience, this world *cannot be real*. Indeed, the three static forces – gravitational Newtonian, Coulombian electric, and Coulombian magnetic – cannot act simultaneously with the same 'intensity' at any space scale. Each one of them prevails at a certain scale of space: the gravitation prevails at the *transfinite scale*, the Coulombian electric force prevails at the *finite scale* of our experience, and the Coulombian magnetic force allegedly prevails at the *infrafinite scale*; but, again, only *allegedly*, for there is no firm evidence of the existence of magnetic monopoles. To conclude: there is not a *real world* where the matter can exist in the form of static ensembles of material points in equilibrium under Newtonian forces at the same space and time scales. However, from the point of view of the Planck's type of conservation laws as above, the brain can be taken as such a world, pending a right natural philosophy, as we shall see here.

The Newton's cosmological model for the beginning of the world assumes a static world of an ensemble of bodies in equilibrium, extended transfinitely. Apparently because of this extension, the particles start 'falling' towards *different centers of force*. Such a 'fall' persists up to the position where the falling particle meets its own 'orb', and at that point it undegoes a "transverse impulse of a just quantity", in the expression of Newton, necessary in order to place the particle along its orbit, in a Keplerian motion. The velocities thus transmitted to the particle by this 'just transverse impulse' – creating the initial velocities of the Kepler motion that starts when the falling particle meets its 'orb' – are imprinted in the elements of the orbit, *viz*., in its shape and orientation, and so we became, in time, able to recognize them, as we do regularly today [(Mazilu, 2020), § 2.3]. Those velocities are so real that they can be even… calculated. However, when it comes to physics, *they cannot be justified*, because we do not have the corresponding experience: no one of us has ever lived through the times of Creation! They are like the mind, as it is characterized by Wilder Penfield in the excerpt from § 1.3 above: we can 'see' them, (by their consequences in contemporaneity!), but we do not comprehend their reality. Newton himself, appealed to an "intelligent Agent", in order to have a justification of the 'just transverse impulse'. It may appear as quite significant, at least for those among us who believe in the existence of a 'physics of mind', as it were, that the *Newtonian initial conditions* for the Kepler motion can be retrieved as a metric tensor by the Einstein's cosmological static field, a manifestation that can even be considered satisfactorily explained by the Wu's and Yang's idea of a static condition of the Yang-Mills field, among others [(Wu & Yang, 1969); see also (Mazilu, 2024), Chapter 3; also Chapter 5, especially § 5.1 and Chapter 4, especially § 4.4].

However, in the world of brain, and under the Planck's quantization procedure, the situation just presented above may be taken as real, at least to a certain extent, showing us what the general natural-philosophical case may be. Let us start with the notion of 'fall': it suggests a trajectory that, according to Thomson's kind of natural philosophy can only be a ray, and a ray in the universe of brain can only be a set of neurons linked with each other 'in series', through synapses. The electricity 'falling along the ray' goes through the synapses of the axon of a certain neuron, which inject messages at a 'just transverse impulse'; or it goes through the dendrites into another neuron which 'enhances or diminishes' the message, according to Hillis' scenario. These *are conserved*, and thus imprinted into the flowing electricity, just like the initial velocities into the current shape of an orbit. This is a natural circumstance in a universe and, accordingly, the Planck's procedure of quantization produces two families of conservation laws that support it (see § 4.7 *infra*). Later we will also show that even the Keplerian orbit itself has an informational analog in the *idea of cycle*, provided we take into consideration the holographic property of the concept of universe in general [(Mazilu, 2024); §5.5]. All these facts will be, in a moment or other, targets of our discussion in the present work.

Right now, though, we think is the proper moment to do some justice to a series of theoretical facts that, starting with Newtonian cosmogony just recalled above, which appeared every now and then as hypotheses used to explain facts of experience. In doing this, we can learn plenty for carrying the physics over into some other worlds. These facts are specifically connected to that 'sideways motion' inside a ray, once brought to our awareness by Joseph James Thomson [see (Mazilu, 2020), Chapter 6; see also (Mazilu, 2024) § 2.2, for a discussion in connection with Newton's cosmogony]. The lesson we draw from this history is this: *the de Sitter background of charge cannot be carried by matter because it is the very cause of motion of the matter*. The motion is 'induced', to use the term of Poincaré – thus implicitly adopting the fact that *the de Sitter background is a Maxwellian inducing fluid* – by its *capability of deformation*. This is why we brought the subject here in connection with Zenaida Uy's and Nicholas Manton's names, in the first place. Let us unfold an argument along this line of the natural philosophy.

As we have mentioned before, we have reasons to settle for the idea of an analogy between the atom in the physical world, and the neuron in the universe of brain. The key concept of this analogy is the nucleus: it plays the same role on both sides of the analogy. In physics, the nucleus was described as a physical structure by Tony Hilton Royle Skyrme, whence the name 'skyrmions' for its constitutive particles. Nicholas Stephen Manton presented a geometrization of the Euclidean 'skyrmions', whereby these are *related to the deformations of matter* (Manton, 1987). As Manton shows, this idea is one of classical inspiration taken from the theory of deformation of *continua*, where the experimental deformations of solids and fluids are described by the so-called *tensor of elongations* [see (Hill, 1968) for the unitary definitions of the measures of strain in terms of the tensor of elongations; see also (Ogden, 1972)]. The most general energetic functional of Nicholas Manton can be written in the form of a function of the three eigenvalues of the tensor of elongations of a continuum; an example, involving the general procedure can be adduced along with a continuum theory of nuclear matter (Mazilu, Ioannou, Diakonos, Maintas, & Agop, 2013). The general physical point of view involves a vector $\boldsymbol{\Phi}$ dictated by the *deformation of matter* in the regular Euclidean space:

$$\boldsymbol{\Phi} \stackrel{\text{def}}{=} \begin{pmatrix} \lambda^1 & 0 & 0 \\ 0 & \lambda^2 & 0 \\ 0 & 0 & \lambda^3 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{4.3.1}$$

with $\lambda^{1,2,3}$ the elongations along three reciprocally orthogonal directions. Here the interior (or dot) product is simply the dot product of regular vectors with matrices. Assuming that *the fundamental property of a continuum background is that of deformation*, *and this property is independent of space scale*, an equation like (4.3.1) is valid at the infrafinite scale of space. Representing this scale by Newtonian differentials, we have a correspondent of (4.3.1) in the form of differential vector:

$$d\boldsymbol{\Phi} \stackrel{\text{def}}{=} \begin{pmatrix} \lambda^1 & 0 & 0 \\ 0 & \lambda^2 & 0 \\ 0 & 0 & \lambda^3 \end{pmatrix} \cdot \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} \tag{4.3.2}$$

Now, we extract from Manton's theory the fundamental observation [see (Manton, 1987), Equation (2.12) and discussion following it, regarding the *non-trivial deformation*, *i.e.* a deformation producing geometrical distorsion] that the flux:

$$d\boldsymbol{\Phi} \wedge d\boldsymbol{\Phi} = \lambda^2\lambda^3 dy \wedge dz + \lambda^3\lambda^1 dz \wedge dx + \lambda^1\lambda^2 dx \wedge dy \tag{4.3.3}$$

is generated by the Zenaida Uy's vector $\boldsymbol{b}_0$ [see I, equation (4.49)] through the surface element $d\boldsymbol{S}$, transcribed here in the form of a ket:

$$\boldsymbol{b}_0 \equiv \begin{pmatrix} \lambda^2\lambda^3 \\ \lambda^3\lambda^1 \\ \lambda^1\lambda^2 \end{pmatrix}, \quad d\boldsymbol{S} \equiv \begin{pmatrix} dy \wedge dz \\ dz \wedge dx \\ dx \wedge dy \end{pmatrix} \tag{4.3.4}$$

Then, the transport of the *elementary space oriented volume* along the vector (4.3.3):

$$i_{d\boldsymbol{\Phi}\wedge d\boldsymbol{\Phi}}(dx\wedge dy\wedge dz) = (\lambda^2\lambda^3 + \lambda^3\lambda^1 + \lambda^1\lambda^2)\cdot(dx\wedge dy\wedge dz) \tag{4.3.5}$$

defines a field of velocities which are referring to matter, if the classical Liouville theorem is true. That is, the variation of the volume element exterior differential ($dx\wedge dy\wedge dz$) along the Zenaida Uy's vector defined in the equation (4.3.4) is, basically, dictated by the differential form (4.3.5). Thus, this relation offers the fundamental transition between the *deformation of the background* and and the aparent *motion of the matter*. This transition is, again apparently, governed by a static Yang-Mills field [see (Uy, 1976); see also I, § 4.6]. We shall have to deepen this transition along the present work, so we stop here with the discussion that concerns its principles, trying to find its classical roots from the point of view raised through a theory of interpretation. First, let us summarize here with a conclusion: assuming that this way we are able to create an interpretative ensemble, then we can conclude that Zenaida Uy was able to describe this ensemble as a kind of Wu-Yang static Yang-Mills field; this spirit of approaching the issue should be continued at any rate. The question is: *was this spirit present in the classical physics*? The answer to this question appears to be affirmative, and we offer now some of its details.

As we said, over an interpretative ensembles, the forces acquire statistical properties, due to the fact that *the distance between ensemble particles* becomes a statistical variate. Especially *the square of distance* between particles can be deemed as a variance of this statistical variate. It is now the moment to consider this issue a little closer. Notice that in discussing it, we need an Euclidean reference frame *depending on the coordinates*, such that: (1) any *gauge length* can be represented as an Euclidean vector

$$\boldsymbol{r} = x\hat{\boldsymbol{e}}_1 + y\hat{\boldsymbol{e}}_2 + z\hat{\boldsymbol{e}}_3 \tag{4.3.6}$$

and (2) the reference frame in any point is *uniquely* associated with the *gauge length*. Assuming an Euclidean geometry, there is but a single reference frame associated with any decomposition of the *gauge length* in the sense of algebraic relation $r^2 = \sum x^2$. This reference frame could be chosen as given by the three columns of the orthonormal matrix:

$$\frac{1}{r^2}\cdot\begin{pmatrix} 2x^2 - r^2 & 2xy & 2xz \\ 2xy & 2y^2 - r^2 & 2yz \\ 2xz & 2yz & 2z^2 - r^2 \end{pmatrix} \tag{4.3.7}$$

where the three numbers (x, y, z) can be arbitrary. So, we have here a *gauge freedom*, if we may say so, because, as a gauge length, *r can vary independently*: over an interpretative ensemble it is a random variate. We extend this gauge freedom at the infrafinite scale, by assuming a certain independence between what we make out geometrically of the gauge length, during the gauging procedure, and the gauge length itself. And, as we geometrically make a vector out of the gauge length, this freedom would mean that the variation *dr* of the gauge length is somehow *independent* of the variations of the 'components' of the vector $\boldsymbol{r}$. In an Ampère generalization of the central forces (see § 3.3 above) this fact should have an overwhelming importance, so that we have to elaborate on it a little bit longer.

Suppose we consider the force *as a function of* x, y, z, *and* r, where $r^2 = x^2+y^2+z^2$. This last quantity must be regarded here as a Poincaré-type *scalar coordinate* (see § 2.5 above) while the first three are *vectorial coordinates*. There are Frenet-Serret equations for the variation of the reference frame given by the matrix (4.3.7), that can be written in the form

$$d\hat{\boldsymbol{e}}_k = \sum_j \Omega_{kj} \cdot \hat{\boldsymbol{e}}_j$$

where $\boldsymbol{\Omega}$ is a skew-symmetric matrix, having the components the solid angles:

$$\Omega_{12} \triangleq \hat{\boldsymbol{e}}_2 \cdot d\hat{\boldsymbol{e}}_1 = 2\frac{xdy - ydx}{r^2}, \quad \Omega_{23} \triangleq \hat{\boldsymbol{e}}_3 \cdot d\hat{\boldsymbol{e}}_2 = 2\frac{ydz - zdy}{r^2}, \quad \Omega_{31} \triangleq \hat{\boldsymbol{e}}_1 \cdot d\hat{\boldsymbol{e}}_3 = 2\frac{zdx - xdz}{r^2} \tag{4.3.8}$$

Thus, the entries of the Frenet-Serret matrix can be assimilated as the components of the *solid angle* in a spherical representation. Just for the record, and also for incidental later purposes, using the spherical polar angles, we have the following entries:

$$\begin{aligned} \Omega_{12} &= 2\sin^2\theta d\varphi, \\ \Omega_{23} &= 2\sin\varphi d\theta + \sin 2\theta \cos\varphi d\varphi \\ \Omega_{31} &= -2\cos\varphi d\theta + \sin 2\theta \sin\varphi d\varphi \end{aligned} \tag{4.3.9}$$

Now, we need to calculate the differential of a certain vector in this reference frame, just as it is done before in differential geometry, for the reference frame related to spherical coordinates.

Then, let us consider the vector force, whose components with respect to the reference frame (4.3.7) are taken as *contravariant components*, just for convenience we should say, in view of the position of indices of the unit vectors of the frame:

$$\boldsymbol{f}(\boldsymbol{r}) \triangleq f^k(\boldsymbol{r})\hat{\boldsymbol{e}}_k \tag{4.3.10}$$

The differential of this vector can be written as

$$d\boldsymbol{f}(\boldsymbol{r}) = Df^k(\boldsymbol{r})\hat{\boldsymbol{e}}_k, \quad Df^k \equiv df^k + f^j\Omega_j^k \tag{4.3.11}$$

where the summation convention over dummy indices of different variances enters the game. This, as it can be seen on our formula, needed a sudden change in the position of one of the indices of the entries of matrix $\boldsymbol{\Omega}$, with the following explanation: if we have to use a general formalism for this geometry, then we need to forget about the Cartesian notation, and adopt an indicial one. According to this idea, a position vector can be written as in the equation (4.3.10), just like any vector, so that the coordinates are to be considered, even if just conventionally for now, as its contravariant components. Thus, we have a formula (4.3.6) entirely analogous to (4.3.10):

$$\boldsymbol{r} \triangleq x^k\hat{\boldsymbol{e}}_k \tag{4.3.6}$$

and therefore, the corresponding equation (4.3.11) sounds like:

$$d\boldsymbol{r} = (Dx^k)\hat{\boldsymbol{e}}_k, \quad Dx^k \equiv dx^k + x^j \Omega_j^k \tag{4.3.12}$$

Calculating the components of this vector, based on equation (4.3.8) results in

$$Dx^k \equiv -dx^k + 2x^k \frac{dr}{r} \tag{4.3.13}$$

but – and this is very important here – with the observance of lower indices rule, and the following consideration of the entries of the matrix $\boldsymbol{\Omega}$:

$$\Omega_j^k \equiv 2\frac{x^k dx_j - x_j dx^k}{r^2} \tag{4.3.14}$$

As a *mnemonic* rule, therefore: for each position, any pair of the two kinds of coordinates (x^k, x_j) can be interpreted as a pair of coordinates on a surface, say a *de Broglie surface*, at the finite scale. Corespondingly, at the infrafinite scale, we have the pair of differentials (dx^k, dx_j). Then the equation (4.3.14) represents the dot product of the finite pair of coordinates thus defined, with the infrafinite pair (dx_j, $-dx^k$), which can be written as an expression involving the fundamental skew-symmetric two-dimensional matrix:

$$\Omega_j^k = \frac{2}{r^2}\begin{pmatrix} x^k & x_j \end{pmatrix}\cdot\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\cdot\begin{pmatrix} dx^k \\ dx_j \end{pmatrix} \tag{4.3.15}$$

The upper and lower indices of this second order tensor, are the indices of the two coordinates in the pair. The problem of definition of the covariant components remains in suspension for now, because, in general, a metric may not be available, in order to follow the classical route.

Now, if the differential forms (4.3.13) are the fundamental differential forms in this world, then the *covariant* components of a force vector, which are known to be generally defined by the elementary mechanical work, are specifically defined by the differential form:

$$DW = f_k(\boldsymbol{r})(Dx^k) \tag{4.3.16}$$

The notation DW is here intended to signify that, just like in the case of coordinates, the elementary displacement is, in general, not an exact differential vector, but acquires terms due to the variation of the scalar coordinate r, which is a statistical variable. Consequently, in view of (4.3.13), this differential form can be expanded into:

$$DW = -f_k(\boldsymbol{r})dx^k + 2[x^k f_k(\boldsymbol{r})]\frac{dr}{r} \tag{4.3.17}$$

which shows that this differential form is not an exact differential even if the covariant force vector is conservative, as in the classical case. Indeed, assuming that the covariant force derives from a potential depending only on the coordinates x^k, but not on the sum of their squares, we can write

$$f_k(\boldsymbol{r}) = \frac{\partial U(x^1,x^2,x^3)}{\partial x^k} \quad \therefore \quad DW = -dU + 2\left(x^k \frac{\partial U}{\partial x^k}\right)\frac{dr}{r} \tag{4.3.18}$$

showing clearly in what conditions *DW* is an exact differential, by comparison with the classical case. That is, everything depends on the *projection of the conservative force along the position vector*. In particular, if the force *acts perpendicularly to the position vector*, the potential is a homogeneous function of degree zero, and we have the case of Ampère's forces, decribed by us previously.

The coefficient of the differential $d(\ln r)$ from equation (4.3.17) is the celebrated *virial of forces*, introduced to theoretical physics by Rudolf Clausius in his attempt of connecting the fields of forces, within the molecular ensembles, with thermodynamics, *via statistics* [(Clausius, 1870); the two articles to be found under this heading in the specialty literature are practically the same regarding their content. However, the French version of the work is particularly suggestive as to what virial means along the lines of the present section of our work: "*a quantity analogous to potential*"]. That attempt was, in fact, one of the classical cases of interpretation, where, like in the later Lorentz case for electricity, the continuum to be interpreted is conspicuously missing. And here is the virial, again, appearing naturally we should say, and most significantly, in close connection with a 'Maxwell continuum', if we may, as described above, in a problem of interpretation indeed – *Thomson's interpretation*.

In the general case, represented by equation (4.3.17), with no reference whatsoever to potential, one can merely say that the differential form *DW* can only be *closed* if its exterior differential is zero: $d\wedge(DW) = 0$: the existence of a potential – which requires that DW should be an *exact* differential – is, in most cases, too much to ask. In quite general conditions, this requirement comes down to

$$df_k(\boldsymbol{r}) \wedge Dx^k = 0 \qquad (4.3.19)$$

whence, according to *Cartan's Lemma 1*, in view of equation (4.3.17) we need to have

$$df_k(\boldsymbol{r}) = \Lambda_{kj} dx^j, \quad x^k df_k(\boldsymbol{r}) = \Lambda \frac{dr}{r} \qquad (4.3.20)$$

for some conveniently chosen symmetric matrix $\boldsymbol{\Lambda}$ and function Λ. Obviously, these two equations cannot be quite independent of each other, for we must have the constraint:

$$\Lambda_{kj} x^k dx^j = \Lambda \frac{dr}{r} \qquad (4.3.21)$$

Just to settle our ideas, choosing $\boldsymbol{\Lambda}$ as a matrix of constants, and Λ as a constant, will give r^2 as a Gaussian, and if the matrix $\boldsymbol{\Lambda}$ is positively defined and the constant Λ is positive, we shall have

$$\frac{1}{r^2} \propto \exp\left(-\frac{\Lambda_{kj}}{\Lambda} x^k x^j \right) \qquad (4.3.22)$$

thus giving us a proper Gaussian distribution to be used, for instance, in the Berry-Klein gauging procedure (Berry & Klein, 1984), or even downright as a density of probability in the spirit of the Jeans' observations above. In general therefore, the equation (4.3.21) suggests a manner of lowering the indices offered by the matrix $\boldsymbol{\Lambda}$, which thus plays the part of a metric tensor. But, before launching ourselves on the way of mathematical speculations, let us read meanings into some facts related to the Planck's procedure of quantization.

There are reasons to believe that in the interpretation of matter in general, the absolute temperature is no more a *sufficient statistic* as taken in the classical case of the ideal gas [for a detailed story of the facts leading to this conclusion, see (Mazilu, Agop, & Mercheș, 2021), the discussion in Chapter 2]. This fact is also obvious in Thomson's interpretative theory, as we noticed before. The idea is that, in view of the Planck quantization procedure, which turns out to be a necessity, some forces *must exist* inside any material physical structure – even ideal gases – *whose virial is constant*. Therefore, no interpretation is apparently possible, since no ensembles of free particles exist. All we can hope within a Planck quantization philosophy, is the existence of a gauging based on a 'partial freedom', as it were, like the Berry-Klein gauging procedure, which asks only for a kind of… 'radial freedom'. As it happens, the condition of constant virial for such forces, means exactly that. Indeed, assume the *physical* situation where we calculate the virial in a *fixed* reference frame, in a coordinate space. If this quantity is constant in the space described with this reference frame, we must have:

$$f_k(\boldsymbol{r})dx^k + x^k df_k(\boldsymbol{r}) = 0 \quad \therefore \quad dW + \Lambda\frac{dr}{r} = 0 \tag{4.3.23}$$

Here we have used the definition of the elementary mechanical work, and the second of the conditions (4.3.20) which is a necessary condition that the differential form (4.3.17) should be closed. Under the constant virial, *DW* should therefore read

$$DW = K\frac{dr}{r}, \quad K \equiv \Lambda + 2C \tag{4.3.24}$$

where *C* is the constant value of the virial. Therefore, the forces of constant virial should be *logarithmic* forces having the magnitude that goes inversely with the distance. If these forces are conservative and central too, they *do not have radial component* in any reference frame whatsoever: any particle under these forces is 'radially free'. This means that its motion towards any other particle from the interpretative ensemble is the motion of a free particle, and therefore the Berry-Klein gauging procedure works. Let us prove this.

Any central conservative force can be described by an equation like

$$\nabla U(\boldsymbol{r}) = f(\boldsymbol{r})\frac{\boldsymbol{r}}{r} \quad \therefore \quad (\nabla U)^2 = [f(\boldsymbol{r})]^2 \tag{4.3.25}$$

Here $f(\boldsymbol{r})$ is the magnitude of forces, while $U(\boldsymbol{r})$ is a potential function. The centrality of forces cannot be declared but in a fixed reference frame, as it is the case with the Newtonian universe, which, because of this property, fell under the Einstein's critique that installed the general relativity. In such a reference frame, the equation (4.3.25) can be solved in a 'spherical' coordinate system. Using adequate notations, this last equation looks like

$$\left(\frac{\partial U}{\partial r}\right)^2+\frac{1}{r^2}\left[\left(\frac{\partial U}{\partial \theta}\right)^2+\frac{1}{\sin^2\theta}\left(\frac{\partial U}{\partial \varphi}\right)^2\right]=[f(\boldsymbol{r})]^2 \tag{4.3.26}$$

which is the expression of the magnitude of forces having the spherical components:

$$f_r(\boldsymbol{r})=\frac{\partial U}{\partial r},\quad f_\theta(\boldsymbol{r})=\frac{1}{r}\frac{\partial U}{\partial \theta},\quad f_\varphi(\boldsymbol{r})=\frac{1}{r\sin\theta}\frac{\partial U}{\partial \varphi}$$

Now, the equation (4.3.26) can be solved by sum-separation method, much in the manner of solving the classical Hamilton-Jacobi equation. Provided, of course, *the magnitude* of force depends exclusively on the distance between particles, as in the case envisioned by Newton for the inverse quadratic forces. In this case we can write (4.3.26) in the form

$$r^2\left[\left(\frac{\partial U}{\partial r}\right)^2-f^2(r)\right]=-\left[\left(\frac{\partial U}{\partial \theta}\right)^2+\frac{1}{\sin^2\theta}\left(\frac{\partial U}{\partial \varphi}\right)^2\right] \tag{4.3.27}$$

and, therefore, we can assume, tentatively, a solution of the form

$$U(r,\theta,\varphi)=R(r)+F(\theta,\varphi)\quad\therefore\quad f_r(\boldsymbol{r})\equiv R'(r) \tag{4.3.28}$$

where the prime means differentiation with respect to the unique independent variable, as usual. The equation (4.3.27) can have such a solution if, and only if

$$r^2\left\{[R'(r)]^2-f^2(r)\right\}=-\left[\left(\frac{\partial F}{\partial \theta}\right)^2+\frac{1}{\sin^2\theta}\left(\frac{\partial F}{\partial \varphi}\right)^2\right]\equiv-\beta^2 \tag{4.3.29}$$

where β is a real constant. Thus we must have

$$r^2\left\{[R'(r)]^2-f^2(r)\right\}=-\beta^2,\quad \left[\left(\frac{\partial F}{\partial \theta}\right)^2+\frac{1}{\sin^2\theta}\left(\frac{\partial F}{\partial \varphi}\right)^2\right]\equiv\beta^2 \tag{4.3.30}$$

The conclusion to be drawn here, is that for *a field of central forces* having the magnitude that goes as the inverse distance between particles the forces *can have no radial component* if

$$f(r)=\pm\frac{\beta}{r}\quad\therefore\quad f_r(r)\equiv R'(r)=0 \tag{4.3.31}$$

The particles in such a field are 'radially free' with respect to each other, and the Berry-Klein gauging procedure may be applied. This property defines free particles, indeed, but *not from classical point of view*, inasmuch as, the two remaining components of the force are, possibly, nonzero. Indeed, the other components of these forces can be further calculated in exactly the same manner of separation of the variables. That is, trying for the second of the equations from (4.3.30) a solution of the form:

$$F(\theta,\varphi)=\Theta(\theta)+\Phi(\varphi)\quad\therefore\quad f_\theta=r^{-1}\Theta'(\theta)\,\&\,f_\varphi=(r\sin\theta)^{-1}\Phi'(\varphi)$$

we have only Thomson forces 'acting sideways':

$$f_\theta(\boldsymbol{r}) = \pm\frac{1}{r}\sqrt{\beta^2 - \frac{\gamma^2}{\sin^2\theta}}, \quad f_\varphi(\boldsymbol{r}) = \pm\frac{\gamma}{r\sin\theta} \tag{4.3.32}$$

Now, we may not know, at this point, too much about the covariant coordinates of position, but certainly we can define some contravariant coordinates in a reference frame defined by equation (4.3.7). We have here the possibility of defining the contravariant coordinates using the *physical condition of constant virial*, if we make out of a Berry-Klein gauge length a geometrical position according to prescription suggested by the matrix (4.3.7), whereby the vector (4.3.6) that has the form

$$\boldsymbol{r} = (\alpha r)\hat{\boldsymbol{e}}_r \pm \alpha r\sqrt{\beta^2 - \frac{\gamma^2}{\sin^2\theta}}\hat{\boldsymbol{e}}_\theta \pm \frac{\alpha r\gamma}{\sin\theta}\hat{\boldsymbol{e}}_\varphi \tag{4.3.33}$$

with α is a constant. The corresponding virial of these forces is, indeed, a constant

$$\boldsymbol{f}\cdot\boldsymbol{r} = \alpha(1+\beta^2) \tag{4.3.34}$$

so that their elementary mechanical work is all invested in the 'sideways displacements'

$$d\boldsymbol{r} = \pm\alpha r\left\{\sqrt{\beta^2 - \frac{\gamma^2}{\sin^2\theta}}\left(\frac{dr}{r} + \frac{\gamma^2\tan\theta d\theta}{\beta^2\sin^2\theta - \gamma^2}\right)\hat{\boldsymbol{e}}_\theta + \frac{\gamma}{\sin\theta}\left(\frac{dr}{r} - \tan\theta d\theta\right)\hat{\boldsymbol{e}}_\varphi\right\} \tag{4.3.35}$$

and amounts to

$$\boldsymbol{f}\cdot d\boldsymbol{r} = \alpha\beta^2\frac{dr}{r} \tag{4.3.36}$$

which, of course, is an exact differential. The displacement (4.3.35) can count as Thomson's 'sideways motion', so that Thomson's general theory has no cracks… Provided, of course, we can say something about those central forces with magnitude going with the inverse length, outside the idea of a fixed reference frame which is the week point of the Newtonian cosmology. And we cannot say but exactly what Joseph James Thomson himself once said [it appears to us as quite instructive to see also the important works of the great teacher, in connection with the structure of light (Thomson, 1920, 1924)].

And one can say, indeed! Classically speaking, the mathematics of a concept of force generated by matter found its significant expression into Poisson equation, which we rewrite here in the form:

$$\nabla^2 U(x,y,z) = 4\pi\rho(x,y,z) \tag{4.3.37}$$

Here $U(x, y, z)$ is the potential of the forces in the medium of density $\rho(x, y, z)$. Originally, that is, to Siméon Denis Poisson himself (Poisson, 1812), this density was the Newtonian density, and his conception instituted the idea that 'the matter of density $\rho(x, y, z)$ generates the (potential of) forces', as in equation (4.3.37). However, along with permeation of the electric forces in our awareness as Coulombian forces, this conception started changing. First, if the medium is electrically active, then today, in the field theory, we conceive that ρ is the *numerical* density of electricity rather than a Newtonian density. The change in conception goes even further, and came, naturally we should say, with the Maxwellian theory of electrodynamics, a theory obviously in need of some change in the old concepts, primarily in those regarding the force. It was, apparently, James Clerk Maxwell the first one who took equation (4.3.37) as *defining the density* of the medium, rather than the potential, for the following *good reason* taken from the mechanics of continua: he proved that the equation of equilibrium of a *system of stresses* in a continuum, is satisfied with the volumetric forces corresponding to a density of this continuum given by (4.3.37). Let us show this briefly.

The equation of equilibrium of a *continuous stress system* in general which, in its simplest form, asserts that the divergence of the *stress tensor*, ***t*** say, is balanced by the density of *static volume forces*, ***f*** say, written in the form (Love, 1944):

$$\nabla \cdot \boldsymbol{t} + \boldsymbol{f} = \boldsymbol{0} \tag{4.3.38}$$

When specifically applied to the stress tensor ***t*** defined by the matrix

$$t_{ij} \overset{\text{def}}{=} \frac{1}{8\pi}\{2(\nabla_i U)(\nabla_j U) - \delta_{ij}(\boldsymbol{\nabla} U)^2\} \tag{4.3.39}$$

the equilibrium equation (4.3.38) is identically satisfied for a *force density **f*** defined by

$$\boldsymbol{f} \overset{\text{def}}{=} \frac{1}{4\pi}(\nabla^2 U) \cdot \boldsymbol{\nabla} U \tag{4.3.40}$$

In other words, this stress system is *statically equivalent* with the system of volume forces of the matter having a density $(\nabla^2 U)/4\pi$, as given by Poisson's equation. As we have shown, the Thomson's condition of 'motion sideways' can be realized dynamically by a central force of magnitude given in equation (4.3.31). This is what we have called a *logarithmic force*, inasmuch as it derives from a logarithmic potential:

$$U(\boldsymbol{r}) = \pm\beta \cdot \ln r \tag{4.3.41}$$

Assume that the Maxwell view is applicable here, and we want to use this view in order to calculate the general continuous stress field, and the system of forces statically equivalent with these stresses. What we need to calculate is, first, the gradient of (4.3.41), then the Laplacian, and finally to plug the results into equations (4.3.39) and (4.3.40). The results are as follows: first we have

$$\nabla U(r) = \pm\beta \cdot \frac{\boldsymbol{r}}{r^2}, \quad \nabla^2 U(r) = \pm\beta \cdot \frac{1}{r^2} \tag{4.3.42}$$

Then the force field statically equivalent with Maxwell stress system is exactly the *Thomson universal force field*

$$\boldsymbol{f}(\boldsymbol{r}) = -\frac{\beta^2}{r^3}\hat{\boldsymbol{r}} \tag{4.3.43}$$

while the stress system itself is given by the matrix from equation (4.3.7) in the form

$$\frac{\beta^2}{8\pi r^2}\begin{vmatrix} \hat{\boldsymbol{e}}_1 & \hat{\boldsymbol{e}}_2 & \hat{\boldsymbol{e}}_3 \end{vmatrix} \tag{4.3.44}$$

the unit vectors being the columns of the matrix (4.3.7). This means that the equilibrium of the interpretative ensembles for such a continuum are described by equation (4.3.38) with the static forces (4.3.43) balancing the stresses (4.3.44), by definition *omnipresent in a continuum*.

In conclusion, let us notice the closing phrase of the excerpt above from Thomson: the 'electric-doublets of constant moment' are essential in the description of the forces (4.3.43). Keeping in mind the statistical nature of these forces, the *dipoles* enter just naturally in the physical theory here, giving a natural-philosophical reason to Planck's approach to quantization [see (Mazilu, 2024), § 1.2]. And, if this statistical theory of forces is described by a static Yang-Mills electrodynamics, for instance in the form of Zenaida Uy's theory recalled by us in the beginning of the present section, then the atomic nucleus has, indeed, its analog in the cellular nucleus. In view of this observation, we can further venture to say that the 'double-helix ladder' of the genetic matter of the world should be described as a 'Planckian property' of the matter in any universe. The real problem in an incidental analogy with the atomic nucleus, would actually be *the similar property of the planetary atom*.

However, within the natural philosophy of J. J. Thomson, the planetary structure that the 'molecule expose' during collision encounters with negative particles has a clear explanation, destined to make the planetary model cope with the Planck quantization procedure. First, we have to recall that a molecule 'is made out of atoms', which, in a collision encounter of the kind described by Thomson, means that the molecule exposes an atom to the incoming negative particle. Then, we need to recall the Thomson's prescription from the end of the excerpt from the opening of this section: central forces going inversely with the third power of distance 'would be exerted by electric-doublets of constant moment with their negative ends pointing to the corpuscles'. In other words, the atomic electron should be a 'negative end of an electric-doublet' points toward the incoming negative particle. It is in these conditions that, in the expression of Thomson: 'the time of collision of a corpuscle moving with a given speed must be constant and independent of the nature of the molecule against which the corpuscle collides'.

4.4. An Update of the Poincaré-Lorentz Theory of Charge

Let us insist a little longer on the idea of a resonator, in order to offer a truth to the classical concept of electricity, as it appears to Poincaré and especially to Lorentz. For once, such a structure can be defined purely geometrically, being necessarily not a physical structure. However, at some point in the course of using this definition for physical purposes, one has to take due notice of the fact that it actually adds up to the idea that *the motion of a reference frame* in a charge sea *creates* pairs of charges, positive and negative, exactly as envisaged by Lorentz himself (Lorentz, 1892). From our perspective, the Lorentz contention means this: the matter is *incidentally* neutral from an electrical point of view – *i.e.* it is vacuum – however, this kind of neutrality is effective only on surfaces. Outside these surfaces, and in their immediate neighborhood, the charges are in a *de Broglie region*, 'moving at constant time along a ray', as de Broglie once said, therefore across the neutral surface in general (de Broglie, 1926 b). In order to illustrate our point of view, it is better to refer our reasoning to the Lorentz's own works, and to the connection of his ideas with some classical works on electricity, nowadays pretty much neglected as obsolete in theoretical physics, in spite of their overwhelming conceptual importance, obvious at a deeper consideration of matters. In order to get a better grip on the subject, let us follow these ideas along with the *classical definition of electricity*.

The experience shows that the matter can carry charges, just the way it carries mass. However in such a case the image of ether, as interpreted by Samuel Earnshaw [see (Mazilu, Agop, & Mercheș, 2021), Chapter 1] seems impossible: in the ether we manifestly have waves not particles. Electromagnetic waves, it is true, but still waves. It is on this occasion, that one can conclude, borrowing the later words of C. G. Darwin, that *the ether is a continuum which needs to be interpreted*. The hard part of this interpretation is that the ether appears as electrically neutral, and no one could see how the wave concept could be reconciled with the experimental idea of charge. It is at this point that Lorentz enters the stage with an idea of incidental electric neutrality. Quoting:

> If, after *arbitrary movements*, the matter is reduced to its *primary configuration*, and if, during these movements, *every element of a surface* which is *steadfastly attached to the matter* was crossed by *equal quantities of electricity in opposite directions*, all of the points of system will be found in their *primary positions* [(Lorentz, 1892), §57; *our translation and emphasis*, *a*/*n*]

Notice, first, that this hypothesis already assumes that an interpretation is in place, for otherwise one cannot describe a *surface* 'attached to *matter*': the two – surface and matter – need to have common *points*, or even regions, in order to be 'attached'. On the other hand, if one takes the 'element of surface steadfastly attached to matter' as referring to an infinitesimal portion of a 'wave surface' for instance, the situation suggested by Lorentz in this excerpt is, indeed, the one envisioned by Louis de Broglie in his condition mentioned above, that we once found 'strange', where the motion along the ray is instantaneous [see (Mazilu, 2020), §2.1]. In view of this, we venture to assume that 'configuration' in the above excerpt means *an ensemble of classical material points*, so that when Lorentz says that an 'element of surface is attached to matter', we have o understand that this element of surface is determined by the positions of at least one material point, playing the part of *chosen positions* on describing a surface. This may be the origin of an hypothesis of Maxwell that attracted the attention of Lorentz:

> By virtue of the *constraints existing in the system*, the positions of the points P at a later instant *t* are *entirely determined once we know the new positions of the circuits* and, for each of them, *the quantity of electricity which*, between the moments t_0 and *t*, *crossed a section* [(Lorentz, 1892), §2; *emphasis added*, *a*/*n*].

Obviously, this is the kind of assumption that also attracted the Poincaré's observation on the character of Maxwell electrodynamics of being an implicit two-fluid theory (see § 2.7 above): *the deformation and inertial motion in conductors are considered by Maxwell on equal footing*, which, obviously, is not the case. However, let us continue here with the line of reasoning of Lorentz himself, for it contains an interesting point of view that meets the Poincaré's own observation.

First, Lorentz finds that the assumption above *is not always satisfied* – we should add: *within the framework of Earnshaw interpretation* – and by now we can even tell why, according to his own findings: inasmuch as the 'circuits' are involved, it becomes a problem of *transport theory*. Indeed, there is a discrepancy here, between the time derivative, and substantial derivative involved in the transport of energy [see (Lorentz, 1892), pp. 423 – 424, §66]. However, Lorentz does not see in this a reason not to go any further with the fluid model for charge, and this shows us just to what extent was he going with the fluid as a model in the interpretation problem. Quoting, indeed:

> If this hypothesis *cannot be admitted in the case of an ordinary fluid*, it *could not be applied to the electric fluid either*. However, this fact does not prevent our equations of motion from being accurate. Indeed, *the mass of this last fluid was supposed to be negligible*, and in calculating the variation δT (*kinetic energy*, *n/a*) only that kinetic energy was considered *which is specific to the electromagnetic movements*; it will suffice therefore that the material points liable of these motions, and *which are not to be confused with the electricity itself*, enjoy the property of returning to the same positions *if for each surface element the algebraic sum of the quantities of electricity* by which it has been crossed, is *0*.
>
> Now, one is entirely free to try on the mechanism that produces the electromagnetic phenomena *any convenient assumption*, and while recognizing the difficulty of *imagining a mechanism that possesses the desired property*, it seems to me that we do not have the right to deny its possibility. [(Lorentz, 1892), §67; *our translation and Italics*]

Notice, incidentally, the important distinction of Lorentz, namely that the material points – the classical 'bodies' of dynamics – 'liable of motion' are 'not to be confused with electricity itself', a distinction which, we may say, being of the nature of that observed by Poincaré, brings forward yet another observation once made by him, about the impossibility of action of matter upon ether (Poincaré, 1900). Also notice that the Lorentz matter thus interpreted, is the *counterpart of the physical universe at large*. Indeed, here we have to assume that 'the mass of *electric* fluid is supposed to be negligible', since the Coulomb forces dominate, while in a regular cosmology based on the general relativistic ideas, it is the charge that is 'supposed to be negligible', for the gravitation forces dominate. We are talking here of static forces, so that, if we take the Newtonian point of view on them, the gravitational mass is understood of being 'negligible'.

The concept of Lorentz matter, therefore, speaks of a universe where the charge is *force-wise dominant*, for the mass, obviously, cannot be negligible in the sense that it is missing: the *dynamics* knows nothing of the concept of *zero mass*, be it inertial or gravitational. Fact is that we cannot 'dismiss the mass' in the construction of a physical theory of the universe, at least not the way we do it nowadays with the charge in the case of physical universe at large. As for the rest of the excerpt above, the most important thing, namely 'that mechanism… possessing the desired property' from the last sentence, was not to be 'assumed' anymore. For, just about the period of time we are talking here, it was *physically accomplished in the form of the field generated via a periodic charge motion* by Heinrich Hertz [see, for instance, the English translations collected in (Hertz, 1893), for the fundamental works which instituted the modern theory of electromagnetic field]. The essential observation is that the physical form generating fields was a dipole, and this is the concept that led to the idea of resonator to Planck.

Continuing our discussion, though, let us notice that Lorentz insisted at length in making the point clear that the interpretation of the electric matter is not a trivial thing. In order to clarify the essential idea, we think it is worth citing again the Lorentz's own words: in characterizing the matter structure by an interpretation based upon the existence of electricity, these words constitute the crowning point of a long ascending path followed by electricity theory starting from the times of Ampère. Quoting, therefore:

> Here is now a system of hypotheses that give the value 0 for this variation (*of the kinetic energy*, *entering the extremum principle of mechanics*, *n*/*a*):
>
> a. There are *two systems of particles participating in electromagnetic motions*, systems that will be indicated by the letters N and N′.
>
> b. Any time *a certain particle* pertaining to one of these systems, *is to be found in the immediate vicinity of a particle of equal mass pertaining to the other system*.
>
> c. The two systems always *have equal movements inversely oriented* or, stating it more exactly:
>
> If two *movements of the same duration* start with *the same initial positions* and do not differ but by the sign of the components of the electric current, and if *P* and *P*′ are points pertaining to systems N and N′ that *coincide in the initial configuration*, the point *P*′ *will reach, in the second mouvement, the same final position the point P reaches in the first movement*.
>
> This obviously implies that at the time of coincidence the points *P* and *P*′ have equal and opposite velocities. Indeed, changing the signs (*of the components of current*, *a*/*n*) will reverse the velocity of the point *P*; but, according to the last hypothesis, this velocity must then become equal to that which the point *P*′ had previously.
>
> Notice again that, in the course of a certain movement, a new particle *P*′ will coincide with a given particle *P*. Two juxtaposed wheels, having equal and opposed rotations of the same axis, may serve as an example. [(Lorentz, 1892), §69; *our translation and emphasis*, *a*/*n*]

We think that with these excerpts from Lorentz, the purpose is served in illustrating the role of the concept of surface in a comprehensive case of interpretation. The continuous Lorentz matter has all of the classically known physical qualities liable to generate forces, according to classical natural philosophy, except one: the *connection between charge and deformation*, which is essential in the natural philosophy of Poincaré, and seems to fit perfectly in the concept of de Sitter continuous background. The qualities able to generate forces, as well known, are the gravitational mass and the charges, electric and magnetic. One can say that with the considerations from the excerpt dight above, regarding the way of adding charges to the concept of interpretation, Lorentz has in store for us one of those physical instances that may have to be imagined by us, in order to make the Louis de Broglie's case: *following a ray in approaching a particle at constant time*. Along with this, however, Lorentz realized what we would like to call a *way of connection between charge and deformation*, along the initial idea of Poincaré (see § 2.7 above). This is the main theme of what follows from this work.

One can take the Lorentz's definition of the electric matter as a testimony of the physical fact that the electricity is not directly connected to motion in the sense of a point particle. Rather, one can say that the motion of a particle through de Sitter background creates Planck resonators, as in the case of Kepler motion. This fundamental concept cannot be, according to the Newtonian view, but only naturally correlated with the planetary atom, in the sense only explained by J. J. Thomson, as we have shown in the previous section of this chapter.

In this connection, it serves our purpose here, recalling some steps of the historical path of an electricity theory, for which the Lorentz approach seems to be the crowning point. This line of reasoning is destined to explain the electric neutrality of interacting material conductors of electricity. It started with Riemann, who, in 1858 tried to accomplish the Gauss' idea of interaction of currents (Gauss, 1833, 1845), by introducing what later came to be known as the Klein-Gordon equation [see (Riemann, 1867b); there are also a few modern translations of this inciting work of Riemann]. Since Riemann used the concept of *retarded mass*, that perhaps appeared as highly speculative at the time, Enrico Betti stepped into argument, with an idea of *cycles of electricity* along a conductor traversed by a current (Betti, 1868). Betti's idea, apparently based on the concept of Fourier series, was criticized by Rudolf Clausius, on mathematical grounds (Clausius, 1868), and the Riemann's line of thought in electrodynamics remained at this level until Lorentz's work has emerged. Lorentz's ideas were undertaken by Einstein, however not along the Riemann-Betti line of reasoning, but along the Maxwell's line, thus leading to special relativity [see (Mazilu, 2020), Chapter 5, §§5.3 and 5.4].

4.5. The Kind of Charge Along Brain-Universe Rays

And now, after this brief review of the status of the classical natural philosophy regarding electricity, let us add to it whatever the phenomenology of brain may be able to bring in. One can see, in the above quotations from Lorentz, especially in the last one, that the interpretation of charge is pending on 'systems of particles participating in electromagnetic motions': any such particle must have a 'motion' associated, pretty much in the sense conveyed by Thomas Gold statement to the effect that the charge must have some 'handedness' (see § 2.3 above). Lorentz may have sensed that this 'handedness' has nothing to do with the dynamics, but did not realize the essence of the Poincaré's idea of the connection between the concept of deformation and the charge. Perhaps, a better explanation of this state of the case would be the vagueness of the geometrical explanation of Poincaré, which can be quite unconvincing for a physicist bent on subtle phenomenological details. This state of the case might be, indeed, able to explain the interpretative details offered by Lorentz in the last of the excerpts above, that prefigurate the later Dirac's idea of magnetic monopole (Dirac, 1931), and Feynman's association of the charge with the arrow of time in motion (Feynman, 1949), among others.

Fact is, that in the phenomenology of charge, an urge was always present, to assume some extra *information* on its definition which was not of the dynamical nature. The notorious case is the one regarding the action of forces at different scales: the forces are not asssociated to the bodies *per se* – except, perhaps, in the close encounters of the nature of a collision – but to their physical qualities, basically mass, and the two charges. They act at a distance with different 'intensities', which depend on the scales of space and time. An information had, therefore, to be added to the definition of forces, in order to 'sanction' this situation, and thus the Newtonian theory of static forces has appeared, with its progenies – the Coulombian forces. At this level, the information was referring to the quantitative characterization of those qualities generating the force, and thus an idea of quantization was present, in a form that prefigurated the Plancks procedure. We shall insist on the different aspects of this theory along this work, but for now, let us see what the phenomenology of brain has to add to the mathematics that serves in unfolding its physics.

The action on a charge, y say, is defined in the cryptographic system above as the *homographic action* of a 2×2 matrix, $\boldsymbol{A}$ say. One essential condition that such a matrix must satisfy is the following:

$$\boldsymbol{A}(\boldsymbol{A}(y)) = y, \quad \boldsymbol{A}(y) \stackrel{def}{=} \frac{\alpha y + \beta}{\gamma y + \delta} \qquad (4.5.1)$$

Here $\boldsymbol{A}(y)$ must, in our opinion, be taken as *that* '*handedness*'*, or the* '*information*' *associated to the charge*. The reason of this equation can be explained, first of all, by the necessities of interpretation: within Feynman's philosophy of interpretation, for instance – to take the modern physical theory which is the closest one to the Hertzian natural philosophy – a charge indicates a location over the space of an instanton. Feynman's interpretation can, therefore, be taken as quite natural if the charge is defined as an intensive magnitude in the sense of John Stroud (see § 2.4 above):

> An intensive dimension is a dimension which can have *any values over its range* – but *only one value at any one point in the space of extension* at which it appears.

Then, the equation (4.5.1) can be read by the following scenario: assume that the matrix $\boldsymbol{A}$ with entries as in equation (3.5.5), for instance, should be the one characterizing the arena covered by the charge. The charge 'indicates' a place in that arena, by a certain coordinate x, representing the 'information' needed in order to characterize its action, which, in view of the Planck's procedure of quantization – the brain is, after all, a universe – can be calculated from the charge itself, by an equation like

$$\boldsymbol{A}(y) = x \tag{4.5.2}$$

This means that the charge itself can be recovered from the 'information' x by the very same matrix:

$$\boldsymbol{A}(x) = y \tag{4.5.3}$$

as the Lorentz's description of the charge seems to imply. This, in our opinion, is the best way to declare that the matrix $\boldsymbol{A}$ is, for instance, 'the mark of association in the brain'. On the other hand, the equation (4.5.3) can be taken as the physical basis of the description of the mind, as Wilder Penfield defined it: an instantaneously established association of the information with the charge, in order to bring the brain in a state of physical working. The association itself involves the idea of a reference frame, for it can be done only by a kind of Shpilker's procedure generalizing the concept of propagation (see I, §5.5).

Continuing, however, only with the algebra involved in equation (4.5.1), when this equation is expanded according to definitions of $\boldsymbol{A}$ and $\boldsymbol{A}(x)$, it provides a condition involving the entries of the matrix and the charge y itself. The same condition must be true for the 'handedness' itself, taken as primary variable instead of the charge, so that we feel compelled to write that condition as:

$$(\alpha+\delta)[\gamma x^2 + (\delta-\alpha)x - \beta] = 0 \tag{4.5.4}$$

So, the only case satisfying the physical condition of reproducing the charge when the 'handedness', or 'information' is transformed back, after the transformation of charge into 'handedness', or *vice versa*, takes place, no matter of the value of this charge, is characterized through a condition to be satisfied by the matrix itself:

$$\alpha+\delta = 0 \tag{4.5.5}$$

This means that the matrix must be traceless, *i.e.*, it should represent an involution. If this condition is not satisfied, then (4.5.4) is valid *only* for the particular cases when the charge y, or the 'handedness' x assumes the values given by the roots of the quadratic equation:

$$\gamma x^2 + (\delta-\alpha)x - \beta = 0 \tag{4.5.6}$$

In the first case, the Hamilton-Cayley equation for $\boldsymbol{A}$ asks that the square of the matrix should be proportional to the identity 2×2 matrix. Indeed, the Hamilton-Cayley equation is:

$$\boldsymbol{A}^2 - (\alpha+\delta)\cdot\boldsymbol{A} + \det(\boldsymbol{A})\cdot\boldsymbol{1} = \boldsymbol{0} \xrightarrow{\alpha+\delta=0} \boldsymbol{A}^2 = -\det(\boldsymbol{A})\cdot\boldsymbol{1} \tag{4.5.7}$$

where the identity matrix $\boldsymbol{1}$ must be considered as the zero power $\boldsymbol{A}^0$, and the matrix $\boldsymbol{A}$ itself is, naturally, $\boldsymbol{A}^1$. The matrix $\boldsymbol{A}$ must be of the second form given in equation (3.5.14), and can be taken as the 'mark of association in the brain'. Since the homographic action of a 2×2 matrix is only defined by its entries up to a common factor – in equation (4.5.1), for instance, only three entries of the matrix are essential in the definition of the action – (4.5.7) implies the first equality from (4.5.1). However, if the condition (4.5.5) is not satisfied, then the first equality from (4.5.1) is valid for just *two possible values of the charge*, corresponding to two values of the 'handedness' or 'information', and these are given by equation (4.5.6): they depend algebraically on three essential parameters constructed out of the entries of the matrix $\boldsymbol{A}$, and are called the *fixed points* of the matrix (see § 2.5 above).

Thus, in this definition of the manner in which a charge indicates a location, we must have at our disposal a matrix $\boldsymbol{I}$, like that from equation (4.1.2) serving for the necessities of communication. Assume now that this matrix is the same for each location of a certain region in space. So, for that region we have a family of matrices (4.1.2) that can be scanned by the two parameters (p, q), taken as coordinates of the matrices. The problem is: where do these parameters come from? And we have a scenario for this occurrence too.

In order to descrbe this scenario, let us take the matrix $\boldsymbol{A}$ in the form (4.1.2), which we copy here for convenience, but with the omission the coordinate indices:

$$\boldsymbol{A} = p \cdot \boldsymbol{1} + q \cdot \boldsymbol{I}, \quad \boldsymbol{I} \overset{def}{=} \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \tag{4.5.8}$$

In order to calculate the fixed points of the homographic action of this matrix, we need to write the equation (4.5.6), and so we need the explicit form of the entries of $\boldsymbol{A}$. This is

$$\boldsymbol{A} = \begin{pmatrix} p + qa & qb \\ qc & p - qa \end{pmatrix} \tag{4.5.9}$$

and the equation (4.5.6) giving the fixed points of $\boldsymbol{A}$ can be written as

$$q \cdot [c \cdot x^2 - 2a \cdot x - b] = 0 \tag{4.5.10}$$

We read on this equation that the matrices of the family (4.5.8) have the same fixed points: those of the involution $\boldsymbol{I}$. We can say that all these matrices 'point in the same directions', so to speak, for they have common eigen-directions. Indeed, in calculating these eigen-directions, we need the eigenvectors of the matrix, therefore its eigenvalues. These are the roots of the characteristic equation, and must be calculated according to the well-known recipe leading to the equation:

$$(p - \lambda)^2 - q^2(a^2 + bc) = 0 \quad \therefore \quad \lambda_{1,2} = p \pm q\sqrt{a^2 + bc} \tag{4.5.11}$$

The first equality here is the condition of compatibility of the equation which defines the eigen-directions for the matrix from (4.5.9), *viz*.,

$$\boldsymbol{A}\cdot\left|e\right\rangle=\lambda\cdot\left|e\right\rangle,\quad \det(\boldsymbol{A}-\lambda\cdot 1)=0,\quad \left|e\right\rangle\overset{def}{=}\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} \tag{4.5.12}$$

So the components (e_1, e_2) of one of the eigenvectors are solutions of the compatible system:

$$(a+\sqrt{a^2+bc})\cdot e_1+b\cdot e_2=0,\quad c\cdot e_1+(-a+\sqrt{a^2+bc})\cdot e_2=0 \tag{4.5.13}$$

while the components of the other eigenvector are solutions of the compatible sistem:

$$(a-\sqrt{a^2+bc})\cdot e_1+b\cdot e_2=0,\quad c\cdot e_1-(a+\sqrt{a^2+bc})\cdot e_2=0 \tag{4.5.14}$$

We can choose any one of the two equations (4.5.13) in order to calculate the components of the first eigenvector, and the very same way we can proceed with the equation (4.5.14) for calculating the second eigenvector. As long as we do not assume any condition of normalization for the two vectors, both of them are only defined up to an arbitrary factor. Notice, however, that their components do not depend on the coordinates (p, q) of the matrix in the family (4.5.8), but only on the quantities a, b, and c. The ratios of these components are the roots of the equation (4.5.10), or of the reciprocals of these roots. Also notice that the two eigen-directions are, generally, not orthogonal: as expected, they are orthogonal only in the particular case of b and c equal, since the lack of ortogonality of the eigenvectors is decided by the difference between b and c. Based on this play with parameters, a few other important properties are also to be observed.

First of all, using equation (4.5.7) and (4.5.11) we have:

$$a^2+bc=1\ \rightarrow\ \boldsymbol{I}^2=-\boldsymbol{1} \tag{4.5.15}$$

and the eigenvalues of the matrix $\boldsymbol{A}$ do not depend on the parameters a, b, c, and in this case they are real: $p\pm q$. On the other hand the property of independence is also satisfied for the case:

$$a^2+bc=-1\ \rightarrow\ \boldsymbol{I}^2=\boldsymbol{1} \tag{4.5.16}$$

but this time the eigenvalues are complex: $p\pm iq$. Moreover, if $b=c$, the matrix $\boldsymbol{I}$ has a symmetric form:

$$\boldsymbol{I}\overset{def}{=}\begin{pmatrix} a & b \\ b & -a \end{pmatrix},\quad a^2+b^2=\pm 1 \tag{4.5.17}$$

Concluding this section: in connection with Yang's idea laid down in the § 3.2, let us consider the equation (4.5.8) in some detail. The involution $\boldsymbol{I}$ has the remarkable quality that its two actions, homographic and linear are reciprocated, that is, for the case of homography, we have:

$$y=\frac{ax+b}{cx-a}\ \Longleftrightarrow\ x=\frac{ay+b}{cy-a} \tag{4.5.18}$$

In the case of linear action, the reciprocation asks further for the non-singularity of the matrix. Any two such numbers (x, y) satisfy the symmetric equation

$$cx \cdot y - a(x + y) - b = 0 \qquad (4.5.19)$$

This would mean, mathematically, that the *information x* and the *charge y* are to be considered the coordinates over a cycle. Along the cycle, whose physical prototype is the classical Kepler ellipse, the charge and the information are reciprocally interconnected. This, in particular, means that if the information is added anywhere along the cycle, it is shared along the whole cycle, in a way determined by the cycle.

5. The Basic Mathematics to Be Used in Brain Physics

We shall limit our considerations here only to a three-dimensional space, which we assume to be also the case of a realm of daily events of our life. While the geometry of the three-dimensional space, in its Cartesian acceptance, is basically a fictitious construct of our intellect, it serves nevertheless in construction of some other geometrical approximations of reality: the optical space of binocular vision, the Maxwell fish-eye optical medium, the Poincaré characterization of charge by deformation, and such like. When it comes to physics, all of these constructs are connected, in a way or another with the concept of a surface in space, which the idea of deformation takes as instrumental in understanding the physics connected to the concept of charge. So, naturally, our first mathematical topic in this chapter is that of surface: it is required by both the physiological principles (for instance, the case of a *Stroud sphere*) and the physics (the *definition of charge*, in general) as well as by a great many other particular items from some other branches of these two sciences.

5.1. The Venue: Surfaces in Space

In view of this declared task, the three-dimensional space will be described as follows. The vectors will be conceived either as entities defined by components in an Euclidean reference frame, or in the Dirac's matrix form. Thus, the position vector for instance, can be written in one of the following two forms:

$$\boldsymbol{x} = x^k \hat{\boldsymbol{e}}_k, \quad |x\rangle = \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} \qquad (5.1.1)$$

The first of these forms is the usual geometric script for the vectors, whereby the position vector is a linear combination of the unit vectors $\hat{\boldsymbol{e}}_k$ of the reference frame. The coefficients x^k of the above linear combination are the contravariant components of the position vector. The second writing – the matrix notation or, as we would like to call it in order to account for its origin, *the Dirac notation* – disregards the existence of the reference frame. It is appropriate in using for calculations in cases where the reference frame does not count, or needs to be constructed out of coordinates. For instance, either in the cases of positions in the same reference frame, or in the cases where the reference frame is the same everywhere in space, as in the Cartan's approach of the Riemannian geometry (Cartan, 1931). However, there is a third case that seems to encompass these two: the case when the base vectors of the reference frame are constructed from coordinates, by the very same functional rule in any point in space. This is, for instance, the case of *a Beltrami reference frame* which is essential in the case of Maxwellian approach to electricity [see (Mazilu, 2020), §6.1], and asks for a physically valid process of establishing the coordinates independently of the geometry, as, for instance, in the § 4.3 above.

In general, the reference frame is purely local: it can vary from point to point due to some physical reasons. Moreover, still due to some physical reasons, the reference frame may not be always orthogonal. In such cases, using the same general matrix notation as in equation (5.1.1), we write

$$\boldsymbol{x} = \langle x|\hat{\boldsymbol{e}}\rangle, \quad |\hat{\boldsymbol{e}}\rangle \equiv \begin{pmatrix} \hat{\boldsymbol{e}}_1 \\ \hat{\boldsymbol{e}}_2 \\ \hat{\boldsymbol{e}}_3 \end{pmatrix}, \quad |\hat{\boldsymbol{e}}\rangle \cdot \langle \hat{\boldsymbol{e}}| = \boldsymbol{g}(\boldsymbol{x}) \tag{5.1.2}$$

Here $\boldsymbol{g}$ is the metric tensor, a matrix that, due to the fact that the reference frame is made out of vectors that are only normalized, not being orthogonal, has *1* as diagonal entries. If the metric tensor is the identity matrix, we have the usual Euclidean space, with the position expressed in Cartesian coordinates.

The basis of differential geometry of space in the approach we use here, *i.e.* in the Élie Cartan's approach, is the observation that, physically speaking, an infinitesimal (or elementary) displacement involves both a variation in the position of a point *per se*, and a variation of the reference frame itself, according to a rule that may vary from one point to another:

$$d\boldsymbol{x} = dx^k \hat{\boldsymbol{e}}_k + x^k d\hat{\boldsymbol{e}}_k \tag{5.1.3}$$

Here the reference frame is understood as composed of a triad of unit vectors, having also a common origin. Thus, by the general geometrical rules, the elementary variations of the unit vectors of the reference frame can be expressed as linear combinations of these very vectors, with some differential coefficients that can be arranged as entries of a 3×3 matrix. Therefore, the evolution of the reference frame can be described by the so-called *Frenet-Serret equations*, written in the 'indicial form':

$$d\hat{\boldsymbol{e}}_k = \Omega_k^j \hat{\boldsymbol{e}}_j \quad \therefore \quad \Omega_k^j = d\hat{\boldsymbol{e}}_k \cdot \hat{\boldsymbol{e}}^j \tag{5.1.4}$$

Here, in order to use the summation rule over dummy indices, we introduced a '*dual*' *reference frame*, given by the unit vectors $\hat{\boldsymbol{e}}^k$. These are unit vectors that by their dot products give the *contravariant* metric tensor, which is the inverse matrix of the metric tensor defined by the usual reference frame. In general, the matrix $\boldsymbol{\Omega}$ has only zeros on the main diagonal if the reference frame is orthonormal. Indeed, by the virtue of definition of the metric tensor, we have

$$dg_i^j = \Omega_i^j + \Omega_j^i, \quad \hat{\boldsymbol{e}}_i \cdot \hat{\boldsymbol{e}}^j = g_i^j \tag{5.1.5}$$

Thus, as just mentioned, we can make the properties of the matrix $\boldsymbol{\Omega}$ even more precise: it is always a skew-symmetric matrix, in the case of an orthonormal reference frame.

Now, with Frenet-Serret relations from equation (5.1.4), the equation (5.1.3) can be written as

$$d\boldsymbol{x} = s^k \hat{\boldsymbol{e}}_k, \quad s^k \equiv dx^k + x^j \Omega_j^k \tag{5.1.6}$$

An edifying example of such a definition of the components of fundamental displacements can be offered by equation (4.3.12) which allows the introduction of a scalar coordinate defined in the spirit of Poincaré (see § 2.5 above) having the meaning of the variance of distance between the particles of an ensemble serving for interpretation. Obviously, both the components of $d\boldsymbol{x}$ as well as those of $d\hat{\boldsymbol{e}}_k$ must be exact differentials. In the framework of exterior calculus, this fact can be expressed by vanishing of the exterior differentials:

$$d \wedge d\boldsymbol{x} = \boldsymbol{0}, \quad d \wedge d\hat{\boldsymbol{e}}_k = \boldsymbol{0} \tag{5.1.7}$$

The whole geometrical construction of Élie Cartan is a mathematical consequence of these two equations, representing simple facts of differentiability. For once, by following the rules of working with exterior differential forms we can find, starting from (5.1.7), the following relations which connect the components of vector $d\boldsymbol{x}$ to the matrix $\boldsymbol{\Omega}$:

$$d \wedge s^k + \Omega_j^k \wedge s^j = 0, \quad d \wedge \Omega_j^k + \Omega_m^k \wedge \Omega_j^m = 0 \tag{5.1.8}$$

Here the Einstein's rule of sumation over dummy indices is observed, with the only difference that the monomials are defined by exterior multiplication, not by the usual numerical product, and the sign '$\wedge$' after differentiation symbol shows that it has to be an exterior differentiation. By obvious reasons, the first of equations (5.1.8) is usually called *compatibility equation*: it gives, indeed, the compatibility condition between the variation of the reference frame and the elementary displacements in space, as described within this reference frame. The second of the equations in (5.1.8) can be termed as the *Maurer-Cartan equation*, borrowing a name which describes the moving coframe of the Lie algebras.

So, in order to get a grip on the idea, consider the classical pair of intrinsic surface vectors we write in the form of differentials:

$$d\boldsymbol{r} = s^1 \hat{\boldsymbol{e}}_1 + s^2 \hat{\boldsymbol{e}}_2, \quad d\boldsymbol{p} = s^2 \hat{\boldsymbol{e}}_1 - s^1 \hat{\boldsymbol{e}}_2 \qquad (5.1.9$$

where ($\hat{\boldsymbol{e}}_1$, $\hat{\boldsymbol{e}}_2$) is an orthonormal reference frame on the surface. These vectors are, obviously, orthogonal to each other, 'Euclidean-wise' as it were, and orthogonal to the normal $\hat{\boldsymbol{e}}_3$ to surface, in any point of the surface where the differential 1-forms s^1 and s^2 are defined. The first one of the vectors (5.1.9 is the elementary displacement on the surface, while the second is usually identified with the vector characterising the *geodesic torsion* on the surface [see (Struik, 1988); Problems 3 and 4, pp. 201 – 202; also the Appendix of that treatise, §7, pp. 213–215]. Let us calculate the symmetric differential of $d\boldsymbol{p}$, in order to compare it with the corresponding differential $d^2\boldsymbol{r}$ of the position vector. We have, from the geometry of surfaces [(Struik, 1988), Appendix, §§5 – 7]:

$$d^2\boldsymbol{p} = (ds^2 + \omega_1^2 s^1)\hat{\boldsymbol{e}}_1 - (ds^1 + \omega_2^1 s^2)\hat{\boldsymbol{e}}_2 + (s^2\omega_1^3 - s^1\omega_2^3)\hat{\boldsymbol{e}}_3 \tag{5.1.10}$$

so that it is obvious that the direction of this vector in the ambient space does not coincide with the direction of $d^2\boldsymbol{r}$, as calculated from the first of the equalities (5.1.9). However, performing those calculations, one can say that the surface components of these two vectors are again perpendicular to each other, replicating the property of the vectors (5.1.9). Indeed, we have

$$d^2\boldsymbol{r} = (ds^1 + \omega_2^{\ 1} s^2)\hat{\boldsymbol{e}}_1 + (ds^2 + \omega_1^{\ 2} s^1)\hat{\boldsymbol{e}}_2 + (\omega_1^{\ 3} s^1 + \omega_2^{\ 3} s^2)\hat{\boldsymbol{e}}_3 \tag{5.1.11}$$

and a comparison between (5.1.10) and (5.1.11) proves our statement.

As far as the normal to surface components are concerned, these are, in both cases, quadratic forms in the components of elementary displacement on the surface. In view of equation (5.1.10), the normal component of $d^2\boldsymbol{p}$ is the second-degree differential form:

$$\hat{\boldsymbol{e}}_3 \cdot d^2\boldsymbol{p} = s^2\omega_1^{\ 3} - s^1\omega_2^{\ 3} = -b_{12}(s^1)^2 + (b_{11} - b_{22})s^1 s^2 + b_{12}(s^2)^2 \tag{5.1.12}$$

The coefficients of this quadratic form are strictly depending on the curvature of surface. The only difference is in the form of their matrices, which, nevertheless are correlated in a reproducible way. To wit: the matrix of quadratic form (5.1.12) is

$$\boldsymbol{a} \equiv \boldsymbol{i} \cdot \boldsymbol{b} \quad \therefore \quad \boldsymbol{a} = \begin{pmatrix} -b_{12} & -b_{22} \\ b_{11} & b_{12} \end{pmatrix} \tag{5.1.13}$$

i.e., the *left* product of the curvature matrix, with the matrix analogous to imaginary unit of the complex numbers. As a matter of fact, we have here just the curvature matrix in its two posible *actions*. First, the *linear action*, characterized, for physics' purposes, by the two eigenvalues, λ say, of the matrix $\boldsymbol{b}$, *i.e.*, the values for which the equation

$$\boldsymbol{b} \cdot d\boldsymbol{r} = \lambda d\boldsymbol{r} \tag{5.1.14}$$

makes sense. The two eigenvalues are geometrically calculated as the extreme values of the quadratic form ($d\boldsymbol{r}\cdot\boldsymbol{b}\cdot d\boldsymbol{r}$), which represents the second fundamental form of the surface:

$$\hat{\boldsymbol{e}}_3 \cdot d^2\boldsymbol{r} = b_{11}(s^1)^2 + 2b_{12}s^1 s^2 + b_{22}(s^2)^2 \tag{5.1.15}$$

On the other hand, the quadratic from equation (5.1.12) can be written as:

$$d\boldsymbol{r} \cdot \boldsymbol{a} \cdot d\boldsymbol{r} \equiv \hat{\boldsymbol{e}}_3 \cdot d^2\boldsymbol{p} \tag{5.1.16}$$

In other words, the two quadratics given by equations (5.1.12) and (5.1.15) describe the two actions of the curvature matrix $\boldsymbol{b}$, connected with the definition of the curvatures: the *linear action* and the *homographic action*. Thereby, the asymptotic directions of the surface are decided by the fixed points of the matrix $\boldsymbol{a}$. Indeed, asymptotic directions of the surface are given by the directions in which the second fundamental form of the surface vanishes. Using the equation (5.1.15) this condition can be arranged in the form:

$$x = \frac{-b_{12}x - b_{22}}{b_{11}x + b_{12}} \quad \therefore \quad x \equiv \frac{s^1}{s^2} \tag{5.1.17}$$

which, considering the definition of matrix $\boldsymbol{a}$ from equation (5.1.13), proves our statement.

The two quadratic polynomials, (5.1.12) and (5.1.15), are *algebraically apolar* with respect to each other: their roots for the ratio x from (5.1.17) are in *harmonic sequence* [(Burnside & Panton), §38]. This helps us in promoting the conclusion that the curvature matrix determines the possibility of introducing physics through the concept of surface, in a theory for which the likes of the matrix $\boldsymbol{a}$ are instrumental. Let us elaborate further on this conclusion.

Our basis of elaboration is the fact that, from physics' point of view, the 2×2 matrices play a role just as important as the coordinates. Let us illustrate this issue for the case of the matrix $\boldsymbol{b}$. The second fundamental form of a surface is a first indication of the 'profile' of that surface: if the matrix $\boldsymbol{b}$ changes, one can see a change in the appearance of the surface. Assume a differential variation of the matrix $\boldsymbol{b}$, which may be due to some physical causes represented, as usual, by forces. Assuming, further, that the variation of the second fundamental form must be expressed by the rules of classical Newtonian differentiation, we have:

$$d\langle s|\boldsymbol{b}|s\rangle = \langle ds|\boldsymbol{b}|s\rangle + \langle s|d\boldsymbol{b}|s\rangle + \langle s|\boldsymbol{b}|ds\rangle = \langle s|d\boldsymbol{b}|s\rangle + 2\langle s|\boldsymbol{b}|ds\rangle \tag{5.1.18}$$

in view of the symmetry of the matrix $\boldsymbol{b}$. Here the differential matrices are those matrices having as entries the differentials of the entries of the original ones. Assume now a gauging for the kets $|ds\rangle$, such that

$$|ds\rangle = \boldsymbol{a}\cdot|s\rangle \tag{5.1.19}$$

in the case of which the equation (5.1.18) becomes

$$d\langle s|\boldsymbol{b}|s\rangle = \langle s|d\boldsymbol{b}|s\rangle + 2\langle s|\boldsymbol{b}\cdot\boldsymbol{a}|s\rangle \tag{5.1.20}$$

The matrix $d\boldsymbol{b}$ becomes dominant only on the gauged vectors: this means that the variation of the second fundamental form is simply given exclusively by the variation of the curvature matrix if, and only if, the matrix $\boldsymbol{b}\cdot\boldsymbol{a}$ is a skew-symmetric matrix:

$$d\langle s|\boldsymbol{b}|s\rangle = \langle s|d\boldsymbol{b}|s\rangle \quad iff \quad \boldsymbol{b}\cdot\boldsymbol{a} = da\cdot\boldsymbol{i} \tag{5.1.21}$$

where da is the differential of a certain physical quantity playing the part of time. This means that the curvature matrix determines the gauge, for we have:

$$\boldsymbol{a} = da\cdot\boldsymbol{b}^{-1}\cdot\boldsymbol{i} \quad \therefore \quad \boldsymbol{a} = \frac{da}{\det(\boldsymbol{b})}\begin{pmatrix} -b_{12} & -b_{22} \\ b_{11} & b_{12} \end{pmatrix} \tag{5.1.22}$$

which is the matrix from equation (5.1.13) up to a factor.

To conclude: on a *portion of surface* satisfying the 'gauging' from equation (5.1.19), with $\boldsymbol{a}$ given by (5.1.13) up to a differential factor, the *variation of the second fundamental form is strictly due the variation of the* *matrix of curvature*. Now, this variation can be induced by physical causes, such as the electricity, or gravitation, or anything else physical. This may be taken as a universal principle ranking equal to the one introduced by Henri Poincaré connecting the charge with the deformation, through the *definition of state*: on the *state of surface* defined by the gauging equation (5.1.19), with the gauging matrix defined by equation (5.1.22), the variation of the second fundamental form of the surface is strictly due to the variation of the curvature matrix. Now, this variation of the curvature matrix is local, making, consequently, the state of surface a local geometrical concept. The state can be given by the charge, as well as by some other physical causes. The deformation *per se*, does not enter the reasoning here, it is only the gauging that does. In other words, the equation (5.1.19) defines a state, but in order to define a *constrained state* in the sense of Poincaré, we still need to define the deformation. This is the task of the following Chapter of this instalment of the present work.

5.2. Accommodating the Charge: Infinitesimal Deformations of Surfaces

After previous brief review of the mathematical instruments destined to accommodate the presence of surfaces in physics, we are pretty much prepared to show how the Lorentz electric matter realizes the Poincaré's thesis that the charge is inherently connected with the concept of deformation. The main point of this exercise is that the deformation is a local phenomenon, and the mathematics at the time of Poincaré was not yet prepared to properly handle such quite involved phenomena. As a matter of fact, the Poincaré's definition of charge was not taken in consideration as such even after the Louis de Broglie's definition of the physical ray, a concept which had the great virtue of pointing out *directly* the true place of interpretation in the economy of theoretical physics. The essential concept in the case of such a ray is that of a *portion of surface* moving inside the ray like the surface of a fluid in a capillary duct.

What one geometrically needs in the case of a de Broglie ray, is the construction of a vector-function $\boldsymbol{z}(u,v)$, describing the deformation of a surface as a function of the coordinates (u, v) on it, according to the vectorial equation (Guggenheimer, 1977)

$$\boldsymbol{r}(\varepsilon) = \boldsymbol{x} + \varepsilon \boldsymbol{z} \tag{5.2.1}$$

where $\boldsymbol{x}$ is the position of the point (u, v) of surface portion in the ambient space. For the construction of $\boldsymbol{z}$, we use the metric form of the surface, in which the deformation is usually assumed to enter the stage of physics. In this case, the deformation is *infinitesimal* if:

$$\frac{ds^2(\boldsymbol{r}, d\boldsymbol{r}) - ds^2(\boldsymbol{x}, d\boldsymbol{x})}{\varepsilon} \xrightarrow[\varepsilon \to 0]{} 0 \tag{5.2.2}$$

where ε is a parameter and $ds^2(\boldsymbol{x},d\boldsymbol{x})$ is the metric form, that is the *first fundamental form of the surface*, at position $\boldsymbol{x}$, calculated on the displacement $d\boldsymbol{x}$. According to (5.2.1), we can write the deformed metric as

$$ds^2(\boldsymbol{r},d\boldsymbol{r}) = ds^2(\boldsymbol{x},d\boldsymbol{x}) + 2\varepsilon(d\boldsymbol{x}\cdot d\boldsymbol{z}) + \varepsilon^2(d\boldsymbol{z}\cdot d\boldsymbol{z})$$

with the obvious notation of the dot product of vectors. Then the deformation of the metric is infinitesimal in the sense of equation (5.2.2), if

$$d\boldsymbol{x}\cdot d\boldsymbol{z} = 0$$

Assuming, as we always have, an 'Euclidean mentality', this equation says that there is always an arbitrary vector $\boldsymbol{q}$ of the ambient space – the symbol $\boldsymbol{q}$ is chosen to remind us of the charge – assisting us in writing $d\boldsymbol{z}$ in the form:

$$d\boldsymbol{z} = \boldsymbol{q}\times d\boldsymbol{x} \tag{5.2.3}$$

The geometric arbitrariness of ancillary vector $\boldsymbol{q}$ can be significantly reduced, if we take notice that, since $d\boldsymbol{z}$ is an exact differential vector, we must have:

$$d\wedge d\boldsymbol{z} = \boldsymbol{0} \quad\rightarrow\quad d\boldsymbol{q}\times^\wedge d\boldsymbol{x} = \boldsymbol{0} \tag{5.2.4}$$

Here '×^' means that, in the vector product, the usual multiplication needs to be replaced by an *exterior multiplication* of the differentials. Using the notation

$$d\boldsymbol{q} \overset{def}{=} j^k\hat{\boldsymbol{e}}_k \tag{5.2.5}$$

where the vector-symbol $\boldsymbol{j}$ is intended to remind us of a *current density* – the condition from equation (5.2.4) can be transcribed in the form

$$(-j^3\wedge s^2)\hat{\boldsymbol{e}}_1 + (j^3\wedge s^1)\hat{\boldsymbol{e}}_2 + (j^1\wedge s^2 - j^2\wedge s^1)\hat{\boldsymbol{e}}_3 = \boldsymbol{0}$$

which, in turn, comes down to the system of equations:

$$j^3\wedge s^1 = j^3\wedge s^2 = 0, \quad j^1\wedge s^2 - j^2\wedge s^1 = 0 \tag{5.2.6}$$

The first two of these equations show that, according to the rules of exterior calculus, $j^3 = 0$ on the state of surface defined by deformation, because s^1 and s^2 are independent in the geometry of a surface described by them. This means that, in order to be appropriate in generating an infinitesimal deformation, the vector $d\boldsymbol{q}$ must be situated in the tangent plane of the surface, *i.e.* it can be taken as *an intrinsic vector*.

Better yet, notice that by its very definition the vector $\boldsymbol{q}$ is a flux, from tensorial point of view [see (Roche, 2000) for an exquisite presentation of the idea, in the particular case of the classical magnetic flux], and so we can say that the portion of surface over which the flux $\boldsymbol{q}$ happens to fall, must act on this vector in such a way that *its component normal to surface is a constant*: the surface 'absorbs' from the whole flux of particles, serving for an interpretation for instance, only those having a constant component normal to surface. If one likes the anecdotes, one can infer that, physically speaking, the surface serves as a physical instrument for the celebrated Maxwell demon! Then, the last equality from equation (5.2.6) says something more about the vector $\boldsymbol{j}$. First, by the *Cartan*'*s Lemma 1*, it can be transliterated into equation:

$$\begin{pmatrix} -j^2 \\ j^1 \end{pmatrix} = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \cdot \begin{pmatrix} s^1 \\ s^2 \end{pmatrix} \therefore \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} j^1 \\ j^2 \end{pmatrix} = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \cdot \begin{pmatrix} s^1 \\ s^2 \end{pmatrix} \tag{5.2.7}$$

According to its 'intrinsic' property, the vector ***dq*** looks like a sort of 'complement' of the infinitesimal displacements *d**x*** on the surface.

For the case of brain, according to Valentino Braitenberg's synthesis [(Braitenberg, 1959); see § 2.6 above], and using the basic idea of Poincaré on the connection between charge and deformation of surface, the third component of the vector ***q*** is constant for the infinitesimal deformation of surface: it characterizes a fiber and, therefore, from physiological point of view, it should be *proportional with the velocity of propagation* along that fiber. This fact can be explained 'tautologically', so to speak, in general terms: a portion of ray is situated in the space of a Lorentz quantum that engulfs it momentarily, wherein the corresponding *portion of a wave surface created by the quantum* inside the ray is deformed by the charge. This explanation goes for a de Broglie ray in the case of optics, just as well as for a neuron in the case of the physics of brain. A problem still remains to be solved, namely to explain, and describe mathematically, of course, how 'a *Lorentz quantum creates a portion of surface* inside a ray'.

This 'similarity' involving the Lorentz quantum, goes even deeper: in view of definition (5.2.5), ***dq*** is an exact differential, and the conditions for its integrability $d\wedge d\boldsymbol{q} = \boldsymbol{0}$ are

$$d\wedge j^1 + \Omega^1_2 \wedge j^2 = 0,\ d\wedge j^2 + \Omega^2_1 \wedge j^1 = 0,\ \Omega^3_1 \wedge j^1 + \Omega^3_2 \wedge j^2 = 0 \tag{5.2.8}$$

Obviously, these equations replicate the similar ones for the components of *d**x***, given in equation (5.1.8) above. Indeed, representing locally the idea of undeformed surface by the condition $s^3 = 0$, the first set of equations from (5.1.8) can be written in detail as:

$$d\wedge s^1 + \Omega^1_2 \wedge s^2 = 0,\quad d\wedge s^2 + \Omega^2_1 \wedge s^1 = 0,\quad \Omega^3_1 \wedge s^1 + \Omega^3_2 \wedge s^2 = 0 \tag{5.1.8}$$

Now, if we use the *Cartan's Lemma 1*, the entries $\Omega^3{}_1$ and $\Omega^3{}_2$ of the matrix $\boldsymbol{\Omega}$, should be the components of a ket vector, say $|\Omega^3\rangle$, representing the variation of the unit normal to surface at the given position, as in equation (5.1.4). Then, according to equation *Cartan's Lemma 1*, the last of the relations above shows that the variation, by infinitesimal deformation, of the unit normal to surface is an intrinsic vector that can be expressed *linearly* in terms of s^1 and s^2, *by a homogeneous relation, involving a conveniently chosen symmetric matrix*:

$$\Omega^3_\alpha = b_{\alpha\beta} s^\beta,\quad b_{\alpha\beta} = b_{\beta\alpha}$$

Written symbolically, this means

$$\left|\Omega^3\right\rangle = \boldsymbol{b}\cdot\left|s\right\rangle,\quad \boldsymbol{b} = \boldsymbol{b}^t \tag{5.2.9}$$

where the upper index '*t*' stands for 'transposed'. Since, intuitively speaking, the variation of the normal to surface means its variation of curvature, the very matrix ***b*** should be taken as related to the *curvature*, – we call it the *curvature matrix* just to mark this meaning, even though this name is assigned occasionally to some other matrices in geometrical treatises – as in the classical theory of surfaces (Flanders, 1989).

Assuming, therefore, that *the curvature of surface is essential in describing its physics*, especially in the problem of electricity, as the Poincaré's theory suggests, even if the surface is not one of equilibrium, *we choose* to read the third of the equalities from equation (5.2.8) *as determining the ancillary vector* $|j\rangle$ *in terms of the curvature*, according to the following relation:

$$|j\rangle = \boldsymbol{A}\cdot|\Omega^3\rangle \tag{5.2.10}$$

Here $\boldsymbol{A}$ is, again, a *convenient symmetric matrix*, introduced in order to satisfy the *Cartan's Lemma 1*, and representing the intuitive idea that the current generating the deformation is somehow related to the variation of curvature, as the experience instructs our intellect, according to Poincaré's idea of describing the charge. Now, when we use (5.2.10), in conjunction with the geometrical definition of $|\Omega^3\rangle$ from equation (5.2.9) and with equation (5.2.7), both written formally as:

$$\left|\Omega^3\right\rangle = \boldsymbol{b}\cdot\left|s\right\rangle,\quad \boldsymbol{i}\cdot\left|j\right\rangle = \boldsymbol{a}\cdot\left|s\right\rangle$$

we get from (5.2.10) the following *local* relation defining the matrix $\boldsymbol{A}$ in terms of the gauging matrix:

$$\boldsymbol{a} = \boldsymbol{i}\cdot\boldsymbol{A}\cdot\boldsymbol{b} \quad \therefore \quad \boldsymbol{A} = -\boldsymbol{i}\cdot\boldsymbol{a}\cdot\boldsymbol{b}^{-1} \tag{5.2.11}$$

Here $\boldsymbol{i}$ is the 2×2 fundamental skew-symmetric matrix from the second equality of equation (5.2.7). The notation is intended to suggest the obvious fact that $\boldsymbol{i}$ is the matrix replica of the imaginary unit from the case of complex numbers: *its square is negative identity matrix*.

Carrying now these observations back to the classical case of second fundamental form, as in the previous section, we can express the matrix $\boldsymbol{a}$ in terms of the matrix of curvatures $\boldsymbol{b}$ and its variation $d\boldsymbol{b}$. Then the matrix $\boldsymbol{A}$ itself, from equation (5.2.11) does not depend, indeed, but only on the curvature and its variation:

$$\boldsymbol{A} = \boldsymbol{i}\cdot\boldsymbol{b}^{-1}\cdot d\boldsymbol{b}\cdot\boldsymbol{b}^{-1} \quad \therefore \quad \boldsymbol{A} = \left\{(dn)\boldsymbol{i} + \boldsymbol{i}\cdot\boldsymbol{E}\right\}\cdot\boldsymbol{b}^{-1} \tag{5.2.12}$$

so that the equation (5.2.7) becomes

$$\left|j\right\rangle \equiv \delta\boldsymbol{b}\cdot\left|s\right\rangle \quad where \quad \delta\boldsymbol{b} \equiv -\begin{pmatrix} \omega^1 & \tfrac{1}{2}\omega^2 + dn \\ \tfrac{1}{2}\omega^2 - dn & \omega^3 \end{pmatrix} \tag{5.2.13}$$

In other words, by infinitesimal deformation as defined here, the curvature matrix gathers a skew-symmetric part, for it loses its matrix symmetry. In a classical view (Lowe, 1980), suggested by the mechanical deformation of the thin plates, this property should be connected with the *torsion* of surfaces. This further suggests the practicality of using the affine theory of surfaces in the fundamental physics, an idea that will be presented in broad strokes at the very end of the present instalment of our work.

Limiting, though, the mathematics to only what we have right now, it says that in the gauging given by equation (5.1.10), *the infinitesimal deformation* adds to the second fundamental form of the reference 'undeformed' surface characterized by matrix ***b***, a quadratic form having the matrix $\delta\boldsymbol{b}$ given in equation (5.2.13). The result of this addition, describes the differential geometry of a surface having the support function *h* for the definition of support function one can consult (Guggenheimer, 1977); especially § 10-5 of this reference – as a quadratic form which *is the sum of two quadratic forms* which are mutually harmonic:

$$h \equiv \left\langle s \middle| \boldsymbol{b} + \delta\boldsymbol{b} \middle| s \right\rangle = \alpha(s^1)^2 + 2\beta s^1 s^2 + \gamma(s^2)^2 + \omega^1(s^1)^2 + \omega^2 s^1 s^2 + \omega^3(s^2)^2 \tag{5.2.14}$$

That is, these quadratics have the roots in a harmonic range, with their characteristic cross-ratio assuming the value of -1, or a value compatible with this one in the natural range of of cross-ratios, since the coefficients are naturally satisfying the algebraic condition:

$$\gamma \cdot \omega^1 - \beta \cdot \omega^2 + \alpha \cdot \omega^3 = 0 \tag{5.2.15}$$

This construction plainly justifies the Lorentz definition of the electromagnetic matter, in that the surface characterized by the support function (5.2.14) *should not be necessarily* '*pegged*', *i.e*. defined by points: it is just an imagined surface in a continuum, that may or may not contain particles, *i.e*. a surface of the kind of those we imagine as being created by the electromagnetic field in ether, or existing within a Lorentz quantum for the description of the plane waves. Since the condition (5.2.15) is also satisfied for the differentials $d\alpha$, $d\beta$ and $d\gamma$, instead of the corresponding finite magnitudes, the surface described by the matrix $\boldsymbol{b} + d\boldsymbol{b}$, satisfies it too in the case of a deformed surface. Roughly, one can say that the two surfaces *delimit an Ampère element*, which can be represented, in general, by the de Broglie's capillary tube. In the physics of brain, this Ampère element would be assimilable to 'a nerve trunk', of the kind described by Lord Adrian of Cambridge in his article from the year 1930 (see § 2.6 here).

Consider now the differential 1-form representing the volume of the cuboid constructed on the three original vectors defining the infinitesimal deformation:

$$(\boldsymbol{x}, \boldsymbol{q}, d\boldsymbol{q}) \stackrel{\text{def}}{=} \begin{pmatrix} x^1 & x^2 & x^3 \\ q^1 & q^2 & q^3 \\ j^1 & j^2 & j^3 \end{pmatrix} \tag{5.2.16}$$

Insofar as all three vectors are variable, we need the variation of this volume. The exterior differential of the quantity (5.2.16) is a 2-form, just like the electric induction or magnetic flux in electrodynamics. This 2-form can be easier calculated by first rearranging the elementary volume 1-form such that it appears as:

$$d \wedge (\boldsymbol{x}, \boldsymbol{q}, d\boldsymbol{q}) = d\boldsymbol{q} \cdot (\boldsymbol{x} \times \boldsymbol{q}) \tag{5.2.17}$$

In this case its exterior differential is simply the exterior product of two differentials:

$$d \wedge (\boldsymbol{x}, \boldsymbol{q}, d\boldsymbol{q}) = d(\boldsymbol{x} \times \boldsymbol{q}) \stackrel{\cdot}{\wedge} d\boldsymbol{q} \tag{5.2.18}$$

where ($\wedge^{\cdot}$) shows that in the exterior product of differential forms we need to use the dot product of vectors. Now, because, according to the usual rules of differentiation, the first factor here can be written as

$$d(\boldsymbol{x}\times\boldsymbol{q})=(q^{3}s^{2}-x^{3}j^{2})\hat{\boldsymbol{e}}_{1}+(x^{3}j^{1}-q^{3}s^{1})\hat{\boldsymbol{e}}_{2}+(x^{1}j^{2}-x^{2}j^{1}+q^{2}s^{1}-q^{1}s^{2})\hat{\boldsymbol{e}}_{3}$$

where $|\hat{\boldsymbol{e}}\rangle$ is an appropriate orthonormal frame. Thus, one can transcribe (5.2.18) as

$$d\wedge(\boldsymbol{x},\boldsymbol{q},d\boldsymbol{q})=q^{3}(-j^{1}\wedge s^{2}+j^{2}\wedge s^{1})+2x^{3}(j^{1}\wedge j^{2})$$

In the right hand side the first paranthesis is zero by (5.2.7); by the same token, calculating the remaining term, we finally can wrap up the final expression of (5.2.18) to the form:

$$d\wedge(\boldsymbol{x},\boldsymbol{q},d\boldsymbol{q})=2x^{3}(j^{1}\wedge j^{2})\quad\therefore\quad d\wedge(\boldsymbol{x},\boldsymbol{q},d\boldsymbol{q})=2x^{3}(ac-b^{2})(s^{1}\wedge s^{2})\tag{5.2.19}$$

Locally, the coordinate x^3 is the *support function* of the surface, which can be used as a space coordinate in case we use the local patch as reference (Rainich, 1925). The formula (5.2.19) defines a physical flux through a state of surface engendered by the vector $\boldsymbol{q}$.

5.3. A Three-Dimensional Phase Space: 'Dynamics' Within Lorentz Quantum

No doubt, we have to maintain in the picture, at least at a certain level of our reasoning, the classical idea of Euclidean space and Cartesian coordinates, in both the problems of the physics *per se*, as well as in the problems related to the physics of brain. Such an example, occurs clearly when related to the *Coll's universal deformation structure* as provided by a metric constructed according to the absolute geometry [(Coll, Llosa, & Soler, 2002); see also (Mazilu, 2024), § 3.4]. There are indications that this universal deformation should be connected to the theory of Yang-Mills fields and charges (see I, § 5.6), which makes it the best candidate in a general definition of fields for a concept of universe [see (Mazilu, 2024), Chapter 4, and especially the § 5.4].

We presented such a theory in this very work, first in connection with a classical theory of charges (I, § 5.4), and then as an incentive in applying this theory in the Wu-Yang development of the Yang-Mills theory of quantized fields (I, § 5.6). What, specifically, might have called for such an association, is quite clear from the description given in our previous instalment. Namely, the Wu-Yang theory of Yang-Mills field was destined to fill in for a missing point of the classical Maxwellian electrodynamics: *the absence of the static fields from the theory*. This condition became critical within the Yang-Mills field theory taken as a gauge theory, where the static conditions expressly call for interpretation – after all, the necessity of these conditions led to the concept of pseudoparticle – and this directed our attention to the eternal problem of theoretical physics at large: the lack of dynamical characterization of the *state of rest*. In the expression of Eugene Wigner (1954), we do not have, in the corpus of classical axioms, a principle sanctioning the kinematic idea of rest. The second principle of dynamics sanctions the state of uniform rectilinear motion; the first principle says that if no forces are involved, the state of uniform rectilinear motion or rest exist, but no principle is saying that:

which, cosmologically, can be seen as a definition of rest, for there is no rest in the universe. According to Wigner, this principle would ask for *forces generating velocities*, not accelerations, so that we presented such a dynamics as *transversal*, in the Newtonian spirit initiated by Wigner himself (Wigner, 1954), most adequate in connecting the propagation along a ray with the transversal motion of an interpretative particle (see § 4.3 above). Here, however, we present this 'dynamics' in the spirit of physics from the second half of the 20th, after the times when the concept of soliton caught up firmly with theoretical physics (Dumachev, 2011). Consider the vector:

$$\left|m_2 - m_3\right\rangle \stackrel{def}{=} \begin{pmatrix} m_2 - m_3 \\ m_3 - m_1 \\ m_1 - m_2 \end{pmatrix} \tag{5.3.1}$$

whose components are obtained from one another by positive permutations of the three indices. This vector is, indeed, orthogonal in the Euclidean sense to the position vector

$$\left|m_1\right\rangle \stackrel{def}{=} \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} \tag{5.3.2}$$

and can be obtained from any 'sample' of this position vector along a ray, simply by application of the basic rotation matrices:

$$\boldsymbol{h}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad \boldsymbol{h}_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \boldsymbol{h}_3 \equiv \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{5.3.3}$$

These matrices are obtained according to the scheme:

$$(\boldsymbol{h}_k)^i_j \stackrel{def}{=} \epsilon^i_{kj} \tag{5.3.4}$$

from the tensor of structure constants of the rotations' group (the Levi-Civita totally skew-symmetric symbol). In this case, the scheme

$$(\boldsymbol{h}^i)_{kj} \stackrel{def}{=} \epsilon^i_{kj} \tag{5.3.5}$$

provides the same result as (5.3.3) up to sign. We presented this Wigner dynamics as a transversal dynamics in the § 4.4 of the first instalment of this work. It was intended to give at least some indications on *two fundamental issues* of the physical theory in general, if not to solve them at all. The first was the issue raised by Eugene Wigner as to the equations of classical dynamics (Wigner, 1954):

$$\dot{m}_1 = \nu(m_2 - m_3), \quad \dot{m}_2 = \nu(m_3 - m_1), \quad \dot{m}_3 = \nu(m_1 - m_2) \tag{5.3.6}$$

where ν is a constant frequency, and a dot over means time derivative as usual. These equations represent a motion perpendicular to the ray, as in the case of light, and can be taken as a model for the cosmogonic 'just transversal impulse' of Newton: if the vector $\boldsymbol{m}$ represents the position on a Thomson tube playing the part of a ray, then the equations (5.3.6) provide the Thomson transversal velocities (see § 4.3) necessary to put the particle falling along the tube on its orbit, when reaching its Newtonian orb. Now, the equations of this motion can be 'decoupled', so to speak, by successive differentiation on time. The conclusion is that each component of the motion described by (5.3.6) is a solution of the very same third order differential equation, *i.e.* the vector $\boldsymbol{m}$ is solution of the same equation:

$$\dddot{\boldsymbol{m}} + 3\nu^2 \dot{\boldsymbol{m}} = \boldsymbol{0} \tag{5.3.7}$$

showing that the components of this vector satisfy the condition defining a ray, indeed, but in a Maxwell fish-eye optical medium [see (Mazilu, 2024), § 1.2]. And the Maxwell fish-eye is the optical medium characterizing a Planck universe, *i.e.*, a Wien-Lummer enclosure. This means, implicitly, that the Newtonian cosmogonic problem asks for a Planck quantization, and the theoretical physics of the last half of the 20th century has shown, in our opinion, that the Newtonian forces are, in fact, the classical expression of this quantization. But there is more to be taken in consideration along this 'Newtonian line', if we may be allowed to use this term, and these facts can be revealed by proceeding to the solutions of the equation (5.3.7). Our main point here is that this equation is fundamental in regularization procedure of the classical Kepler problem [see (Mazilu, Agop, & Mercheș, 2021), Chapter 4].

In order to solve the equation (5.3.7), we can search for constant integrals of this motion as this motion is described by the system (5.3.6). And, in case we are starting from the differential system (5.3.6), it is worth trying to find, based on it, exact differentials that could offer us a physical interpretation to the time parameter of continuity. The most obvious method involves constructing linear forms equivalent to the system (5.3.6), in coordinates according to the following procedure. We can derive algebraically an exact differential equivalent to the system (5.3.6):

$$\frac{a dm_1 + b dm_2 + c dm_3}{(c-b)m_1 + (a-c)m_2 + (b-a)m_3} = \nu dt \tag{5.3.8}$$

with constant (a, b, c). Such exact differentials can exist under the conditions

$$c - b = \lambda a, \quad a - c = \lambda b, \quad b - a = \lambda c \tag{5.3.9}$$

with λ a parameter. This means that the left-hand side of the equation (5.3.8) is an exact differential only for the cases in which λ has as values the roots of the cubic equation:

$$\lambda(\lambda^2 + 3) = 0$$

representing the condition of compatibility of the linear system (5.3.9). This algebraic equation is also the characteristic equation for the differential equation (5.3.7), assuming the constant ν unity. In terms of the three roots of this equation, that is 0 and $\pm i\sqrt{3}$, the following three *complex integrals* can be constructed with respect to some initial conditions at the time $t = 0$:

$$\begin{aligned} m_1 + m_2 + m_3 &= m_1^0 + m_2^0 + m_3^0 \\ m_1 + j^2 m_2 + j m_3 &= e^{i\nu t\sqrt{3}}(m_1^0 + j^2 m_2^0 + j m_3^0) \\ m_1 + j m_2 + j^2 m_3 &= e^{-i\nu t\sqrt{3}}(m_1^0 + j m_2^0 + j^2 m_3^0) \end{aligned} \tag{5.3.10}$$

Here *j* is the *cubic root* of unity, intended as a counterpart of the classical *i*, which is the *square root* of negative real unit in the case of usual complex numbers. The three complex variables from the left-hand side of this system are related – and in quite a few ways at that – to the name of Paul Appell, and they have a tremendous importance of principle, both from physical [(Appell, 1893), p. 351] as well as from purely mathematical point of view (Appell, 1877).

Start by noticing that the first of the integrals (5.3.10) is a linear constant of motion. Another constant of motion is quadratic, and can be obtained from the product of the last two of them, *i.e.*:

$$(m_2 - m_3)^2 + (m_3 - m_1)^2 + (m_1 - m_2)^2 = const \tag{5.3.11}$$

The 'trajectory' is then to be found in the intersection of this quadric with the real plane given by the first equality from equation (5.3.10), therefore it also belongs to the quadric

$$m_2 m_3 + m_3 m_1 + m_1 m_2 = const \tag{5.3.12}$$

and thus, belongs to a sphere. Indeed, according to equation (5.3.7), the charge can be itself represented as a genuine periodical process having as components the solutions of the differential equation

$$\ddot{\boldsymbol{m}} + 3\nu^2 \boldsymbol{m} = \boldsymbol{c} \tag{5.3.13}$$

where $\boldsymbol{c}$ is an arbitrary constant vector introduced by three constants of integration. Such a phase space obviously generalizes, by dimension at least, the phase plane of a regular harmonic oscillator: as one can easily see, if we settle for a plane of coordinates in (5.3.6), we get a two-dimensional harmonic oscillator. The solution of (5.3.13), on the other hand, is offered by the velocity vector:

$$3\nu^2 \boldsymbol{m} = \boldsymbol{c} + \boldsymbol{a}\cos(\nu t\sqrt{3}) + \boldsymbol{b}\sin(\nu t\sqrt{3})$$

with $\boldsymbol{a}$ and $\boldsymbol{b}$ some initial conditions. It is located on the homogeneous quadratic cone, having the equation

$$[(\boldsymbol{b}\times\boldsymbol{c})\cdot\boldsymbol{m}]^2 + [(\boldsymbol{c}\times\boldsymbol{a})\cdot\boldsymbol{m}]^2 - [(\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{m}]^2 = 0 \tag{5.3.14}$$

The coefficients are here decided by the initial conditions.

We can even complicate a little the equations of motion (5.3.6), admitting a gauging where the velocity $\dot{\boldsymbol{m}}$ has also a component along the ray. Such a component would correspond to a propagation phenomenon along a light ray. This extension can be made in the spirit of a unitary description of the light phenomenon, which would thus include both the propagation – measured always along the ray – as well as the light motion proper – measured orthogonally to the ray – in describing the light. Mention should be made that such a situation corresponds to the motion of an electric charge in the field of a magnetic pole (Poincaré, 1896). Then the equations of motion corresponding to those from (5.3.6) are:

$$\frac{dm_1}{lm_1 + p(m_2 - m_3)} = \frac{dm_2}{lm_2 + p(m_3 - m_1)} = \frac{dm_3}{lm_3 + p(m_1 - m_2)} \tag{5.3.15}$$

where *l* and *p* are two parameters representing the 'amounts' in which the motion is decomposed *along the ray* and *perpendicular* to it, respectively. The integration procedure described above, leads to a differential form a little more complicated than (5.3.8), *viz*.:

$$\frac{adm_1 + bdm_2 + cdm_3}{[la + p(c-b)]m_1 + [lb + p(a-c)]m_2 + [lc + p(b-a)]m_3} = \nu dt$$

which can be considered an exact differential:

$$\frac{adm_1 + bdm_2 + cdm_3}{n(am_1 + bm_2 + cm_3)} = \nu dt \quad \therefore \quad am_1 + bm_2 + cm_3 = Ae^{n\nu t} \tag{5.3.16}$$

if, and only if, *a*, *b*, *c* are solution of the linear algebraic system given by:

$$(l-n)a + p(c-b) = 0$$

and its positive permutations. This system has nontrivial solutions only if the constants *l*, *p* and *n* satisfy the algebraic equation:

$$(l-n)[(l-n)^2 + 3p^2] = 0$$

which offers three possibilities of construction of the differentials representing the corresponding kinematics. They are given by the system of values:

$$l = n \quad \therefore \quad a = b = c$$
$$l - n = ip\sqrt{3} \quad \therefore \quad a = jc; b = j^2c$$
$$l - n = -ip\sqrt{3} \quad \therefore \quad a = j^2c; b = jc$$

Formally, then, nothing changes with respect to the preceding simpler case: it is just that we have here to do with a *harmonic of the frequency ν*, rather than with the frequency itself.

The analogy with the classical case can still be taken to work in this case, because there is a 'hidden dynamics' involved here, and this is, we think, the right place to bring about the name pf Paul Appell, mentioned above in connection with the equation (5.3.10). This dynamics appeared for the first time in 1893 [see (Appell, 1893), on page 351], but only as an exercise. Quoting:

> A point is moving in space, under the action of a force whose components *X*, *Y*, *Z* are functions of *x*, *y*, *z*, which verify the relations

$$\frac{\partial X}{\partial x} = \frac{\partial Y}{\partial y} = \frac{\partial Z}{\partial z}, \quad \frac{\partial X}{\partial z} = \frac{\partial Y}{\partial x} = \frac{\partial Z}{\partial y}, \quad \frac{\partial X}{\partial y} = \frac{\partial Y}{\partial z} = \frac{\partial Z}{\partial x} \tag{5.3.17}$$

Prove that the integration of the equations of motion is reduced to quadrature. [(Appell, 1893), *Exercise 16, p. 351, our translation*]

The proof is simple: first, one has to define a *complex position vector*, having as components the three complex coordinates as in equation (5.3.10). Then we need to define a *complex force vector*, having as components the three corresponding complex quantities, constructed from the real components of force in the same manner the coordinates are constructed. Obviously, the principles of analysis allow us to infer that, if the real forces are functions of real position, the complex forces must be functions of complex positions. Therefore, using our notations for the coordinates along the ray, in the following table constructed by the rules just mentioned:

$$\begin{gathered} x^1 = m_1 + m_2 + m_3, \quad x^2 = m_1 + jm_2 + j^2 m_3, \quad x^3 = m_1 + j^2 m_2 + jm_3 \\ X^1 = X + Y + Z, \quad X^2 = X + jY + j^2 Z, \quad X^3 = X + j^2 Y + jZ \end{gathered} \tag{5.3.18}$$

every variable of the second line should be a function of the variables from the first line. Then notice that, under the conditions (5.3.17), each of the components of complex force thus defined – assumed conservative, of course – is a function only of the corresponding complex coordinate from the first line. Therefore, the differential equations of motion can be written as

$$\frac{d^2 x^1}{dt^2} = X^1(x^1), \quad \frac{d^2 x^2}{dt^2} = X^2(x^2), \quad \frac{d^2 x^3}{dt^2} = X^3(x^3) \tag{5.3.19}$$

and can be solved by integrating twice, indeed. The property is transmitted as such over to the real corresponding quantities, because the transformations (5.3.18) are always nonsingular. So, the Appell's result is proved.

Following the historical development of the theoretical physics, one can say that the above coordinates ***m***, or their complex form ***x***, describing the condition of analyticity, are destined to offer the theoretical platform allowing us to understand and describe the transition from classical to quantum mechanics. We shall follow here this line closely. First notice that the equation (5.3.6) defines the necessary momenta associated to these coordinates, by a 'Newtonian cosmological recipe', we might say. However, the observation that the above 'dynamics' is in many respects analogous to Poincaré's classical description of the motion of a charge in a magnetic pole (Poincaré, 1896), may be taken to suggest something more important: the universal applicability of the theory. To wit: the *idea of helix*, for instance, characterizing the nuclear matter in all of its determinations, suggests that the transition between the transversal motion and the longitudinal motion is of essence: a band is always created, for example, on the wall of an extant ray. This means a connection between the parameters *l* and *p*, an idea worth following in the case of nervous matter.

5.4. A Transition Between Realities: the Physical Necessity of Surfaces

It is a common habit in theoretical physics, which started with Louis de Broglie and Erwin Madelung, that when it comes to the concept of interpretation, the regular procedure of approach in the wave mechanics, is its constant appeal to the theory of fluids: the particles of a fluid are taken as the usual interpretative particles of matter. This conclusion, shaped gradually along the history of physics was, indeed, realized and best 'cast into modern equations', so to speak, by Roman Jackiw and his group, with remarkable inferences for the benefit of the modern theoretical physics (Jackiw, Nair, Pi, & Polychronakos, 2004).

The classical Lagrange formulation of the hydrodynamics of an ideal fluid asks for considering some 'labels' in order to 'tag' the particles of the fluid. As long as these particles maintain their identity during the flow of fluid, the labels could be given by the initial conditions of the motion of particle individually, which is, indeed, the classical case from which it all has started. Then, according to Jackiw, the core of hydrodynamical description of matter stays in the fact that it is equivalent to *the noncommutativity of the coordinates* of the particles (Jackiw, 2002), expressed through a non-Abelian scaling (Jackiw, 2004). The necessary groupal relations are, in principle, given by infinitesimal transformations of the form

$$X^i(t,\boldsymbol{x}) = x^i + \theta^i{}_j \cdot A^j(t,\boldsymbol{x}) \tag{5.4.1}$$

where A^j are scaling potentials, and θ_{ij} are entries of a skew-symmetric matrix. Now, such a matrix can be effectively constructed based upon the structure constants of any three-dimensional Lie algebra, using the scheme provided by the equations (5.3.3–5) for the rotation group algebra $\mathfrak{so}(3)$. Pending a geometrical and physical interpretation of the situation, an example shall be given presently, based upon the structure constants of the $\mathfrak{sl}$(2,R) algebra, in order to use in the introduction of surfaces in the theory of Planck rays.

In this particular case we shall take here the structure constants as provided by the values that can be extracted, for instance, from the structural equations (3.3.17) of this algebra, which provide the following values:

$$C^1_{12} = 1, \quad C^3_{23} = 1, \quad C^2_{31} = -2$$

any other one being zero. The identities $(\boldsymbol{c}^k)_{ij} \equiv C^k_{ij}$, counterpart of those from equation (5.3.5) from the case of the rotation group algebra, define the entries of the following skew-symmetric matrices, the analogs of the matrices (5.3.3) from that case:

$$\boldsymbol{c}^1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \boldsymbol{c}^2 = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ -2 & 0 & 0 \end{pmatrix}, \quad \boldsymbol{c}^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \tag{5.4.2}$$

Just like the matrices from equation (5.3.3), these matrices give a basis of linear independent matrices in the space of 3×3 skew-symmetric matrices, *i.e*. any skew-symmetric matrix, of the kind we are seeking in constructing the equation (5.4.1), can be uniquely written in the 'coordinate' form:

$$\boldsymbol{\theta} \overset{def}{=} v_k \boldsymbol{c}^k = \begin{pmatrix} 0 & v_1 & 2v_2 \\ -v_1 & 0 & v_3 \\ -2v_2 & -v_3 & 0 \end{pmatrix} \tag{5.4.3}$$

The coordinates of these matrices, as geometrical objects, are the parameters v_k. The commutation relations of the matrices (5.4.2) are

$$[\boldsymbol{c}^1,\boldsymbol{c}^2] = -2\boldsymbol{c}^3, \quad [\boldsymbol{c}^2,\boldsymbol{c}^3] == -2\boldsymbol{c}^1, \quad [\boldsymbol{c}^3,\boldsymbol{c}^1] = -(1/2)\boldsymbol{c}^2 \tag{5.4.4}$$

and they reveal a closed algebraic system of dimension three. Obviously, if we take $(1/2)\boldsymbol{c}^2$ instead of $\boldsymbol{c}^2$, then the commutation relation become those of the Euclidean rotation group:

$$[\boldsymbol{c}^1,\boldsymbol{c}^2] = -\boldsymbol{c}^3, \quad [\boldsymbol{c}^2,\boldsymbol{c}^3] == -\boldsymbol{c}^1, \quad [\boldsymbol{c}^3,\boldsymbol{c}^1] = -\boldsymbol{c}^2 \tag{5.4.5}$$

We have here the case of a genuine rotation group related to what we would like to call *a Shchepetilov-type frame* for the case of 3×3 matrices (Shchepetilov, 2003).

A generalization of the equation (5.47) of the first instalment of this work, was given in (Cartan, 1927), defining the so-called *normal coordinates* **v** which enter the following differential equations involving an obvious matrix $\boldsymbol{\theta}$ through the equation (5.4.3):

$$\frac{d}{dt}\omega^k = dv^k + (\boldsymbol{c}^k)_{ij} v^i \omega^j \tag{5.4.6}$$

From the point of view of the differential geometry, this is the equation (5.1.3) or equation (5.1.6), pending a few natural identifications in terms of the vectors. What we mean by this, is that such an equation is able to provide a meaningful connection between the Poincaré's vectorial and scalar coordinates. The normal coordinates in question, are subject to the differential equation of evolution:

$$0 = dv_k + v_i C^i_{kj} a^j \tag{5.4.7}$$

so that if v_k are, again, *normal coordinates*, as in the Cartan's designation, they need, this time, to be covariant coordinates corresponding to the contravariant ones from the equation (5.4.6).

Unlike in the case of rotation group, one must take due notice of the difference between the two sets of matrices involved in the bilinear forms from the last two equations above. The first one of them, *viz*., equation (5.4.6), is defined, indeed, by the matrix from equation (5.4.3), but the equation (5.4.7) is defined by the matrices, $\boldsymbol{h}_i$ say, establishing another set of three linearly independent 3×3 matrices, which can be expressed by the structure constants of the $\mathfrak{sl}$(2,R) algebra, *via* an equation analogous to (5.3.4):

$$(\boldsymbol{h}_k)^i_j \stackrel{def}{=} C^i_{kj} \tag{5.4.8}$$

Explicitly these matrices are:

$$\boldsymbol{h}_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}, \quad \boldsymbol{h}_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \boldsymbol{h}_3 = \begin{pmatrix} 0 & 0 & 0 \\ -2 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} \tag{5.4.9}$$

and they represent, indeed, an $\mathfrak{sl}(2, \mathrm{R})$ Lie algebra: their commutation relations are the same as those of the group algebra in the standard form given by the chosen structure constants. The bilinear forms from equation (5.4.7) can then be written as the components of a covariant vector, having a great importance *as a transition vector* between two algebras otherwise usually intransitive. These components are

$$\xi_k \equiv a^i (\boldsymbol{h}_k)^j_i v_j \tag{5.4.10}$$

We call them 'transition forms' because these bilinear forms can play a double role: on one hand, as components of an Euclidean vector, and, on the other hand as the components of a $\mathfrak{sl}(2,\mathrm{R})$ vector. Indeed, the quadratic form representing the dot product of $|v\rangle$ and $|\xi\rangle$ in the sense of the $\mathfrak{sl}(2,\mathrm{R})$ algebra is null:

$$v_1\xi_3 + v_3\xi_1 - 2v_2\xi_2 = 0 \tag{5.4.11}$$

showing that the two vectors – the covariant $\boldsymbol{v}$ and $\boldsymbol{\xi}$ – can be, in this instance, interpreted as *coefficients of the two binary quadratic forms*, apolar with respect to one another. They can be taken as characterizing, for example, the deformation of a wave surface inside a de Broglie ray, as in the § 5.2. On the other hand, *the Euclidean dot product* between contravariant $|v\rangle$ and $|\xi\rangle$ is also null:

$$\xi_1 a^1 + \xi_2 a^2 + \xi_3 a^3 = 0 \tag{5.4.12}$$

showing that the two vectors are perpendicular in the usual Euclidean sense. An important case of normal Cartan coordinates is offered by the coefficients of the second fundamental form of a surface. The proof of the previous relations is based on the explicit form of the quantities (5.4.10):

$$\xi_1 = a^2 v_1 + 2a^3 v_2, \quad \xi_2 = -a^1 v_1 + a^3 v_3, \quad \xi_3 = -2a^1 v_2 - a^2 v_3 \tag{5.4.13}$$

These quantities are defined by a connection between the ensembles and the Cartesian frames. The equation (5.4.7) is essential! It provides the Euclidean momenta:

$$dv_j + \xi_j = 0 \tag{5.4.14}$$

or, in detail, along the $\mathfrak{sl}(2,\mathrm{R})$ geodesics:

$$\frac{dv_1}{d\theta} + a^2 v_1 + 2a^3 v_2 = 0, \quad \frac{dv_2}{d\theta} - a^1 v_1 + a^3 v_3 = 0, \quad \frac{dv_3}{d\theta} - 2a^1 v_2 - a^2 v_3 = 0 \tag{5.4.15}$$

This system can be solved by exponentiation, for we have

$$\frac{d}{d\theta}|v\rangle = \boldsymbol{Q}\cdot|v\rangle, \quad \boldsymbol{Q}^3 + q\cdot\boldsymbol{Q} = \boldsymbol{0}, \quad q \equiv 4a^1a^3 - (a^2)^2 \tag{5.4.16}$$

which fits perfectly into the *regularization scheme*. This result was first obtained by Baker (1901). The exponential is simply determined by the first and second powers of the Baker's matrix $\boldsymbol{Q}$. This matrix is:

$$\boldsymbol{Q} \stackrel{def}{=} \begin{pmatrix} -a^2 & -2a^3 & 0 \\ a^1 & 0 & -a^3 \\ 0 & 2a^1 & a^2 \end{pmatrix}, \quad \boldsymbol{Q}^2 = \begin{pmatrix} (a^2)^2 - 2a^1a^3 & 2a^2a^3 & 2(a^3)^2 \\ -a^1a^2 & -4a^1a^3 & -a^2a^3 \\ 2(a^1)^2 & 2a^1a^2 & (a^2)^2 - 2a^1a^3 \end{pmatrix} \tag{5.4.17}$$

The exponentiation gives:

$$e^{t\boldsymbol{Q}} = \boldsymbol{1} + \frac{\sin(t\sqrt{q})}{\sqrt{q}}\boldsymbol{Q} + \frac{\cos(t\sqrt{q})}{q}\boldsymbol{Q}^2 \tag{5.4.18}$$

In view of (5.4.16), this matrix satisfies the third order differential equation characterizing the regularization scheme:

$$\frac{d^3}{dt^3}e^{t\boldsymbol{Q}} + q\frac{d}{dt}e^{t\boldsymbol{Q}} = \boldsymbol{0} \tag{5.4.19}$$

This equation is to be compared with the equation (5.3.7), and this comparison tells us that the concept of a portion of surface serving in the construction of a de Broglie ray is a consequence of the requirement of analyticity, as defined in the Appell-Miles form (Miles, 1954). Inasmuch as this kind of equation is instrumental all along the regularization procedure of the classical Kepler problem, and as this procedure is scale-transient, the concept of de Broglie ray, itself, is instrumental at any scale and in any kind of Poincaré coordinates: vectorial as well as scalar coordinates. Thus, the 'Euclidean momenta' defined by equation (5.4.16) are actually a kind of *bona fide* Appell coordinates representing the background geometry of a proper continuum, just like the coordinates $\boldsymbol{m}$ and $\boldsymbol{x}$ from the previous § 5.3.

5.5. A Statistical Realization: the Uncertainty Relations

As we have seen in the § 2.5, the fundamental Fresnel principle in the physical theory of light involves an 'arousing' of forces by light within matter; this process can further qualify as *a quantum measurement of the forces thus created*. This situation is an expression of the Hertz's natural philosophy, according to which an interpretative particle *indicates a position*, and the wave-mechanical point of view is specified by the assumption that this position can never be measured precisely. As far as we aware, such a point of view was expressed just once, and in a radical view at that, by Vladimir Rojansky, along the general idea of association of quantum-mechanical operators to coordinates and times. In short, Rojansky's idea was that such operators exist according to fundamental relations of commutation imposed by the *second quantization* procedure, but *they can have no* *eigenvalues*. This idea is quite radical, indeed, inasmuch as it involves the second quantization procedure, that is, it disregards completely the original Planck's procedure of quantization. However, it has some merits that make it quite correct. Quoting:

> A *precise measurement of the position* of a particle cannot be made with a microscope. We hazard the guess that *such a measurement cannot be made with any apparatus*. We then recall that *the eigenvalues of the operator associated to a dynamical variable are the possible results of a precise measurement of this variable*. And we are thus led to conclude that the operator *X*, which is to be associated with what we may vaguely call the *x* coordinate of a particle, *must have no eigenvalues*. This argument, extended to include *y*, *z*, and perhaps also *t,* is the main point of this note [(Rojansky, 1955); *emphasis added*, *a*/*n*]

Taking the phrase 'precise measurement of the position', as defining the concept of location of an interpretative particle, in the sense defined by Hertz, one can see right away that Vladimir Rojansky expresses, in a way, an exactly opposite alternative to the classical one. However, we find Rojansky's approach significant insofar as he considers the *coordinates as dynamical variables* in the sense of second quantization (Snyder, 1947) in order to use the commutation relation to describe the lack of eigenvalues. This conception fits best our idea of physical charge connected to a certain interpretative particle able to support the Newtonian forces.

The main incentive of our choice of this line of thinking comes, nevertheless, from a development of Rojansky's idea, involving the *scale transitions*: the existence of functions with *double orthogonality* (Slepian & Pollack, 1961). These are functions orthogonal – according the a dot product involving their integration, necessary for the Fourier analysis – both on a *finite interval*, as well as on an *infinite one*, speaking in real terms. They can be realized by a Fourier analysis on finite intervals, which is essential for a characterization of the commutation relations by uncertainty (Silvert, 1970 b). One of the most important conclusion of this reasoning involves the connection between the wave function itself, and the probability amplitude generated by this function according to the well-known Max Born's recipe of interpretation of the wave function. Quoting:

> It appears that the above set of uncertainty relations will prove useful in situations where previous forms of these relations give only the trivial result $\Delta p \geq 0$. The *form of these relations also sheds light on the nature of the uncertainty principle itself*, which proves quite simply *to relate the Fourier components of the wave functions*, ψ, *to the Fourier components of the probability density*, $|\psi|^2$. [(Silvert, 1970 a); *emphasis added*, *a*/*n*]

According to this view, the wave function is the most important tool of our knowledge, insofar as, on one hand it is connected to the manner of defining the probability and, on the other hand, it should be a scale transcendent function. One of the main issues to be clarified here, is the connection between the coordinate and its corresponding momentum, which assumes the very definition of the wave function. Our idea is that the physical magnitude satisfying this requirement *is associated to the charge*.

The Wigner-type 'dynamics' from § 5.3 is connected, as already mentioned, with the poorly-considered problem of principle in the classical dynamics, as formulated by Wigner himself: to *express the fact that under no forces a material point remains at rest*. As we have seen, such a 'dynamics' can be useful from quite a few points of view. In the first place, it is quite adequate as a 'transverse dynamics', if we may, as applied to a de Broglie's ray. The development of this dynamics along the lines from the § 5.3, leads to the conclusion that its coordinate description is appropriate in the scale transitions involved in the regularization procedure. If this is the case, then we are encouraged to seek for a possibility of accomplishing a 'Gibbs program', as it were, of defining the fields according to the different scales of space and time (see § 2.5). As it turns out, this realization is closely connected with the transition from classical mechanics to quantum mechanics in quite a logical manner, involving the statistics.

The whole idea is that, the 'coordinates of rest', denoted $|m\rangle$ in the § 5.3 or, equivalently, their complex linear extensions denoted $|x\rangle$ there, are a kind of *special coordinates* satisfying the Appell-Miles conditions of a *three-dimensional type of analyticity* [see § 5.3 above; see also (Miles, 1954)]. They are, therefore, quite proper for a quantum construction using the concept of Lorentz quantum, for the realization of which we shall use the uncertainty relations, but not in the Heisenberg's original form. Our guidance in this endeavor is taken from an exquisite work of John Lighton Synge on the most general form of the uncertainty relations (Synge, 1971). The reason of this choice of ours rests with a mathematically significant fact, easiest to understand with Synge's approach to uncertainty relations: the construction of a reference frame along a ray in its cross-sectional space is *statistically dominated by the variance of some quantum observables*. Therefore, such an option is quite understandable in view of the observation that we take as fundamental in physics, namely that the variance function of a statistical distribution is instrumental in the Planck's procedure of quantization: Synge's remark is that the variance can provide a statistical geometry in the transversal space of a certain ray. Let us present his theory in its broad strokes (Synge, 1971).

The essential observation that makes the variance a fundamental statistic in characterizing the uncertainty is, according to Synge, the possibility of defining statistically a set of unit vectors, therefore the possibility of constructing a 'statistical geometry', so to speak. This possibility was implicit in the Schrödinger's work referring to an amendment of the original uncertainty relations of Heisenberg's (Schrödinger, 1930 a), which, further, facilitates an interpretation of electricity in the spirit of the definition of photon, as given by Gilbert Newton Lewis

[(Schrödinger, 1930 b); see § 2.3 above]. This amendment comes quite naturally with the *conditional definition* of a quantity in a given direction, for instance in the case of light along a ray, or in the case of the current of charges along a conductor. In view of its importance of principle in connection with a Gibbs-type program of theoretical physics, we present this theory here as associated with a gauge theory of the classical mechanics of material point (Debbasch, 1993).

Fabrice Debbasch had the idea – apparently continued and improved over time, judging by his own recent works – that the inertial types of forces on a classical material point, can all be represented in a kind of gauge theory by constructing an invariant Lagrangian entirely analogous to the classical kinetic energy for the free particle. Here we present the Debbasch's construction of the Lagrangian, since this construction turns out to be universal: it is valid within the Euclidean geometry presented in § 5.3, as well as in the $\mathfrak{sl}(2,\mathrm{R})$ geometry presented in the § 5.4. Thus, in a given reference frame the action itself involves a Lagrangian which is a quadratic form in the most general gauge rates of the rotation group of the medium described in that reference frame. Let us construct those rates. The action of rotations upon a certain position in matter can be represented as the action of a matrix on a *ket*:

$$|r'\rangle = \boldsymbol{R}\cdot|r\rangle \tag{5.5.1}$$

where $\boldsymbol{R}$ is the rotation operator, so that we can write:

$$|dr'\rangle \equiv d(\boldsymbol{R}\cdot|r\rangle) = \boldsymbol{R}\cdot|dr\rangle + d\boldsymbol{R}\cdot|r\rangle$$

This can be written in the form

$$|dr'\rangle = \boldsymbol{R}\cdot(|dr\rangle + \boldsymbol{A}\cdot|r\rangle);\ \ \boldsymbol{A} \equiv \ \boldsymbol{R}^{-1}\cdot d\boldsymbol{R} \tag{5.5.2}$$

thus making obvious the source of non-invariance of an incidental Lagrangian. Defining in general, *i.e*. regardless of the form of $\boldsymbol{A}$, the differential of position vector by:

$$|Dr\rangle = |dr\rangle + \boldsymbol{A}\cdot|r\rangle \tag{5.5.3}$$

the invariant Lagrangian is a quadratic form, just like its ancestor, the kinetic energy of particle:

$$L \overset{\text{def}}{\equiv} \frac{1}{2}\langle Dr\,|\,Dr\rangle \tag{5.5.4}$$

provided $\boldsymbol{A}$ has the following transformation property:

$$\boldsymbol{A}' = \boldsymbol{R}\cdot\boldsymbol{A}\cdot\boldsymbol{R}^{-1} - d\boldsymbol{R}\cdot\boldsymbol{R}^{-1} \tag{5.5.5}$$

Using the exponential group of rotations:

$$\boldsymbol{R} = \exp(\Omega^k \boldsymbol{h}_k) \tag{5.5.6}$$

where $\boldsymbol{h}_k$ are the three skew-symmetric base matrices of the rotation algebra, given in equation (5.3.3), so that $\boldsymbol{A}$ from equation (5.5.2) is a 'pure gauge' field, defined by

$$\boldsymbol{A}\cdot|r\rangle \equiv \boldsymbol{\Omega}\times\boldsymbol{r} \tag{5.5.7}$$

The rate from equation (5.5.3) becomes

$$\left|Dr\right\rangle = \left|dr\right\rangle + \boldsymbol{\Omega} \times \boldsymbol{r} \tag{5.5.8}$$

And, therefore, the invariant Lagrangian (5.5.4) can be written in the form

$$L = \frac{1}{2}\left\langle dr \mid dr \right\rangle + d\boldsymbol{r} \cdot (\boldsymbol{\Omega} \times \boldsymbol{r}) + \frac{1}{2}(\boldsymbol{\Omega} \times \boldsymbol{r})^{2} \tag{5.5.9}$$

leading to the following form of the second principle of this dynamics:

$$d^{2}\boldsymbol{r} = \boldsymbol{r} \times d\boldsymbol{\Omega} + 2(d\boldsymbol{r} \times \boldsymbol{\Omega}) + (\boldsymbol{\Omega} \times \boldsymbol{r}) \times \boldsymbol{\Omega} \tag{5.5.10}$$

Clearly all the inertial forces due to a substratum appear in this equation of motion: the Coriolis and the centrifugal forces are given by the second and third terms from the right hand side rerspectively.

However, this is not our main contention here: rather, the essential observation is that the place of the basic rotation matrices $\boldsymbol{h}_k$ in defining the rotation matrix $\boldsymbol{R}$ which, in turn, defines such a 'rotating' gauge, can be taken very well by the $\mathfrak{sl}(2, R)$ matrices from equations (5.4.2) or (5.4.9), describing the situation of a surface state in the case of a ray, in general. In other words, if necessity occurs of a physics involving more general inertial fields, we have at our disposal the algebra $\mathfrak{sl}(2, R)$ in two different instances. Ther matrices equivalent to R in these two situations are obtained by exponentiation, using, for instance the method of exponentiation given in § 5.4 above, when the matrix Q was calculated, as in equation (5.4.18). This is, for instance, the case of a gauge involving the charge *via* the Planck quantization, as presented by us in § 3.2.

Therefore, limiting the presentation to only the statistical side of the physics – which, concretely, means no scale transition, we must mention here the Synge's way of approach, by a generalization of the uncertainty relations due, as we said to Schrödinger. Synge's observation is relatively simple, allowing to treat the statistics geometrically – a routine in the theory and practice of the statistics itself, as a matter of fact – but with the explicit intervention of physics. Quoting:

> The familiar process of finding, *relative to a vector x*, the expectation $\bar{A}$ of the self-adjoint linear operator A, and then using the deviation $A-\bar{A}$ to obtain a dispersion ΔA, *may be interpreted as a projection leading to a unit vector e*, orthogonal to x [(Synge, 1971), Abstract; *emphasis added*, *a*/*n*]

The proof is fairly simple, and we will not insist on it here. Rather, we use this opportunity in order to discipline the future developments in this very work. The most important observation here is that the transversal Wigner motion, responsible for the definition of rest, must have a statistical component, and this component is due to the charge. Indeed, according to Katz's natural philosophy of charge, there are two pairs of unit vectors that can be generated stochastically, in a process involving the 'extraction' of the charge from the de Sitter continuum that makes the background for all existences in the physical universe. This statistics justifies the Lorentz's definition for the electric matter (see § 4.4), giving the possibility of describing the basic surface – a

Stroud sphere for instance – by its deformations. This means that Synge's scheme must be conceived within the algebra of matrices, not in that of the complex numbers. Such a description of the charge was conceived quite a while ago, in the form of a special algebraic structure [see (Donth, 1988); also (Donth & Lange, 1986)]

6. Under the Spell of Planck: the Theoretical Physics of Charge and Matter

Planck's original quantization is referring to the physical universe we inhabit: this means *a world of light and matter*. Along this line of reasoning, as a universe, the brain should then be considered *a world of charge and matter*. Even the 'matter' has different meaning in the two worlds, which is an obvious fact, if only we start from the observation that the rays are naturally provided in the brain, by the very organization of matter, while in the physical universe they need to be 'constructed' and, moreover, they are fictitious. The rays are, however, real in an imaginary experience of the brain universe: one can say that the correlation real-fictitious has different terms, depending on the universe we are considering. In this respect, whithin the brain universe the Maxwell hypothesis (see § 4.4 above) may be closely respected *ad litteram*, insofar as everything is in place in this universe, exactly as in the case of classical electrodynamics.

Our reader may have raised the legitimate question about the conservation laws: the Planck's equation (3.1.1) has nothing to do *directly* with a conservation law in the classical sense of the word. It makes a stationary value of the magnitude u characterizing a physical system depend on many – ideally *all* – of the details of a universe containing the physical system described by u. The case in point, taken to sustain this statement of ours, is that of the light, of course: assuming that the universe we inhabit is a Wien-Lummer enclosure, the Planck's equation (3.1.1) correlates the different measurements of its spectral density of energy.

However, mathematically speaking, the stationarity of u establishes a Riemannian geometry of its fluctuations compatible with this condition [the equations (3.1.3) *ff*]. If there are conservation laws in this physics, they can only be expressed through some invariants of the geometry thus constructed. This instance involves the quantization in a fundamental way in physics, a way in which it was involved *only once* in the history of humanity, in a case that we considered in specific details above, with the benefit of some learning points. This is the case of the BKS theory, that called into question, for the first time in the modern history, the exact form of the conservation laws for energy and momentum (Bohr, Kramers & Slater, 1924). In spite of the fact that this theory was rejected and, apparently, discarded, it had the power of a natural philosophy, for it produced, directly we might say, the modern quantum and wave mechanics.

The present chapter is dedicated to the causes and consequences of *Planck's differential equation* (3.1.2), in order to see what the brain universe can teach us, either in the matters generating it, or in the matters that are created by it. The way we can assimilate these teachings is, first of all, through some conservation laws. Let us see how these appear in connection with Planck's differential equation, and what is their meaning.

6.1. Two Important Suggestions from Theoretical Physics

Within the previous discussion, the explanation of fields $|e\rangle$ and $|b\rangle$ goes along with the concept of interpretation: in order to apply the definition of Hertz for a material particle, a charge must indicate a location, in the first place. According to Feynman's philosophy of interpretation, 'the charge is either here, or there, or anywhere else within the space of an instanton, but wherever it is, it indicates a location in that space'. The fields $|e\rangle$ and $|b\rangle$ though, are considered to act *between* 'here and there, or here and anywhere else, or even there and anywhere else', taken as locations existing *simultaneously* in the space of the instanton. The simultaneous existence involves a static ensemble of charges can be understood in a special logic. More to the point, this should be a *Hegelian logic*, allowing us to think that, in the fields $|e\rangle$ and $|b\rangle$, the motion and displacement must be 'sublated and, at the same time, preserved'. So, $|e\rangle$ and $|b\rangle$ are, primarily at least, an expression of 'immediacy', as it were [we have used terms borrowed from the English rendition by George Di Giovanni of the Hegel's *Science of Logic*, especially *The Doctrine of Being* (Hegel, 2010)]. In this take, the transformation (3.5.1) simply means the way in which the interpretation *per se emerges* into physics from its immediacy. In fact, this is the whole point of the Ampère conception of infinitesimal current elements [see (Mazilu, 2024), §1.5].

In physics' terms, these statements can be expressed by turning to the arguments once presented by Ernst Katz, on the occasion of discussion of his natural philosophy of charges (Katz, 1965). One must notice that the Maxwell tensor of the electromagnetic field is invariant with respect to duality rotations, thereby involving just one variable, *the duality angle* in the description of charges [see (Katz, 1965); equations (13–15)]. As a consequence, the Newtonian character of the static electromagnetic forces is thereby expressed by a difference of 90° between the duality angle of electric charges and that of magnetic charges [see (Katz, 1965), equation (22–23)]. However, the electric and magnetic forces are not alone in the physical universe – the universe we inhabit. As a matter of fact an old theorem of Earnshaw on static electricity assures us that there is no equilibrium state of a purely electric matter: even intuitively, for equilibrium one must also consider the Newtonian force of gravitation [see (Mazilu, 2020), §5.1]. This makes the Katz's condition on the phase-difference of electric and magnetic charges irrelevant for a universe concept, in general. That is to say that, in general, *the angle difference between the two kind of charges must be arbitrary*, not just 90°: it can take any value, in principle, depending on the universe, or even on parts of universe. This means, on the other hand, that the duality transformation must be generalized to a transformation depending on some other parameters too, which is exactly what we just accomplished with the invariance of the Maxwell tensor above (see Chapter 3 above, especially § 3.5).

At this point we stop for an acknowledgment: in view of the idea of quadratic realization of the statistics in the transversal space of a ray (see § 5.5 right above), in unfolding the present instalment of this work, we found particular inspiration, and encouragement in fact, in the physical theory of the accelerators [see for a comprehensive presentation (Lee, 2021), especially Chapter 2 of the book]. It is in this theory that an invariant occurred for the first time (Courant & Snyder, 1958), which a decade later (Lewis, 1967) has been linked with the idea of quantization along the line of Planck's procedure. The fundamental – in our opinion, obviously – realization of the theoretical work on accelerators just cited can be suggestively presented along the following lines.

Let us work on constructing the most general solution of a second order differential equation, having an Yang exponential as generic solution of the type given in equation (3.2.1). To wit: the exponential we denoted $X(\alpha)$ is just a particular type of solution of second-order differential equation. Thus, we are entitled to talk of the solution of the second-order equation:

$$X''(\alpha)+e^2X(\alpha)=0 \tag{6.1.1}$$

which must be of the most general form, with the amplitude depending on the independent variable too:

$$X(\alpha)=w(\alpha)\cdot e^{i\psi(\alpha)} \tag{6.1.2}$$

The two functions $w(\alpha)$ and $\psi(\alpha)$ are then to be determined from the system of equations, obtained when introducing the form (6.1.2) of the signal into equation (6.1.1):

$$\frac{w''(\alpha)}{w(\alpha)}+e^2=\left[\psi'(\alpha)\right]^2,\ 2\frac{w'(\alpha)}{w(\alpha)}+\frac{\psi''(\alpha)}{\psi(\alpha)}=0 \tag{6.1.3}$$

The second of these equations can be integrated right away to give:

$$w^2\cdot\psi'=const \tag{6.1.4}$$

so that the amplitude of the general signal (6.1.2) is a solution of the *Ermakov-Pinney equation*:

$$w''+e^2w=\frac{R^2}{w^3} \tag{6.1.5}$$

where R is an arbitrary constant, assumed real for the moment. The equation (6.1.4) can be solved right away for ψ, because the square of amplitude satisfies a third-order linear differential equation:

$$(w^2)'''+4e^2(w^2)'=0 \tag{6.1.6}$$

which can be proved by direct calculation using (6.1.5). This last equation appears to be the key in finding the general solution (6.1.2): on one hand, finding its solution means finding $w(\alpha)$ directly, which, calculated from a solution of the linear equation (6.1.6), retains just the indeterminacy of the square root. On the other hand, inserting directly such a solution in (6.1.4) for a given value of the constant – which, by the way, is a second indeterminacy of this problem – provides the phase by integration, thereby introducing a third indeterminacy. This will provide the signal (6.1.2) in the most general form, depending on three constants, but in an apparently arbitrary way, that is, not as we presented it in the § 5.3.

The whole point of the physics involved in this theory, stays in the connection of the physical magnitudes represented by w and X, and their variations. More to the point, there is a *joint invariant* of the two magnitudes involving their rates too (Courant & Snyder, 1958). In order to see this, we can proceed directly, by building a first integral to (6.1.5), along the following algebraical way. First multiply (6.1.5) by $2w'$, after which we integrate. This produces the constant:

$$I \stackrel{\text{def}}{=} (w')^2 + e^2 w^2 + \frac{R^2}{w^2} = const \tag{6.1.7}$$

Obviously, this constant plays the part of a Hamiltonian for the equation (6.1.5), taken as an equation of motion. But there is more to this procedure than it appears at such a superficial sight [see (Mazilu, 2020), especially §5.3]. Consider, indeed, from the same point of view, the two equations (6.1.1) and (6.1.5), jointly. Using (6.1.4) we find an equation for the ratio of the two functions:

$$\frac{d^2}{d\psi^2}\left(\frac{X}{w}\right) + R^2\left(\frac{X}{w}\right) = 0 \tag{6.1.8}$$

In other words, the ratio (X/w) behaves like a harmonic oscillator with a new frequency, given by the coefficient of the forcing term from equation (6.1.5). Then, applying the previous procedure of integration and going back to the original independent variable, we find the invariant function:

$$I \equiv R^2 \frac{X^2}{w^2} + (X \cdot w' - w \cdot X')^2 = const \tag{6.1.9}$$

Except the equation (6.1.6) these are, in broad strokes, the basic points of both the theory of accelerators (Courant and Snyder, 1958), as well as of the theory of quantization *per se* (Lewis, 1967). The invariant (6.1.9) is the one that was discovered by Harold Ralph Lewis (1967) in connection with the quantization procedure for the nonlinear harmonic oscillator [see also (Symon, 1970) for a correlation between the two important moments in the physics of the last century]. Lewis insisted on the idea that the equation (6.1.5) is an *auxiliary equation* and came with a lengthy argument in making this point [(Lewis, 1967); see also the longer article (Lewis, 1968) where the author unfolds the theory 'at ease', so to speak, and acknowledges the priority of Courant's and Snyder's work from 1958 on the subject (Footnote #2 of the paper just cited)].

Fact is that the particle physicists were interested mainly in the *propagation of the particle beams*, and this is, at least as far as we are concerned, the main point of the whole theory. Thus, originally, Courant and Snyder postulated a matrix of the form we gave here in equation (3.5.8) as a condition of conservation of the Maxwell tensor, working on it by using w as an auxiliary function, indeed. In time, the method was refined with the recognition that the quantization may play an important part in such a scenario after all (Symon, 1970). So, finally the theory of transverse motions of the beams in accelerators got the nice closed form existing today [(Qin & Davidson, 2006, 2009); see also (Lee, 2021), Chapter 2]. However, the Lewis' approach to quantization has another fundamental meaning, again, at least as far as *we are concerned*: he constructed the invariant (6.1.9) from the integral of the classical action, *over cycles in the phase plane*. This means that the Planck's constant can be identified with the value of this invariant, and the quantization procedures applied to different material systems provide different 'quantization constants'. The original Planck's constant is an expression of quantization only for the electromagnetic interaction, and only for the Maxwellian electromagnetism (Boyer, 1969)! The quantization of other systems produces some other 'Planck's constants', as it were.

Assume, now, that the equation (6.1.4) is the expression of the second of the Kepler's laws of the planetary motions – *the area law*. The phase ψ is then the angle, ϕ say, of orientation of the position vector with respect to the center of force, in the plane of motion. Then, if the charge satisfies an oscillator equation in the time defined by the second of the Kepler's laws, the invariant (6.1.9) constructed with the charge and radial coordinate of the motion is a Newtonian force. Thus, one can conclude that these forces are the first expression ever of the procedure of quantization, 'legalized' as it were, by the natural philosophy of Max Planck from the year 1900.

The particle physicists' approach in working with the quadratic form (6.1.9) endorses, in fact, the declared 'objective' of Harold Ralph Lewis, namely of considering the Ermakov-Pinney equation as ancillary, while the harmonic oscillator equation is taken as fundamental from a physical point of view. However, the working procedure involves the quadratic form (6.1.9) that we assume to be the quantum in this case, based on the way it is deduced by Harold Lewis. Importantly enough, though, it is taken as a quadratic form *in the phase plane* of the *transversal* variable X. So, keeping in mind that the transversal motion has two degrees of freedom, this fact complicates the problem of description of the transversal motion. In any case, we can at least say that *one* of the two degrees of motion, has the quantum:

$$I = \begin{pmatrix} X & X' \end{pmatrix} \cdot \begin{pmatrix} (w')^2 + \dfrac{R^2}{w^2} & -ww' \\ -ww' & w^2 \end{pmatrix} \cdot \begin{pmatrix} X \\ X' \end{pmatrix} \qquad (6.1.10)$$

and Lewis' objective is thus explicitly served by the dependence of the matrix of this quadratic form on the auxilliary function $w(\alpha)$. We shall adopt this approach here, but with the important change in emphasis, as already noticed above. Namely, if any particle of the incidental interpretive current of particles along the ray, is considered as having two charges, electric and magnetic – it is a *dyon*, as we have shown beforehand in § 4.5 – there are *two phase planes* in this physics to be recognized in the transversal dynamics of the current of particles: one for the electric charge of the dyon, the other for the magnetic charge. Each one of these planes represents a geometrical degree of freedom. There will be, therefore, two different quanta of the form (6.1.10), and our problem is to construct a correspondence between them, in order to build a geometry of the two transversal variables, X and Y say, based on their phase planes (X, X') and (Y, Y').

At this point, it is worth our while insisting on the equation (6.1.9) from the classical point of view of Newtonian forces. Let us transcribe that invariant, with the understanding that X represents the charge, Q say, while *w* represents the standard deviation of the distance between the interpretative particles over their ensemble, *r* say. Then the invariant is:

$$I \equiv R^2 \frac{Q^2}{r^2} + (Q \cdot r' - r \cdot Q')^2 = const \tag{6.1.11}$$

One can see, indeed, that this invariant has the classical Newtonian force, especially if the term in paranthesis is absent. This happens when the charge dictates the variance of the interpretative ensemble: the term is zero over the ensemble if $Q/r = constant$. Tentatively, one can say that this last condition defines the 'information' to be attached to the charge: it is practically, the standard deviation of the distance between material points over the interpretative ensemble. It is important to notice, though, that this constant has the remarkable form of the Newtonian potential.

6.2. The General Theory of a Quantum

Let us summarize once more the results thus far, regarding the concept of a quantum. According to its use in physics, this concept has a *twofold meaning*, if we may be allowed an extension of the common language: *as a quantity*, the quantum must be the value of *a quadratic form*, constant under a certain gauging of its variables; on the other hand, *as a physical object* the quantum can only be a 'chaotic' ensemble of waves as defined by Hendrik Antoon Lorentz: *a Lorentz quantum*. In this last instance, the quantum is closer to the old definition given by Bernhard Riemann for a 'part of a manifold'. Let us recall that notion. Quoting:

> *Parts of a manifold* distinguished by a characteristic or a boundary are called *quanta*. Their comparison regarding quantity takes place in case of *discrete extensions by counting*, in case of *continuous extensions by measurement*. The measuring consists in a superposition of the extensions to be compared; for measuring, therefore, a means is required to carry one extension as a meter for the other. If this is lacking, one can only compare two extensions if one is a part of the other, and even then only *to decide for the more or less*, *not for how much*. The investigations which may be made of them in this case represent a general part of the doctrine of extensions, independent of measures, where the extensions are not regarded as independent of location, and not as expressible by a unity, *but as domains in a manifold*. For several parts of mathematics, *especially for the treatment of multivariate analytic functions*, such investigations have become a necessity, and their absence is probably a major cause that the famous Abel theorem and the achievements of Lagrange, Pfaff, Jacobi for the general theory of differential equations were unfruitful for so long. For the present purpose it suffices to emphasize two points from this general part of the doctrine of the extended quantity, where nothing else is assumed, other than what is

already contained in the concept of quantity, of which the first is *the generation of the concept of a multiply extended manifold*, and the second concerns *the reducing of the positional determinations in a given manifold to quantitative determinations*, thus making clear the essential characteristic of an n-fold extension. [(Riemann, 1867 a); *our translation and emphasis*, *a*/*n*]

On one hand, this excerpt from Riemann, raises the problem of connection between the matrix of a quantum *conceived as a quadratic form*, and the physics of a Lorentz quantum. On the other hand, it raises the problem of connection of the variation of the quadratic form and the calibration of its variables. The present section of our work is dedicated to this last issue. So, with no further ado, let us start with an old-style analytic theory of the quadratic forms.

Assume, first, that the quadratic form represents a quantum as a quantity, denoted by *h*, as initially done by Planck. This last mention is intended to show the possible origin of this quantitative characterization: taken as the value of such a quadratic form, it can be, incidentally, the Planck quantum. On the other hand, the symbol *h* may be taken as meaning Gold's '*h*andedness', thus indicating that such a quadratic form may measure some information content. This 'information' comes, analogously, with the information content of the *second fundamental form* of a surface, whereby this quadratic form contains information *ad litteram* on the local shape of the surface by comparison with the tangent plane. We write the quadratic form as:

$$h \equiv \langle x|\boldsymbol{h}|x\rangle \tag{6.2.1}$$

where the symbol $\boldsymbol{h}$ for the tensor of this quadratic form is, likewise, intended to replicate the symbol for the quantum, or for the 'handedness', in the case of charge. In view of the observations from the § 5.5 above, we think it proper to give it, in fact, the name of *quantum tensor*. The ket $|x\rangle$ is a 'position' in the *transversal twofold* of the beam of particles making, for instance, a ray, as in § 5.3.

Assume, further, that the variation of this quadratic form can be calculated as in the classical theory of differentials, that is, by the rules of usual Newtonian differentiation. Start, therefore, with the basic equation representing the differential of the quantum according to the classical rules of differentiation, whereby the algebraic expression of the differential involves three distinct terms:

$$dh = \langle x|d\boldsymbol{h}|x\rangle + \langle dx|\boldsymbol{h}|x\rangle + \langle x|\boldsymbol{h}|dx\rangle \tag{6.2.2}$$

Assuming, further, that $|dx\rangle$ is defined by a *gauging* in the form

$$|dx\rangle = \boldsymbol{a}\cdot|x\rangle \tag{6.2.3}$$

where $\boldsymbol{a}$ is a matrix with differential entries, the equation (6.2.2) becomes

$$dh = \langle x|(d\boldsymbol{h} + \boldsymbol{a}^t\boldsymbol{h} + \boldsymbol{h}\boldsymbol{a})|x\rangle \tag{6.2.4}$$

Notice the specific definition (6.2.3) in this idea of gauging: it realizes a connection between the finite and infrafinite measures of positions in the transversal twofold. First of all, by such a definition the emphasis is placed on the scale transitions: from finite to infinitesimal measures. Secondly, this way, the emphasis is shifted upon the *differentiability of the matrices* that characterizes this transition. So the matrices should have *physical meaning* too in this geometry, just like the coordinates, which we take as being, indeed, the case.

With this last observation in mind, we go for a few mathematical details regarding the result contained in equation (6.2.4). Notice, in the first place, that for a skew-symmetric matrix product $\boldsymbol{h}\cdot\boldsymbol{a}$, of the quantum tensor with the gauging matrix, the variation (6.2.4) of the quadratic form of the quantum is strictly defined by the variation of the quantum tensor. Indeed, if $\boldsymbol{h}\cdot\boldsymbol{a}$ is a skew-symmetric matrix, we have $(\boldsymbol{h}\cdot\boldsymbol{a})^t \equiv \boldsymbol{a}^t\cdot\boldsymbol{h} = -\,\boldsymbol{h}\cdot\boldsymbol{a}$, the sum of the two matrix products in equation (6.2.4) is the matrix having zeros as entries, and thus the equation for the variation of the quantum reduces to:

$$dh = \langle x | d\boldsymbol{h} | x \rangle \tag{6.2.5}$$

If we know the symmetric quantum matrix $\boldsymbol{h}$, this condition helps in constructing the very matrix $\boldsymbol{a}$ serving for gauging. In other words, the variation of the quantum in a neighborhood of a certain point of the transversal manifold is controlled by the gauging equation (6.2.3), with the matrix $\boldsymbol{a}$ given by an equation of the form:

$$\boldsymbol{a}^t \cdot \boldsymbol{h} = (da)\boldsymbol{i}, \quad \boldsymbol{i} \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{6.2.6}$$

Here da is a differential factor, and the skew-symmetric matrix $\boldsymbol{i}$ has the property:

$$\boldsymbol{i}^2 = -\boldsymbol{1} \tag{6.2.7}$$

where $\boldsymbol{1}$ is the identity matrix. Thus, the matrix $\boldsymbol{i}$ is the counterpart of the imaginary unit of the complex numbers. Solving (6.2.6) for $\boldsymbol{a}$ produces the matrix:

$$\boldsymbol{a} = -(da)\boldsymbol{h}^{-1}\boldsymbol{i} \quad \therefore \quad \boldsymbol{a} \equiv \frac{da}{\Delta}\begin{pmatrix} \beta & \gamma \\ -\alpha & -\beta \end{pmatrix} \tag{6.2.8}$$

where α, β, γ are the entries of the quantum tensor $\boldsymbol{h}$:

$$\boldsymbol{h} \equiv \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} \tag{6.2.9}$$

and $\Delta \equiv \det(\boldsymbol{h})$. The gauging matrix $\boldsymbol{a}$ thus defined is, of course, only dependent on the quantum tensor, not on its variation, and replicates the property (6.2.7) of the matrix $\boldsymbol{i}$, which, in turn, replicates the property of the imaginary unit from the case of complex plane of numbers. More specifically we have, by a simple calculation or, even simpler, using the Hamilton-Cayley equation:

$$\boldsymbol{a}^2 = -(da)^2\boldsymbol{1} \tag{6.2.10}$$

The theory can be constructed based on the same principles as those of the construction of second fundamental form of the surfaces, that is, the quantum is analogous to the 'height' of the surface as measured by its support function. The wavelength, to be sure, is an example of such a quantum: it has been used almost exclusively as such in optical physics!

Now, a few more words on the gauging condition of the transversal twofold, contained in its definition by equation (6.2.3). As we said, that equation associates the differentials of the coordinates around a point in the transversal twofold, with the differential properties contained in the structure of the matrix $\boldsymbol{a}$. Fittingly, there are in this particular construction – and we take here 'particular' as meaning that the construction *is based on differentials* as in the classical geometry – still other forms of the matrix $\boldsymbol{a}$, that allow for a definition of the variation of the height of surface *via* the variation of its associated tensor alone. Indeed, the equation (6.2.4) shows that we can define the very variation of the quantum tensor $\boldsymbol{h}$ according to one of the two possibilities:

$$d\boldsymbol{h} + \boldsymbol{h} \cdot \boldsymbol{a} = \boldsymbol{0} \quad or \quad d\boldsymbol{h} + \boldsymbol{a}^t \cdot \boldsymbol{h} = \boldsymbol{0} \tag{6.2.11}$$

Actually, these conditions turn out to be completely equivalent, in view of the symmetry of the height tensor and the properties of the classical differential operation used here. For, in this case, the matrix $d\boldsymbol{h}$ respects, in fact, the algebraical symmetry of the original matrix $\boldsymbol{h}$. Therefore, the whole argument of the case can be taken as showing that *we must have symmmetry* instead of *skew-symmetry* for the product of matrices involved in the two expressions from the above equation (6.2.11), that is:

$$\boldsymbol{ha} = \boldsymbol{a}^t \boldsymbol{h} \tag{6.2.12}$$

Should a variation of the quantum tensor occur, whereby the symmetry of this tensor is lost, this last condition is broken, and the matrix $\boldsymbol{a}$ can be defined in two different ways at will by equation (6.2.11). As it stands now, however, the matrix $\boldsymbol{a}$ is uniquely defined, for instance, by the first of the relations (6.2.11) as:

$$\boldsymbol{a} = -\boldsymbol{h}^{-1} d\boldsymbol{h} \quad \therefore \quad \boldsymbol{a} = -(dn)\boldsymbol{1} - \boldsymbol{E} \tag{6.2.13}$$

where the following notations have been used:

$$n \equiv \ln\sqrt{\Delta}, \quad \boldsymbol{E} \stackrel{def}{=} \begin{pmatrix} -(1/2)\omega^2 & -\omega^3 \\ \omega^1 & (1/2)\omega^2 \end{pmatrix} \tag{6.2.14}$$

The matrix $\boldsymbol{E}$ has the differential forms of the $\mathfrak{sl}(2,R)$ coframe from equation (3.1.3) as entries. In view of the symmetry of the quantum tensor $\boldsymbol{h}$, these differential forms can be transcribed here as:

$$\omega^1 = \frac{\alpha d\beta - \beta d\alpha}{\Delta}, \quad \omega^2 = \frac{\alpha d\gamma - \gamma d\alpha}{\Delta}, \quad \omega^3 = \frac{\beta d\gamma - \gamma d\beta}{\Delta} \tag{6.2.15}$$

This coframe is to be met, as such, in the physical theory of the de Broglie waves associated with optical rays in the case of light [(Mazilu, 2020); see Chapter 3 there, especially equation (3.3.15); the whole §3.3 contains the essential algebraical properties of the $\mathfrak{sl}(2,R)$ manifold, based on these differentials]. The coframe (6.2.15) satisfies the Maurer-Cartan equations:

$$d \wedge \omega^k = C^k_{ij} \omega^i \wedge \omega^j \tag{6.2.16}$$

where the structure constants are taken as

$$C^1_{12} = 1, \quad C^3_{23} = 1, \quad C^2_{31} = -2 \tag{6.2.17}$$

the rest of them being zero. This algebraic structural arrangement for the $\mathfrak{sl}(2,R)$ algebra is taken as standard in the present work, in matters of discussion of the essential properties of the Riemann manifolds of negative curvature in low dimensions. What we have in mind when saying 'low dimensions', are some typical examples that will occur herein with the dimensions 2, 3, and 5.

Now, just for a future theoretical benefit, we need to bring in an important observation: instead of equation (6.2.5), for this case we can have – by using for instance the first of the conditions (6.2.11) in equation (6.2.4), and then the second of (6.2.11) in the result thus obtained – the following condition:

$$dh = \langle x | \boldsymbol{a}^t \boldsymbol{h} | x \rangle \quad \therefore \quad dh = -\langle x | d\boldsymbol{h} | x \rangle \tag{6.2.18}$$

In other words, in this case, the variation of quantum has the same magnitude as that from equation (6.2.5) but is opposite in sign. One can say that there are two possibilities of gauging of the transverse twofold, not just the one given by the equation (6.2.3): if in equation (6.2.3) we use for $\boldsymbol{a}$ the matrix (6.2.13), then the variation of quantum has the same magnitude as that resulting from a gauging with a matrix $\boldsymbol{a}$ from the equation (6.2.8), but with opposite sign. The conclusion can be drawn, that if there is a posibility of simultaneous gauging in the transversal twofold, the variation dh of the symmetrically distributed around zero: it is, for instance, normally distributed or Cauchy, which appears as quite natural from the Planck's quantization procedure (see §3.2).

6.3. A Vector Space of Physical Qualities

The first problem to be solved here, before anything else, is: how can an address on a joint action region be created? In order to see it, let us return to the algebra of matrices $\boldsymbol{A}$ from equation (4.1.2). It will help if we notice that the involution $\boldsymbol{I}$ can be considered as a matrix $\boldsymbol{A}$ of coordinates $p = 0$ and $q = 1$, the only one having a null trace from the family of these matrices:

$$\boldsymbol{I} \equiv \boldsymbol{A}_{01}, \quad \mathrm{tr}(\boldsymbol{A}_{01}) = 0 \tag{6.3.1}$$

As long as we consider the non-null trace as an essential condition in defining the family of matrices $\boldsymbol{A}_{pq}$, the matrix $\boldsymbol{A}_{01}$ does not belong to this family, for the mathematical condition for a non-null trace exacts $p \neq 0$. However, it can be used as a 'mark' of the family – which can be used as a kind of 'postal code', as it were, in communicating the messages according to Hillis' scenario – in the sense that it commutes with the whole family, in view of the fact that the composition formulas for the coordinates of a matrix in the family, given in equation (4.1.3), are symmetric in these coordinates:

$$\boldsymbol{A}_{01}\cdot\boldsymbol{A}_{pq}=\boldsymbol{A}_{pq}\cdot\boldsymbol{A}_{01}\equiv p\cdot\boldsymbol{A}_{01}+q\cdot\boldsymbol{A}_{01}^{2} \tag{6.3.2}$$

Therefore, the essential condition of 'secrecy' of communication from equation (4.1.6), is satisfied for the product of this matrix with anyone of the matrices of the family, and in this sense the matrix $\boldsymbol{A}_{01}$ can be taken as a 'mark' of this family. Such a 'mark' has three 'coordinates' that can serve to its 'location' and, according to this observation, it belongs to a three-dimensional linear space. This can be shown as follows.

Notice first, that the matrix $\boldsymbol{I}$ can be written as a product of other two involutive matrices, $\boldsymbol{I}_1$ and $\boldsymbol{I}_2$ say, in the form:

$$\boldsymbol{I}_1=\begin{pmatrix} a_1 & b_1 \\ c_1 & -a_1 \end{pmatrix},\quad \boldsymbol{I}_2=\begin{pmatrix} a_2 & b_2 \\ c_2 & -a_2 \end{pmatrix},\quad \boldsymbol{I}=\boldsymbol{I}_1\cdot\boldsymbol{I}_2=\begin{pmatrix} a_1a_2+b_1c_2 & a_1b_2-b_1a_2 \\ c_1a_2-a_1c_2 & c_1b_2+a_1a_2 \end{pmatrix} \tag{6.3.3}$$

However, this expression of $\boldsymbol{I}$ cannot be taken as operative but under the condition of null trace of the resulting product of the two matrices from the last equality here. That is, the conditions:

$$2a_{1,2}a+b_{1,2}c+c_{1,2}b=0,\quad 2a_1a_2+b_1c_2+c_1b_2=0 \tag{6.3.4}$$

must be satisfied. The first two conditions from this equation are naturally satisfied, for they are identities in view of the special form of the entries of $\boldsymbol{I}$ from equation (6.3.3); this can be easily verified by direct calculation. The third condition in (6.3.4), however, which expresses the fact that $\boldsymbol{I}$ is an involution, must be observed when the choice is made for the two factors $\boldsymbol{I}_1$ and $\boldsymbol{I}_2$ of $\boldsymbol{I}$, for the result of their product must be an involution. This condition sets a constraint on the *homographic actions* of those two factors. In order to see this, consider those numbers preserved by the homographic action of $\boldsymbol{I}$. There are two such numbers, the roots of equation:

$$\frac{ax+b}{cx-a}=x\quad\therefore\quad cx^2-2ax-b=0 \tag{6.3.5}$$

So, these two numbers are common to the whole family $\boldsymbol{A}_{pq}$: any matrix of the family has *the same* numbers preserved by its homographic action, as one can see right away by a direct calculation based on equation (4.1.2), which defines the family. This condition is a consequence of the commutativity of $\boldsymbol{I}$ with the whole family $\boldsymbol{A}_{pq}$ engendered by the pair of matrices (**1**, **I**). So, from the point of view of communications, the two *fixed numbers*, representing the roots of the equation (6.3.5), *can play the part of a public key* – a 'postal code', as we said above – for the family (4.1.2).

Now, the two involutive factor-matrices, $\boldsymbol{I}_1$ and $\boldsymbol{I}_2$, have, each one of them, particular fixed numbers too, given by the roots of the corresponding equations analogous to (6.3.5). Then the last condition from (6.3.4) represents the fact that these two pairs of numbers are in a *harmonic sequence*, and each of them is in harmonic sequence with the pair of fixed points of $\boldsymbol{I}$. A harmonic sequence of two pairs of numbers means that one of their six possible cross-ratios is -1 [see (Burnside & Panton, 1960), for an introduction to the issue of *cross-ratios*, rendered in this classical treatise under the name *anharmonic ratios*; see especially §38, and §68 Example 16 of that work; see also (Needham, 2001), Chapter 3, §§ V and VI, for more advanced analytical issues on this subject].

At this point it is worth noticing again that there is an essential advantage of choosing the involution $\boldsymbol{I}$ and its factors – if the parameters are out for a choice, of course! – such that the conditon:

$$a^2 + bc = 1 \tag{6.3.6}$$

is satisfied. In other words, we assume that the family (4.1.2) is located on a *quadratic surface* in the space coordinated by parameters (a, b, c). For a future convenience, we need to notice further that this condition is invariant to all the choices that are obtained from a given one by the transformation:

$$a \leftrightarrow -a, \quad b \leftrightarrow m \cdot b, \quad c \leftrightarrow m^{-1} \cdot c \tag{6.3.7}$$

where, in general, m can even be complex. So, in such an instance, we are free to choose *as an address* for a ‘postal code’ in a Hillis’ kind of scenario, any real number for a, combined with any complex number of modulus $|m| = \sqrt{(b/c)}$. Therefore, keeping in mind that a JAR needs itself an address, this address may be represented as a ‘postal code’ made out of a pair of parameters a and m over the surface (6.3.6), much in the way John Stroud once wanted. This will help us later, when resolving for the matrix $\boldsymbol{I}$ in different physical conditions.

Meanwhile let us just remark that the matrix $\boldsymbol{I}$ was given, and under these very conditions at that, in § 3.5 above, in equation (3.5.10), for instance. This observation should be able give us the opportunity of a *physical interpretation of a matrix* $\boldsymbol{A}_{pq}$. For now, though, let us consolidate, in this view, the algebraical results just obtained above, into a *bona fide* algebra.

Let us denote uniformly, by analogy with the unit vectors ($\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$) from the case of regular Euclidean space, with ($\boldsymbol{I}$, $\boldsymbol{J}$, $\boldsymbol{K}$) the triad of matrices ($\boldsymbol{I}$, $\boldsymbol{I}_1$, $\boldsymbol{I}_2$) satisfying the conditions (6.3.6). These three matrices comply with the algebra characterized by the following multiplication table, that can be derived directly from equation (6.3.3) under condition (6.3.6):

$$\begin{array}{ccc} \boldsymbol{I}^2 = -\boldsymbol{1}, & \boldsymbol{J}^2 = \boldsymbol{1}, & \boldsymbol{K}^2 = \boldsymbol{1} \\ \boldsymbol{I} = \boldsymbol{J} \cdot \boldsymbol{K}, & \boldsymbol{J} = \boldsymbol{K} \cdot \boldsymbol{I}, & \boldsymbol{K} = \boldsymbol{I} \cdot \boldsymbol{J} \\ \boldsymbol{I} \cdot \boldsymbol{J} + \boldsymbol{J} \cdot \boldsymbol{I} = \boldsymbol{0}, & \boldsymbol{J} \cdot \boldsymbol{K} + \boldsymbol{K} \cdot \boldsymbol{J} = \boldsymbol{0}, & \boldsymbol{K} \cdot \boldsymbol{I} + \boldsymbol{I} \cdot \boldsymbol{K} = \boldsymbol{0} \end{array} \tag{6.3.8}$$

Based on this multiplication table, which, obviously, defines a non-commutative algebra, we can prove that the three involutions are *linearly independent*, that is:

$$u\boldsymbol{I} + v\boldsymbol{J} + w\boldsymbol{K} = \boldsymbol{0} \quad \Leftrightarrow \quad u = v = w = 0 \tag{6.3.9}$$

The proof of this condition of linear independence is quite simple, in fact: just left-multiply by $\boldsymbol{I}$ the first equality here, in order to get

$$-u \cdot \boldsymbol{1} + v \cdot \boldsymbol{K} - w \cdot \boldsymbol{J} = \boldsymbol{0}$$

Taking now the trace of this matrix we get $u = 0$; then, repeating the same procedure by the multiplication with $\boldsymbol{J}$ and $\boldsymbol{K}$ respectively, instead of $\boldsymbol{I}$, and taking the traces along, we find $v = w = 0$, and our statement is proved: the three involutive matrices are, indeed, linearly independent, and they can be taken as a basis for a three-dimensional 'space' of involutive matrices. Let us, therefore, reveal now the most important kind of 'metric property' of this 'space' that may be able to open some understanding of the essence of the classical physics from the point of view of the concept of interpretation. This fact is expected to clarify a connection between the analyticity of functions (see § 5.3) and the staftistics involved in a quantum measurement with a ray (see § 5.5)

As we said, any involutive matrix can be written, in a unique way, as a linear combination of the three involutions respecting the multiplication table (6.3.8). Thus, for an arbitrary $\boldsymbol{X}$, we have:

$$\boldsymbol{X} = u\boldsymbol{I} + v\boldsymbol{J} + w\boldsymbol{K} \quad \Leftrightarrow \quad \boldsymbol{X}^2 = (-u^2 + v^2 + w^2)\cdot \boldsymbol{1} \tag{6.3.10}$$

The quadratic form in the right hand side of the last equality plays the part of a *norm of the involution*, helping us in designing an interpretation of a Planck medium in a universe in general, therefore as a differentia of the concept of universe. The interpretation, however, involves *particles in equilibrium under the natural Newtonian forces* represented by the involutions (6.3.10) of null norm [see (Mazilu, 2024), §§ 5.3 and 5.4]. Having two charges – electric and magnetic – these particles are classical in essence (Schwinger, 1969), and came to be accepted as *dyons*, but in the quantum theory of fields (Witten, 1979), since the classical physics does not recognize them: as any interpretative particles, they are simply fictitious.

Mention should be made – to our own benefit this time, – of what Edward Witten specially emphasizes in the work just cited above: the modern field quantization does not work for the charges *per se*, because, classically, they are not constrained by something like a quantization condition; *they can be any real number*. In order to make the quantization of charges effective, we need to supplement the algebra of charges with a space inversion, followed by charge conjugation condition of fields. We see in this condition a reflection of the classical Lorentz requirement in defining the charges [see (Lorentz, 1892), §69]. Along the same line, the Feynman *definition of positron* (Feynman, 1948), makes use of a condition of the same nature: if we are referring a pair annihilation to *the charges instead of particles*, in the world lines formalism, the positron can be seen as an electron "directed backwards in time". As expected, such a conclusion is supported exclusively by a wave-mechanical view of the world. This very fact, plus the holographic principle, allow us to accommodate the Thomas Gold's view, but referring to charges (see § 2.3 above): 'a certain « handedness » of the charges' is effective, even in the world of our experience.

6.4. Generating an Address on a JAR: Neurophysiological Charges

The right residing place of the dyons is, actually, in the classical physics or, more properly expressing the idea, there is no classical physics at all. We have to see physics simply as … physics, a science helping the natural philosophy: not classical, not quantal, not relativistic, etc – just physics. The case of dyons, just mentioned, will be carried all along the unfolding of the present story and, starting from the highly inspiring work of Julian Schwinger just cited above, it illustrates the interpretation *as a general concept of physics*, no matter how we happen to label it: classical, quantal, wave-mechanical, and such like. Let us illustrate the case of dyons for an arbitrary universe, with application for the brain and physical universe.

Respecting the old Schwinger's symbolism, we represent a dyon by an involutive matrix of *electric charge* e and *magnetic charge* g, in the form:

$$\boldsymbol{D} \stackrel{def}{=} e\cdot\boldsymbol{J}+g\cdot\boldsymbol{K} \tag{6.4.1}$$

where we use the symbols of the § 6.3 above. Obviously, $\boldsymbol{D}$ here stands for *d*yon. Then two such dyons make a message matrix $\boldsymbol{A}_{pq}$ by their product:

$$\boldsymbol{D}_1\cdot\boldsymbol{D}_2=(e_1\cdot\boldsymbol{J}+g_1\cdot\boldsymbol{K})\cdot(e_2\cdot\boldsymbol{J}+g_2\cdot\boldsymbol{K})=(e_1e_2+g_1g_2)\cdot\boldsymbol{1}+(e_1g_2-g_1e_2)\cdot\boldsymbol{I} \tag{6.4.2}$$

Consequently, Schwinger's 'combinations … invariant under the redefinitions produced by a duality rotation' are, in fact, the coordinates (p, q) of a matrix $\boldsymbol{A}_{pq}$, within a family given by equation (4.1.2). They are two bilinear forms in the charges (e, g) of our experience:

$$p=e_1e_2+g_1g_2,\quad q=e_1g_2-g_1e_2 \tag{6.4.3}$$

This is, therefore, how one constructs the families of matrices serving for communication in the brain, according to a modern procedure involving a public key cryptosystem: *a product of two dyons*. Notice that for two arbitrary dyons, there are two families of commutative matrices, for the dyons (6.4.1) do not commute:

$$\boldsymbol{D}_2\cdot\boldsymbol{D}_1=(e_1e_2+g_1g_2)\cdot\boldsymbol{1}-(e_1g_2-g_1e_2)\cdot\boldsymbol{I} \tag{6.4.4}$$

One can see that only the coordinate q changes the sign with respect to the one in (6.4.3).

One necessary explanation: *what does a dyon represent*? We have here two kinds of charges: the ones, coordinates of dyons, denoted (e, g) from the absolute de Sitter common background of the universes, while others are the coordinates of the products of two such dyons, and are bilinear forms in the coordinates of such dyons. We can expect these to be dyons within the specific background of a given universe. The brain constitutes such an example, whereby the 'nervous matter' creates charge from the common background of the possible universes of our experience.

Notice, indeed, in view of the previous observations, that the Schwinger's 'invariant parameters' (6.4.3) can be taken as charges too, but of the kind created in the brain, therefore a *physiological kind of charges*, so to speak. They are created, for instance, by a neuron from classical charges, and are, algebraically speaking, bilinear forms of the kind of the coefficients of Newtonian forces. The charges e and g exist, and are created, for instance, by the motional activity of the human being. The physiological charges (p, q), on the other hand, are created by the nervous cells out of the background of the brain universe. They only constitute *unique coordinates of a neuron*, independently of any joint action region: a genuine 'cell' address in a Hillis' type scenario.

Therefore, to round up the conclusions of the previous § 4.1, in view of the present observations we can state that any matrix representing a cell is a product of two dyons, each one of them 'chosen' by matter – specifically, the 'nervous matter' in the case of man as a physical system – from the *two cycles* of the Ernst Katz's phenomenology of the charge [see (Katz, 1965), Figure 1]. The cryptosystem we just illustrated in § 4.1., based on the homographic action of the matrices, asks for the *a priori* existence of the matrices in order to be possible to present a message. In view of this occurrence, it will be necessary to solve another important problem, this time it is of the physical nature: the problem of the physical connection between the physical charges and the physiological charges. In other words, given an *a priori* matrix (4.1.2), we need to know when the coordinates of a matrix $\boldsymbol{A}$ in its own family can be expressed in the form (6.4.3), with e's and g's arbitrary. To this end, we first need to solve (6.4.3) for $|e^2\rangle$ in terms of $|e^1\rangle$; in an obvious notation for the components of the kets, this gives:

$$e^2 = \frac{1}{(e^1)^2 + (g^1)^2}(pe^1 - qg^1),\ \ g^2 = \frac{1}{(e^1)^2 + (g^1)^2}(qe^1 + pg^1) \tag{6.4.5}$$

or symbolically:

$$|e^2\rangle = \frac{1}{\langle e^1|e^1\rangle} \cdot \boldsymbol{A} \cdot |e^1\rangle,\ \ |e\rangle \stackrel{\text{def}}{=} \begin{pmatrix} e \\ g \end{pmatrix},\ \boldsymbol{A} \stackrel{\text{def}}{=} \begin{pmatrix} p & -q \\ q & p \end{pmatrix} \tag{6.4.6}$$

The matrix $\boldsymbol{A}$ here, obviously, belongs to a group: it can be the identity matrix, for $p = 1$ and $q = 0$. The algebraical form of the matrix is preserved when changing the lines of the matrices:

$$|g^2\rangle = \frac{1}{\langle g^1|g^1\rangle} \cdot \boldsymbol{A}^t \cdot |g^1\rangle,\ \ |g\rangle \stackrel{\text{def}}{=} \begin{pmatrix} g \\ e \end{pmatrix},\ \boldsymbol{A}^t \stackrel{\text{def}}{=} \begin{pmatrix} p & q \\ -q & p \end{pmatrix} \tag{6.4.7}$$

It still remains a matrix which belongs to a group, just the skew-symmetrical part changes its sign. In other words, the transformation of the *physical charges* is given by a matrix of the form:

$$\boldsymbol{A} \stackrel{\text{def}}{=} p \cdot \boldsymbol{1} + q \cdot \boldsymbol{I}_0,\ \ \boldsymbol{I}_0 \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{6.4.8}$$

where p and q are *physiological charges*, no matter of their sign. In a word, a family of matrices having as entries physiological charges, represents transitions between what we would like to call *Katz cycles*.

The situation is quite different when we deal with the composition of the matrices $\boldsymbol{A}$ themselves, that is when, according to our philosophy, we are dealing with a connection between two JAR. Indeed, here we cannot have $\boldsymbol{A}$ as a matrix of a transformation group anymore, but of an algebra. Indeed, the equation (4.1.4) in the conditions of definition of the table (6.3.8) become

$$p_1 p_2 - q_1 q_2 = \alpha, \; p_1 q_2 + q_1 p_2 = \beta \tag{6.4.9}$$

so that solving like in (6.4.5) gives

$$p_2 = \frac{1}{p_1^2 + q_1^2}\left(\alpha p_1 + \beta q_1\right), \; q_2 = \frac{1}{p_1^2 + q_1^2}\left(\beta p_1 - \alpha q_1\right) \tag{6.4.10}$$

Thus, formally, we have instead of (6.4.6), the equation

$$|p_2\rangle = \frac{1}{\langle p_1 | p_1 \rangle} \cdot \boldsymbol{B} \cdot |p_1\rangle, \; |p\rangle \stackrel{\text{def}}{=} \begin{pmatrix} p \\ q \end{pmatrix}, \; \boldsymbol{B} \stackrel{\text{def}}{=} \begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix} \tag{6.4.11}$$

and instead of (6.4.7), the equation:

$$|q_2\rangle = \frac{1}{\langle q_1 | q_1 \rangle} \cdot {}^t\boldsymbol{B} \cdot |q_1\rangle, \; |q\rangle \stackrel{\text{def}}{=} \begin{pmatrix} q \\ p \end{pmatrix}, \; {}^t\boldsymbol{B} \stackrel{\text{def}}{=} \begin{pmatrix} -\alpha & \beta \\ \beta & \alpha \end{pmatrix} \tag{6.4.12}$$

where the upper index t means *transposed with respect the second diagonal*, since by this operation the matrix is obtained from the correponding one given in equation (6.4.11). Notice indeed, that these matrices cannot have the identity among them; also notice that they are a particular case of dyons as given by formula (6.4.1), with the particular choice:

$$\boldsymbol{J}_0 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; \boldsymbol{K}_0 \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{6.4.13}$$

whose product is $\boldsymbol{I}_0$.

We can extract the following rules, here, for a future reference: any transformation between charges is provided by dyons which do not have as entries physical charges, but exclusively physiological charges. More specifically, we have the following two cases:

(1) the transformations between the *physical charges* – equations (6.4.6) and (6.4.7) – are provided by dyons with physiological charges as entries, *i.e.*, having as entries addresses within families of commutative matrices. These dyons belong to a *group of matrices*, representing, for instance, ensemble of neurons acting jointly, or JARs, as we have called them. Notice that the entries of these dyons are generated by the coefficients of forces of Newtonian type, defined by the requirement of interpretation *via* an equilibrium (see § 4.3 above).

(2) the transformations between *physiological charges* – *i.e.*, between addresses over commutative families: equations (6.4.11) and (6.4.12) – is given by dyons having as entries adresses of matrices in the families. These dyons belong to an *algebra*, *not to a group*.

The dyons, therefore, associate pure electric with pure magnetic charges, independently of the space and time relative positions of these. The association is a matter of 'mind'! It might be of great interest for 'coding' problems that, classically speaking based on equation (6.4.13), there are four types of dyons. Recall that the physiological charges cannot be strictly of a kind: purely electrical or purely magnetic; they may mix the two cases, with the clear benefit of physiology. Thus, we may have dyons:

a) purely electrical:

$$\boldsymbol{V}_{ee} \equiv e^1 \boldsymbol{J}_o + e^2 \boldsymbol{K}_o = \begin{pmatrix} e^2 & e^1 \\ e^1 & -e^2 \end{pmatrix} \qquad (6.4.14)$$

b) electric-magnetic; this is the classical case, revealing two types:

$$\boldsymbol{V}_{eg} \equiv e\boldsymbol{J}_o + g\boldsymbol{K}_o = \begin{pmatrix} g & e \\ e & -g \end{pmatrix}, \; \boldsymbol{V}_{ge} \equiv g\boldsymbol{J}_o + e\boldsymbol{K}_o = \begin{pmatrix} e & g \\ g & -e \end{pmatrix} \qquad (6.4.15)$$

c) magnetic-magnetic:

$$\boldsymbol{V}_{gg} \equiv g^1 \boldsymbol{J}_o + g^2 \boldsymbol{K}_o = \begin{pmatrix} g^2 & g^1 \\ g^1 & -g^2 \end{pmatrix} \qquad (6.4.16)$$

With an appropriate geometrical theory of surfaces, these dyons may be made responsible for the four biochemical 'words', involved in the physical structure of the nucleic acids from the cells' nuclei: adenine, cytosine, guanine, and thymine (Gamow & Yčas, 1956).

6.5. Concerning the Actions of Brain Matter *on* Charge

Max Planck's quantization procedure, allows an idea of great importance for the imaginary experience of brain: the physical action of the brain matter, exerted on messages along the rays, *can be mathematically expressed as a homographic action* of the 2×2 matrices assumed, just for simplicity now, as having real entries. Thus, in the equation (3.1.1), the energy density u from the original Planck's equation must be replaced by a message, y say – in order to respect the previous notation from equation (3.2.7) and the following ones. This means that we virtually accept that the message is, statistically speaking, a variate Cauchy distributed, having therefore a verifiable mean, but with an undecided extension in variance or standard deviation – the known characteristics of such a distribution. Let us describe *the actions of a neuron*, corresponding to this philosophy.

In order to accomplish the Hillis' scenario of communication in the brain universe, we need to identify and describe that operation of a 'cell' by which this physical structure 'enhances of diminishes' the message it receives. Now, if the brain is considered a universe, the Planck's quantization procedure allows us to conceive such an operation in its utmost generality, keeping in mind the modern physiological achievements regarding the anatomy of the brain. Indeed, according to the modern understanding of the anatomy of neurons, the physiology distinguishes two kinds of actions of brain matter on the charges: first, is the *action of neuron's soma* on the collected *dendritic charges*, providing, after some 'digestion', as it were, a *unique result* to be directed along the axon. Secondly, there is the *action of dendrites and synapses*, along the axon for instance, in order to *extract* different messages from a unique one propagating along the axon, or *insert* different messages into it.

In Cartan's mathematical philosophy [see (Mazilu, 2024), §§ 1.3 and 1.4], the first type of action, *i.e*., that of the neuron's soma, producing a unique charge to be directed along the axon, can be represented by an equation entirely analogous to (3.1.1) from the case of Planck's quantization, that is realizing a correspondence of the type 'many to one':

$$y_0 = \frac{\alpha y + \beta}{\gamma y + \delta} \tag{6.5.1}$$

This equation tells us that the neuron's soma collects different messages y – ideally, an infinity – from its dendrites, transforming all of them into a *stationary message* y_0, to be directed along the axon *via* some dendrites or synapses towards the synapses of some other neurons, *via* their synapses. For a given set of values (α, β, γ, δ), there are just two possibilities of axon messages y_0: the two fixed points of the matrix realizing the homographical correspondence (6.5.1). However, if the set of values (α, β, γ, δ) is variable, there is just one such value, with an infinity of possibilities of its realization. The situation, in this case, is entirely analogous to a geometrical porism of the Steiner's type, once used by Harry Bateman to describe the atomic structure (see § 3.5 above).

From the same point of view of the brain universe, along an axon there are different dendrites or synapses which extract or insert different messages y, from or into a unique one y_0, circulating along the axon, and carrying the information provided by the neuron's soma. This kind of operation of the brain matter can be properly modeled with an equation of the same form as (6.5.1) but realizing a correspondence of 'one to many':

$$y = \frac{\alpha y_0 + \beta}{\gamma y_0 + \delta} \tag{6.5.2}$$

Let us recall once again: as in the original Planck's case, this mathematical philosophy of ours only works under assumption that the parameters (α, β, γ, δ), representing the coefficients of the transformation from equation (6.5.2) are taken as variable. This is what we assumed, in fact, with the 'porismical' case of equation (6.5.1). Obviously, this may physically happen as a result of an interaction of the system described by the two parameters (y, y_0) with the matter of brain, within a space where this interaction takes place. And, like in the Planck's original case, we shall asume, momentarily just for simplicity, such a variation to be continuous, thus allowing us to construct a metric geometry.

A short digression on the difference between physiology and physics is worth mentioning, just for the sake of a proper perspective: there is a neurophysiological difference between equation (6.5.1) and equation (6.5.2), of which the Planck's original procedure of quantization cannot account. Namely, in handling the equation (6.5.1) we must take in consideration the fact that it may be involved in the 'coding' *per se* – for the charges y are coming *towards the nucleus*, from 'upstream of it', as it were – thus containing, incidentally, a part that only transmits the genes 'downstream' [see (Kandel, 2006), Chapter 5, for a pertinent documentation of the matter electrical in the case of neurons]. The equation (6.5.2), on the other hand, cannot be submitted to such an observance: inasmuch as y_0 being an information 'downstream' of the nucleus, it only may involve 'non-coding' information of the kind once qualifying as 'junk' genetical material (see § 4.2 above). This is an important property of the life contained in the brain universe – the fundamental property of life, we should say – which the *physical universe cannot possess*. It is only building on it that we may be able to understand the 'objective existence of mind'.

Coming back to our equations, for the first case – the one of the collection of charges by soma – we have the equation (3.1.2) specific to this universe:

$$dy_0 = 0 \quad \therefore \quad dy = \omega^1 y^2 + \omega^2 y + \omega^3 \tag{6.5.3}$$

with the coframe given by equation (3.1.3), transcribed here for convenience:

$$\omega^1 = \frac{\alpha d\gamma - \gamma d\alpha}{\alpha\delta - \beta\gamma}, \quad \omega^2 = \frac{\alpha d\delta - \delta d\alpha + \beta d\gamma - \gamma d\beta}{\alpha\delta - \beta\gamma}, \quad \omega^3 = \frac{\beta d\delta - \delta d\beta}{\alpha\delta - \beta\gamma} \tag{6.5.4}$$

On the other hand, for the second case of action on charges – the extraction of charges from the unique charge carried by brain matter along an axon – the equation (6.5.2) can be written in the form:

$$y_0 = \frac{\delta y - \beta}{-\gamma y + \alpha} \tag{6.5.5}$$

and then the scheme (6.5.3) can be applied, leading, as before, to:

$$dy_0 = 0 \quad \therefore \quad dy = \bar{\omega}^1 y^2 + \bar{\omega}^2 y + \bar{\omega}^3 \tag{6.5.6}$$

The new coframe can be obtained from (6.5.4) by the formal replacements:

$$\alpha \leftrightarrow \delta, \quad \beta \to -\beta, \quad \gamma \to -\gamma \tag{6.5.7}$$

so that, for its components, we have the expressions:

$$\bar{\omega}^1 = \frac{\gamma d\delta - \delta d\gamma}{\alpha\delta - \beta\gamma}, \quad \bar{\omega}^2 = \frac{\delta d\alpha - \alpha d\delta + \beta d\gamma - \gamma d\beta}{\alpha\delta - \beta\gamma}, \quad \bar{\omega}^3 = \frac{\alpha d\beta - \beta d\alpha}{\alpha\delta - \beta\gamma} \tag{6.5.8}$$

The equations (6.5.3) and (6.5.6) are *Planck-type equations* like (3.1.2) above, but *for the brain universe*. They give us the opportunity of introducing the conservation laws in two different manners. First of all, *via* the coframes (6.5.4) and (6.5.8) that generate two $\mathfrak{sl}(2,R)$ Riemannian geometries. As it turns out, the components of the corresponding frames are Killing vectors of this algebra, and therefore the components of the coframes are conserved rates along the corresponding geodesics of the Riemannian metrics. Secondly, independently of the metric, the equations (6.5.3) and (6.5.6) generate two families of *conservation laws* in this universe, that can be interpreted according to classical natural philosophy of Newton. Let us insist here on these last conservation laws, by the way of closing these mathematical preliminaries.

We need to recall here an old geometrical theory of Cartan regarding the Riccati differential equation (Cartan, 1951) in order to make the position of the stationary values more meaningful. According to Cartan's theory, the integrability condition of the Planck's equation, written here generically in the form:

$$dY = \omega^1 Y^2 + \omega^2 Y + \omega^3 \tag{6.5.9}$$

is that its right-hand side should be an exact differential – the left-hand side already is – which, according to the rules of exterior calculus, means:

$$d \wedge \left(\omega^1 Y^2 + \omega^2 Y + \omega^3\right) = 0 \tag{6.5.10}$$

Here the sign '$d\wedge$' means exterior differentiation of the differential form that follows the sign '$\wedge$'. After performing the due calculations, by using the equation (6.5.9) itself in order to eliminate dY, this equation comes down to the following equations of structure for the coframe, given before in equation (6.2.16):

$$d \wedge \omega^1 = \omega^1 \wedge \omega^2, \; d \wedge \omega^2 = -2(\omega^3 \wedge \omega^1), \; d \wedge \omega^3 = \omega^2 \wedge \omega^3 \tag{6.5.11}$$

As these are the Maurer-Cartan structure equations characteristic of the $\mathfrak{sl}$(2,R) algebras, one can say that for this algebraical structure the condition (6.5.10) is always satisfied, which means that the equation (6.5.9) is always integrable. The coframes (6.5.4) and (6.5.8) are, therefore, such coframes, but the conservation law represented by equation (6.5.9) is by no means unique: let us adjoin to it some families of such laws.

6.6. Conservation Laws Generated by Quantization

The last observation of the previous section can be expanded, in order to obtain families of conservation laws generated by the Planck's procedure of quantization. In drafting this section of the present instalment of our work we benefited from the guidance of some distinguished works that rounded up, geometrically speaking, some conclusions of the important epoch that followed the (re)discovery of solitons in the 20th century. The first, and foremost of these works is (Sasaki, 1979). It shows that, he concept of soliton radically changed the ideas on conservation laws (Scott, Chu, & McLaughlin, 1973), allowing a nice geometrization of physics (Crampin, 1978) along the lines that follow in this section and the ones that come along after it.

For what follows right here, we write the equation (6.5.3) in the form defining a linear differential

$$\gamma^1 \stackrel{def}{=} dx - \omega^1 x^2 - \omega^2 x - \omega^3 \tag{6.6.1}$$

where, obviously, the action of matter on charges is understood as soma action on dendritic charges in order to create a new unique charge to be transmitted along the axon. This differential is exact in view of the structure of $\mathfrak{sl}$(2, R) algebra, shown in § 6.5. Indeed, for the exterior differential of (6.6.1) we have in general (Sasaki, 1979):

$$d \wedge \gamma^1 = -\gamma^1 \wedge \delta^1, \quad \delta^1 \stackrel{def}{=} 2\omega^1 x + \omega^2 \tag{6.6.2}$$

Let us prove this relation. The proof proceeds as follows: taking the exterior differential of γ^1, we get

$$d \wedge \gamma^1 = -(d \wedge \omega^1)x^2 - (d \wedge \omega^2)x - (d \wedge \omega^3) - dx \wedge (2\omega^1 x + \omega^2) \tag{6.6.3}$$

Now, assume that the coframe algebra is such that the structural equations of the $\mathfrak{sl}(2, R)$ algebra are of the form that we use regularly [see (Mazilu, 2024), § 1.4, equation (1.4.16)] which we transcribe here in the form:

$$d \wedge \omega^1 = \omega^1 \wedge \omega^2, \quad d \wedge \omega^2 = -2(\omega^3 \wedge \omega^1), \quad d \wedge \omega^3 = \omega^2 \wedge \omega^3 \tag{6.6.4}$$

In this case, we have for the first three terms from the right hand side of the equation (6.6.3):

$$(d \wedge \omega^1)x^2 + (d \wedge \omega^2)x + (d \wedge \omega^3) \equiv (\omega^1 \wedge \omega^2)x^2 - 2(\omega^3 \wedge \omega^1)x + (\omega^2 \wedge \omega^3) \tag{6.6.5}$$

In order to evaluate the exterior product from the right hand side here, we notice that

$$(\omega^1 x^2 + \omega^2 x + \omega^3) \wedge (2\omega^1 x + \omega^2) \equiv -(\omega^1 \wedge \omega^2)x^2 + 2(\omega^3 \wedge \omega^1)x - (\omega^2 \wedge \omega^3) \tag{6.6.6}$$

so that (6.6.5) becomes:

$$(d \wedge \omega^1)x^2 + (d \wedge \omega^2)x + (d \wedge \omega^3) = -(\omega^1 x^2 + \omega^2 x + \omega^3) \wedge (2\omega^1 x + \omega^2) \tag{6.6.7}$$

and, with this, the right hand side of (6.6.3) turns out to be the differential form of equation (6.6.1) up to a sign. So, the equation (6.6.2) is confirmed.

Thus, the equation (6.6.1) tells us that if $\gamma^1 = 0$, it is exact, *i.e.*, from the theoretical physics' point of view we have a conservation law. In § 6.5 we just assumed two specimens of such law, given in equations (6.5.1) and (6.5.2) which, at least intuitively, are able to ratify mathematically some physiological facts. The mathematical reason of these laws is quite simple: the invariance group of a Riccati differential equation is the group generated by the homographic action of the matrices. Thus, for instance in equation (6.5.1), y_0 and y can be cogitated as two solution of the same Riccati equation. The neurophysiology just comes here to establish that y_0 is constant, and thus it is declared a conservation law. The same happens for the equations (6.5.2); we shall return to this issue a little later in this work.

But the present mathematics actually shows that these are not the only conservation laws under that condition: δ^1 is also a conservation law. Indeed, we have

$$d \wedge \delta^1 = 2(\omega^1 \wedge \omega^2)x - 2(\omega^3 \wedge \omega^1) + 2(\omega^1 x^2 + \omega^2 x + \omega^3) \wedge \omega^1 \tag{6.6.8}$$

where in the process of exterior differentiation we used the structural equations (6.6.4) and the condition $\gamma^1 = 0$. Proceeding with the calculation in the right hand side of (6.6.8) just like before, the result turns out to be null, and so δ^1 is a conservation law, indeed. Going further along this line, we can exhibit another family of conservation laws, by noticing that, if $\gamma^1 = 0$, so is

$$\gamma'^1 \stackrel{def}{=} dy + \omega^3 y^2 + \omega^2 y + \omega^1, \quad y = x^{-1} \tag{6.6.9}$$

Therefore, *mutatis mutandis*, we have also the family of conservation laws, generated if $\gamma'^1 = 0$ by equation:

$$d \wedge \gamma'^1 = \gamma'^1 \wedge \delta'^1, \quad d \wedge \delta'^1 = 0, \quad \delta'^1 \stackrel{def}{=} 2\omega^3 y + \omega^2 \tag{6.6.10}$$

The two sets of conservation laws generated this way are not essentially different, for one set is a mathematical consequence of the other.

However, if instead of (6.6.1), we will consider the case of extraction or insertion of charges by synapses from or into the charge circulating along the axon, as in equation (6.5.6), then we shall have:

$$\gamma^2 \overset{def}{=} dy - \bar{\varpi}^1 y^2 - \bar{\varpi}^2 y - \bar{\varpi}^3 \tag{6.6.11}$$

with the coframe given as in equation (6.5.8). This Planck equation certainly leads, in general, to a different set of conservation laws, for instead of (6.6.2) we have:

$$d \wedge \gamma^2 = -\gamma^2 \wedge \delta^2, \quad d \wedge \delta^2 = 0, \quad \delta^2 \overset{def}{=} 2\bar{\varpi}^1 y + \bar{\varpi}^2 \tag{6.6.12}$$

which is, obviously, an entirely new set of conservation laws.

The two sets of conservation laws just gotten, *viz*. (6.6.2) and (6.6.12), can be interpreted as a counterparts of the Newtonian cosmogonic case involving the forces of universal gravitation that he discovered in a static state serving for the interpretation of the physical universe [see (Mazilu, 2024), §2.2]. In other words, Newton was right in extending cosmologically his gravitation law, but the quantum theory of Planck shows that, when it comes to physics, this extension is rather a physical property of 'nervous matter'. Only in this capacity can it be considered as a differentia of a concept of matter, and this says everything on the manner of constructing such a concept! We shall come on this issue quite a few times along this work.

7. Under the Spell of Chebyshev: the Properties of Dyons

Let us start with the theory of Newtonian forces, as it was applied by Ampère to currents: this assumes central forces which are then proportional to the Newtonian ones, with factors of proportionality that must be functions of the ratios of coordinates [(Mazilu, 2024); §1.5, equation (1.5.4)]. We assume that the ensemble of these factors of proportionality can be organized as a threefold, say with the help of three coordinates, (q_1, q_2, q_3) say, which are intended to remind us that Coulomb used charges when discussing the Newtonian forces. Therefore, in an ensemble of identical fictitious particles in equilibrium, serving the purpose of interpretation, the charge, in its dual definition, electric and magnetic, equals gravitational mass, up to a sign [(Mazilu, 2020), §5.1]. In order to throw some light on the problem, let us show where the classical duality transformation used by Julian Schwinger to help in defining the dyons, can find its place with the general definition of a dyon from the §3.5: a fictitious particle represented as a 2×2 matrix.

7.1. The Crampin's Theorem

Before a deeper consideration involving the subject of dyons in connection with this decomposition of the action of the background of a universe, we need to clarify here the idea of the *obligatory presence of a surface*. On one hand, this is claimed, by John Stroud, to serve for writing those 'files of memory' of the 'library' where the man learns the facts of the mind. On the other hand, it is claimed by Lorentz, in constructing an image of the electric matter that led to the theory of relativity. It is easy then, to grasp the idea that the matrix ***G*** accomplishing a connection between the plane waves and ray waves (see § 3.6 above), is prone to picture one of those dipoles necessary to the background of a Planck universe. According to this view, the dipole, as a fundamental physical structure of the universe, is necessary in order to accomplish a *connection between the free waves and matter waves* in a universal way, not just from the electromagnetic point of view, as Planck was compelled to assume for the prototype quantization (see § 5.5 above). For once this is a general property of the universe – a *differentia* of the concept, as it were – of the kind once claimed by the Kirchhoff's laws of radiation.

The conclusions above in § 3.6, show that the triangular matrix ***N*** is of essence in the construction of an Iwasawa decomposition for physical necessities, and this fact is more transparent if we write the decomposition (3.6.10) in the form:

$$\boldsymbol{G} \overset{def}{=} \boldsymbol{T}\cdot\boldsymbol{K}^{-1},\quad \boldsymbol{T} \overset{def}{=} \boldsymbol{N}\cdot\boldsymbol{A},\quad \boldsymbol{T}\equiv\begin{pmatrix} y^{-1} & xy \\ 0 & y\end{pmatrix} \tag{7.1.1}$$

Here the matrix ***T*** reproduces the property of triangularity of ***N***, and then, an algebraical theorem results, as represented by equation (Crampin, 1978):

$$\boldsymbol{T}^{-1}\cdot d\boldsymbol{T} = \boldsymbol{K}^{-1}\cdot d\boldsymbol{K} + \boldsymbol{K}^{-1}\cdot\boldsymbol{\Theta}\cdot\boldsymbol{K},\quad \boldsymbol{\Theta} = \boldsymbol{G}^{-1}\cdot d\boldsymbol{G} \tag{7.1.2}$$

This equation can be easily demonstrated, as a consequence of the definition of matrix ***G*** from equation (7.1.1). The definition produces the theorem from (7.1.2) by an algebraical combination with the following matrix differentiation rules:

$$d\boldsymbol{G} = d\boldsymbol{T}\cdot\boldsymbol{K}^{-1} + \boldsymbol{T}\cdot d\boldsymbol{K}^{-1},\quad d\boldsymbol{K}^{-1} = -\boldsymbol{K}^{-1}\cdot d\boldsymbol{K}\cdot\boldsymbol{K}^{-1} \tag{7.1.3}$$

applied to the definition of $\boldsymbol{\Theta}$ from the second equality (7.1.2).

Now, we can take the Crampin's formula from (7.1.2) as showing how the concept of surface enters the physics of charges, since ***G*** locally describes, indeed, a surface, and so does the matrix of differentials $\boldsymbol{\Theta}$. Thus, it is quite important to calculate the differentials of the parameters of Iwasawa decomposition: x, y and ϕ in terms of the initial form (3.6.5) of the matrix. It is this case that shows that the concept of surface becomes instrumental! In view of (3.6.9) and (7.1.1) we can write:

$$\boldsymbol{T}^{-1}\cdot d\boldsymbol{T}-\boldsymbol{K}^{-1}\cdot d\boldsymbol{K}=\begin{pmatrix} -y^{-1}dy & y^{2}dx-d\phi \\ d\phi & y^{-1}dy \end{pmatrix} \tag{7.1.4}$$

while, on the other hand, (3.6.5) provides the matrix of differentials

$$\boldsymbol{\Theta}=\begin{pmatrix} -\omega^{2}/2 & -\omega^{3} \\ \omega^{1} & \omega^{2}/2 \end{pmatrix} \tag{7.1.5}$$

where (ω^1, ω^2, ω^3) is the coframe from equation (3.1.3). An observation is in order here, in connection with this last form of the matrix: *it is an involutive matrix*. This property is assured by the same fact that warrants the Iwasawa decomposition: the matrix **G** must have a unit determinant, *i.e.* is a proper SL(2, R) matrix. Otherwise the matrix (7.1.5) would not be a traceless matrix, and the things get complicated. One can say that the Iwasawa decomposition is, in a way the guarantee that the matrix **G** is a matrix of unit determinant.

For now, though, coming back to the main streak of presentation, we get

$$\boldsymbol{K}^{-1}\cdot\boldsymbol{\Theta}\cdot\boldsymbol{K}=$$
$$\frac{1}{2}\begin{pmatrix} -\omega^{2}\cos\theta-(\omega^{1}-\omega^{3})\sin\theta & -(\omega^{1}+\omega^{3})-(\omega^{3}-\omega^{1})\cos\theta-\omega^{2}\sin\theta \\ (\omega^{1}+\omega^{3})-(\omega^{3}-\omega^{1})\cos\theta-\omega^{2}\sin\theta & \omega^{2}\cos\theta+(\omega^{1}-\omega^{3})\sin\theta \end{pmatrix} \tag{7.1.6}$$

where $\theta\equiv 2\phi$. When accounting for the conditions (7.1.4), (7.1.5) and (7.1.6), the Crampin's definition of the matrix **G** from equation (7.1.1) generates the following system of differential equations, allowing us to introduce the idea of surface along the lines of the Lorentz's theory of electricity:

$$\begin{aligned} 2y^{-1}dy &= \omega^{2}\cos\theta+(\omega^{1}-\omega^{3})\sin\theta \\ 2y^{2}dx &= d\theta-(\omega^{1}+\omega^{3})+(\omega^{1}-\omega^{3})\cos\theta-\omega^{2}\sin\theta \\ d\theta &= \omega^{1}+\omega^{3}+(\omega^{1}-\omega^{3})\cos\theta-\omega^{2}\sin\theta \end{aligned} \tag{7.1.7}$$

The general philosophy of this construction can be expressed thus: the action of the matrix **G** can be described in terms of the properties of an instanton, whose coframe is constructed on account of these equation.

A temptation for a speculation presents itself right away, that shall make the mathematics a little bit easier: namely, the third of the equations from (7.1.7) can be integrated right away as it is. Indeed, that equation can be written as a Riccati purely differential equation:

$$d(\tan\phi)=\omega^{1}\tan^{2}\phi-\omega^{2}\tan\phi+\omega^{3} \tag{7.1.8}$$

and along the geodesics becomes

$$\frac{d\tan\phi}{a^{1}\tan^{2}\phi-2a^{2}\tan\phi+a^{3}}=dt \tag{7.1.9}$$

where t is the affine parameter of the geodesics. This equation makes sense as a transformation of tangents (the so-called *Bäcklund transformation*) but only under condition:

$$\Delta \overset{def}{=} a^1 a^3 - (a^2)^2 > 0 \tag{7.1.10}$$

in which case the transformation has, indeed, the generic form of a Bäcklund transformation:

$$a^1 \tan\phi = a^2 + \sqrt{\Delta}\cdot\tan(t\sqrt{\Delta}) \tag{7.1.11}$$

The second part of the temptation presented to us for speculation, consists of combining the first and the third of equations from (7.1.7) in order to get and equation for y. This equation is:

$$\frac{2}{y}\frac{dy}{d\theta} = \frac{2a^2\cos\theta + (a^1 - a^3)\sin\theta}{a^1 + a^3 + (a^1 - a^3)\cos\theta - 2a^2\sin\theta} \tag{7.1.12}$$

independently of the time of geodesics. The integration can be performed according to Example 2.558-2 from (Gradshteyn & Ryzhik, 2007). The general result is

$$y^2 = const\cdot\frac{a^1}{(a^1\cos\phi - a^2\sin\phi)^2 + [a^1a^3 - (a^2)^2]\sin^2\phi} \tag{7.1.13}$$

However, one can arrange the algebra of Iwasawa decomposition so as to have

$$y^2 = Q^2\left(\omega^1 + \omega^3 + (\omega^1 - \omega^3)\cos\theta - \omega^2\sin\theta\right) \tag{7.1.14}$$

where Q is a real constant. In this case the parameter y is simply the standard deviation over the ensemble of charges. Finally, the third part of the temptation consists of combining the second and the third of the equations (7.1.7) in order to get an equation for x. The general result, using, however, the equation (7.1.14), is the equation

$$x = -\left\{(a^1\cos\phi - a^2\sin\phi)^2 + [a^1a^3 - (a^2)^2]\sin^2\phi\right\}^2 \tag{7.1.15}$$

up to an additive constant.

The bottom line of this theory regarding the Iwasawa decomposition is that the parameters of this decomposition *are connected to the interpretation of the continuum supporting the waves*: the Crampin's theorem shows that the parameters x and y are properties dictated by some quanta connected to the charge of the interpretative particles [the quadratic forms from equations (7.1.14) and (7.1.15)]. This property corresponds to the fact that the pseudoparticles of the theoretical physics are expressions of the action of the continua on both the waves and the matter.

7.2. A Defining Property of Dyons: Producing a Quantum

Now we come to the interpretation of the results of §3.6. They too, carry the mark of charge: if the charge acts, it is by the intermediary of a matrix, which endorses two possible kinds of action. Both these natural actions have replicas as physical phenomena in the brain universe. The bottom line here, is that the Iwasawa parameters can tell us something about the kinds of action of continua in connection with the physical structure of the interpretative particles. The theory goes along the following strokes of reasoning.

The dominating line of the matrix $\boldsymbol{G}$ can be physically decided if we take the entries of the matrix as Katz's components of the static charges, magnetic or electric [see I, §§5.3 and 5.4]. For instance if the entries γ and δ are the components of a magnetic charge of Ernst Katz, then the *magnetic charge dominates* the Iwasawa decomposition, by its magnitude and direction. Then, the first line of the matrix can be taken as representing an electric Katz charge, and, therefore, an *electric charge dominated* Iwasawa decomposition can be constructed quite analogously. Formally, this kind of domination may not mean too much, but it is of essence from physical point of view. For, remember that, according to Katz's natural philosophy of charges, the magnetic charges are represented by *poles of pieces of matter*, while the electric charges are represented by *poles of pieces of vacuum*. So, in the light of this philosophy, Iwasawa decomposition with parameters (3.6.12) would mean that the matter prevails upon light, while in a decomposition where the first line dominates, the light prevails over matter. Leaving aside, for the moment, what the statement 'light, or matter, prevails' may mean physically, let us describe the Planck's background of a universe from this point of view.

As we have seen in §3.5, the basic matrix allowed by a conservation of information content in the form of the Maxwell tensor, is a matrix $\boldsymbol{A}$ of the general form given in equation (3.5.14). Its Iwasawa decomposition giving the different actions as in the previous equations (3.6.10) and (3.6.11), has the parameters given by equation (3.6.12) as

$$x = \frac{\alpha(\beta+\gamma)}{\alpha^2+\gamma^2} = \frac{\lambda}{\mu}\cos\theta,\ y^2 = \alpha^2+\gamma^2,\ \tan\phi = -\frac{\gamma}{\alpha} = \tan\varphi_2 \tag{7.2.1}$$

Notice that for a *skew-symmetric dyon* there is no displacement if it represents a particle, nor propagation if it represents a wave: it is a *static dyon*, *immovable*. Such a dyon can be represented by a special matrix, with either the magnetic line of the matrix prevailing in the decomposition, or with the electric line prevailing, but both being charges of the same magnitude:

$$\boldsymbol{D} = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \tag{7.2.2}$$

This kind of immovable dyon can only give transformations between *physical charges*, as we have shown in §6.4, equation (6.4.6).

Should we have used the matrix $\boldsymbol{B}$ from equation (3.5.14) in calculating the parameters of its Iwasawa parameters, we would have instead of (7.2.1) the parameters:

$$x = \frac{\alpha(\gamma-\beta)}{\gamma^2+\alpha^2} = \frac{\lambda}{\mu}\cos\theta, \quad y^2 = \gamma^2+\alpha^2 = \mu^2, \quad \tan\phi = -\frac{\gamma}{\alpha} = \tan\varphi_2 \tag{7.2.3}$$

and instead of (7.2.2) the corresponding static dyon would be:

$$\boldsymbol{D} = \begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix} \tag{7.2.4}$$

As we have shown in the §5.3, these matrices represent a classical possibility of choice of the dual charges as once given by Katz for just about the same theoretical reasons as those of Schwinger (Katz, 1965), but starting from considerations regarding the Maxwell tensor, just like us before in §3.5. One can say that Katz gave a procedure for the construction of a 'classical dyon', as it were. This kind of immovable dyon can only give transformations between *physiological charges*, as we have shown in §6.4, equation (6.4.11)

As an overall conclusion, all these dyons involve physiological charges. However, those from (7.2.2) are a *group* acting only between physical dipoles, while those from (7.2.4) are an *algebra* acting exclusively between physiological charges. Along this line an observation is worth recalling: the product of two dyons (7.2.4) is always a dyon (7.2.2). Consequently, one can conclude that the physiological charges are the ones deciding the evolution of the physical charges, along with the evolution of the physiological charges, of course. And these last charges can have an equation of motion of Heisenberg type, that can be revealed in the following manner which, as we shall show here in due time, appeals to the concept of surface, as conceived by Stroud (see §2.4).

The so-called *Lax formulation* for an operation of differentiation d of matrix $\boldsymbol{L}$, is determined by a *Lax pair* ($\boldsymbol{L}$, $\boldsymbol{M}$) satisfying the differential equation (Lax, 1968):

$$d\boldsymbol{L} = \left[\boldsymbol{M}, \boldsymbol{L}\right] \qquad (7.2.5)$$

Now, there is a theorem showing that the first integrals of such a differentiation are the traces of the different powers of $\boldsymbol{L}$, therefore implicitly, algebraical combinations of these powers. That is, the eigenvalues of the matrix are such first integrals [(Goriely, 2001), § 2.12]. In the case of 2×2 matrices there are just two such first integrals, and one can take advantage of this situation to give a unique first integral if the matrix is an involution, for then it has a null trace. In such a case, the only first integral cannot be but the trace of the square of $\boldsymbol{L}$:

$$\mathrm{tr}(\boldsymbol{L}^2) = 2I \qquad (7.2.6)$$

where I denotes the invariant. According to Vladislav Dumachev (2010), if $\boldsymbol{L}$ is a dyon of the kind (7.2.4), and $\boldsymbol{M}$ is an involution with zero diagonal entries, we have in equation (7.2.5) a genuine equation of motion for dyons. Indeed, a dyon can always be put in the form of a unique linear combination with variable coefficients, of as many dyons we can imagine. Taking (u, v) in a linear combination involving only two such fixed dyons (7.2.4), let us write the matrix $\boldsymbol{L}$ in the form of the linear combination, and $\boldsymbol{M}$ as a particular dyon (7.2.2), having zero on the principal diagonal:

$$\boldsymbol{L} \stackrel{\text{def}}{=} \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_2 & -\alpha_1 \end{pmatrix} u + \begin{pmatrix} \beta_1 & \beta_2 \\ \beta_2 & -\beta_1 \end{pmatrix} v, \ \boldsymbol{M} \stackrel{\text{def}}{=} \begin{pmatrix} 0 & m \\ -m & 0 \end{pmatrix} \qquad (7.2.7)$$

where m is a differential. In this case, the invariant I is given by the quadratic form:

$$I = \langle \alpha \,|\, \alpha \rangle \cdot u^2 + 2\langle \alpha \,|\, \beta \rangle \cdot uv + \langle \beta \,|\, \beta \rangle \cdot v^2 \qquad (7.2.8)$$

where we put

$$\langle \alpha \,|\, \alpha \rangle = \alpha_1^2 + \alpha_2^2, \ \langle \alpha \,|\, \beta \rangle = \alpha_1\beta_1 + \alpha_2\beta_2, \ \langle \beta \,|\, \beta \rangle = \beta_1^2 + \beta_2^2 \qquad (7.2.9)$$

as usual. The proof of this statement is a matter of calculation. Indeed, the equation (7.2.5) can be written as

$$\begin{pmatrix} \alpha_1 du + \beta_1 dv & \alpha_2 du + \beta_2 dv \\ \alpha_2 du + \beta_2 dv & -(\alpha_1 du + \beta_1 dv) \end{pmatrix} = \begin{pmatrix} 2m(\alpha_2 u + \beta_2 v) & -2m(\alpha_1 u + \beta_1 v) \\ -2m(\alpha_1 u + \beta_1 v) & -2m(\alpha_2 u + \beta_2 v) \end{pmatrix} \tag{7.2.10}$$

or, in the equalities of matrix entries,

$$\begin{aligned} \alpha_1 du + \beta_1 dv &= 2m(\alpha_2 u + \beta_2 v) \\ \alpha_2 du + \beta_2 dv &= -2m(\alpha_1 u + \beta_1 v) \end{aligned} \tag{7.2.11}$$

This system can be solved by matrix exponentiation. First, solve for (du, dv) to get:

$$\begin{pmatrix} du \\ dv \end{pmatrix} = \frac{2m}{\alpha_1\beta_2 - \alpha_2\beta_1} \begin{pmatrix} \beta_2 & -\beta_1 \\ -\alpha_2 & \alpha_1 \end{pmatrix} \cdot \begin{pmatrix} \alpha_2 & \beta_2 \\ -\alpha_1 & -\beta_1 \end{pmatrix} \cdot \begin{pmatrix} u \\ v \end{pmatrix} \tag{7.2.12}$$

Formally, in the spirit of notations from (7.2.9), we have:

$$|du\rangle = 2m \cdot \boldsymbol{I} \cdot |u\rangle, \ \boldsymbol{I} \stackrel{\text{def}}{=} \frac{1}{\langle\alpha|\cdot\boldsymbol{i}\cdot|\beta\rangle} \begin{pmatrix} -\langle\alpha|\beta\rangle & -\langle\beta|\beta\rangle \\ \langle\alpha|\alpha\rangle & \langle\alpha|\beta\rangle \end{pmatrix} \tag{7.2.13}$$

where $\boldsymbol{i}$ is the 2×2 skew-symmetric matrix, as defined in equation (6.2.6). The matrix $\boldsymbol{I}$ is an involution whose square is the negative identity matrix, and the notation is meant to remind us of the matrix thus denoted from the §§ 5.1 and 5.2. The whole point of this exercise is the fact that the invariant I from equation (7.2.8) is the first integral of the system of linear differential equations (7.2.13). This integral, should be able to tell us how a universe – the brain or the physical universe, in particular – handles the charges, or the information coming to us *via* the light the quotidian light phenomenon.

With a proper choice of the differential factor $2m$, the equation (7.2.13) assumes the solution:

$$|u(\phi)\rangle = (\cos\phi \cdot \boldsymbol{1} + \sin\phi \cdot \boldsymbol{I}) \cdot |u(0)\rangle, \ 2m \equiv d\phi \tag{7.2.14}$$

where $\langle u(0)| \equiv (u_0, v_0)$, is a constant vector. In the exponentiation here, we used the algebraical identity:

$$\langle\alpha|\alpha\rangle \cdot \langle\beta|\beta\rangle = (\langle\alpha|\beta\rangle)^2 + (\langle\alpha|\cdot\boldsymbol{i}\cdot|\beta\rangle)^2 \tag{7.2.15}$$

Two such classical dyons can be used to make a Planck resonator, in a universe mixing a part from the category of matter, like planetary structures, and a part from the category of light, like the charges.

7.3. A Geometrical View of Sensory Patches

The theoretical case of charges in physics is a difficult matter, mostly because they are usually considered as *a priori* quantized by the very measurement: every charge in the universe we inhabit is taken as a multiple of a quantum having the magnitude of the electron charge. We intend to make this case from the geometrical point of view mentioned in the above excerpt from Thomas Gold, but before going any further in doing this job, we must confess the inspiration taken from some classical works on modern cosmology [see (Lyttleton & Bondi, 1959, 1960), and (Hoyle, 1960)]. These works were chosen by us as being representatives for the attempts of considering different possible quanta of charge, for the positive and negative electric charges of the universe we inhabit. A further inspiration was taken from their modern theoretical echo represented by the multiplicity of vacua (Jackiw & Rebbi, 1976). In short, our contention is that a pair of arbitrary quanta of charges is connected with a continuous family of vacua.

The Feynman context in wave mechanics (Feynman, 1948) does not seem to be just a matter of pure chance discovery, and that quality of it can be documented in many ways. Case in point: if we adopt an Einsteinian view in cosmology [(Mazilu, 2024), §5.1], whereby the time remains arbitrary in a universe, the time scale transition is also supported by a homographic transformation, as in the SL(2,R) realization from equation (3.3.15) (see also I, §2.4). This means that between time and charge we should also have a homographic transformation. We entertain the opinion that the celebrated Feynman's rule of association of charges with the time arrow (Feynman, 1949), taken by Thomas Gold as a quintessential example in constructing an arrow of time [see also (Gold, 1962) for detailed explanations], must be then generalized to a statement like: 'the homographic connection between charges is replicated by a similar connection between the time moments'. Such a generalization, however, asks for an invariant determination of the homography, making much sense from a physical point of view: in a concept of universe some charges remain invariant by the homography, just like the electron and proton charges in the universe we inhabit. A universe will be, therefore, described by a family of homographies that have these charges as invariants. This theory has been conveyed in the § 2.5. We have, therefore, to start from the point where we have left off the theory in the § 2.5.

In order to make our point, notice that equation (3.2.11) only means the determination of the density of probability by sampling. For once, the problem arises regarding the determination of the transformation of charge, in general, and this may assume the following formulation. In the universe we inhabit there is a quantum of charge *e*, in the sense that there are two fixed charges (*e*, –*e*), and a neutral charge *0*. Incidentally, this last one may be taken as the vacuum charge just as well as the charge of a composite electrically neutral structure. Any homographic transformation of charge that leaves these three charges unchanged, cannot be realized according to the scheme (3.2.11), but by a multiple of the identity matrix. However, the situation changes significantly if we assume the existence of universes where the two fixed charges are given by two entirely different quanta, (e_1, e_2) say. Any matrix acting homographically on the charges must leave these values unchanged, together with the cross-ratio:

$$k \equiv (q,q_0;e_1,e_2) = \frac{q-e_1}{q-e_2} : \frac{q_0-e_1}{q_0-e_2} \tag{7.3.1}$$

where q and q_0 are the charges corresponding to one another through homography. Thus, any transformation of charges must be realized by a family of matrices depending on just an arbitrary parameter, λ say. Indeed, if (α, β,

γ, δ) are the entries of such a matrix, then the two fixed values and the invariant cross-ratio k define the matrix realizing the homography up to an arbitrary factor:

$$\frac{\alpha}{e_1 - e_2 k} = \frac{\beta}{(k-1)e_1 e_2} = \frac{\gamma}{1-k} = \frac{\delta}{e_1 k - e_2} \equiv \lambda \tag{7.3.2}$$

This means that the *homographic* action of the matrix is uniquely determined, and this action means the following: the charges of the pair (e_1, e_2) are left invariant, for *they are fixed charges*. On the other hand, the cross ratio of *any* pair of charges – which correspond to each other through the transformation – with respect to the pair of fixed charges, is also invariant, namely the parameter k of transformation. Thus, the two eigenvalues of the matrix are determined up to an arbitrary factor λ by equations:

$$\frac{\lambda_1}{\lambda} = (e_1 - e_2)k, \quad \frac{\lambda_2}{\lambda} = e_1 - e_2 \tag{7.3.3}$$

which give the eigenvalues according to the position of k with respect to 1, assuming k real: the above situation corresponds to $k > 1$; for $k < 1$, λ_1 and λ_2 just exchange places in the equation above.

If we disregard the arbitrary multiplicative constant from the definition (7.3.2) of its entries, a matrix can be written in the form of a *linear combination* of two singular matrices:

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \sim \begin{pmatrix} e_1 & -e_1 e_2 \\ 1 & -e_2 \end{pmatrix} - k \begin{pmatrix} e_2 & -e_1 e_2 \\ 1 & -e_1 \end{pmatrix} \tag{7.3.4}$$

In algebraical parlance this is a *linear pencil* of matrices generated by the two singular matrices, representing an ensemble of pairs of reference frames, when the two fixed points vary arbitrarily. Such a variation can be take as mimicking a process described by Chen-Ning Yang for the pairs in a supraconductor state (see § 3.2 above).

In general if, according to the ideas of Félix Cernuschi (1936), taken in the form later proposed by Alphonsus John Fennelly (1974), every particle is considered a universe – not quite a Gödel universe, but a universe nevertheless, according to a well-defined concept in the sense of Planck's procedure of quantization – two such units of charge must exist in the sense of Yang, with the vacuum charge defined appropriately. This definition of the vacuum must be such that the classical definition of the null charge is a particular case. For instance, it should correspond to that concept of electricity according to which a body is taken as *a priori* as charged positively, and the negativity or the positivity of an incidental charge is defined with respect to its level: if the charge of the incidental particle is smaller than the charge of the body then it is negative, while if the charge of incidental particle is greater than the charge of the body, it is positive [see (Poincaré, 1901), §4]. Under the Planck's procedure of quantization, *i.e.*, if the body can be taken as a universe, this definition does not necessarily require the case of unique fluid theory for electricity.

We shall show now that between e_1 and e_2 there should always be a homographic relation: after all, this is a well-established mathematical theorem. To start with, let us notice that in *coordinates* (α, β, γ, δ) a point represents a matrix. The quadratic form:

$$\alpha\delta - \beta\gamma = 0 \tag{7.3.5}$$

which is a quadric geometrically, more specifically, a one-sheeted hyperboloid, is simply the locus of points of coordinates (α, β, γ, δ) representing the ensemble of singular matrices. Geometrically, we have here a one-sheeted hyperboloid, which is a *doubly ruled* surface: it has *two systems* of generators described by the parameters e_1 and e_2 – the fixed points of the invariant representation (7.3.2) of the matrices. This fact can be ascertained as follows: take, for instance, the first singular matrix from the linear pencil (7.3.4). The representation (7.3.2) gives the point of coordinates

$$\frac{\alpha}{e_1} = \frac{\beta}{-e_1 e_2} = \frac{\gamma}{1} = \frac{\delta}{-e_2} \tag{7.3.6}$$

One can take notice right away that, for (α, β, γ, δ) variables, there are *two* systems of planes whose intersections can count as *two* straight lines – the generators or the rulings of our quadric – at the intersection of which the point (7.3.6) is located on the quadric in question, namely:

$$\begin{matrix} \alpha e_2 + \beta = 0 \\ \gamma e_2 + \delta = 0 \end{matrix} \quad and \quad \begin{matrix} \alpha - e_1\gamma = 0 \\ \beta - e_1\delta = 0 \end{matrix} \tag{7.3.7}$$

So these equations describe the two systems of generators of the hyperboloid, whose parameters are the fixed points, e_1 and e_2, of the homographic action in this world. Now, assume a generic point of coordinates (a, b, c, d) say, in the space of parameterd, located inside or outside the quadric (7.3.5), anyway not on the quadric itself, that is: $ad - bc \neq 0$. Thus, such a point represents a *nonsingular* matrix. Its *polar plane* with respect to the quadric (7.3.5), meets this very quadric along a hyperbola having the equation

$$d\alpha - c\beta - b\gamma + a\delta = 0 \tag{7.3.8}$$

We are interested on the intersection of this plane with one of the rulings of either system (e_1) or (e_2); let us choose (e_2) to settle our ideas. The coordinates of this point are given by the system of three linear equations in four unknows (α, β, γ, δ), given the two equation from the left hand side of equation (7.3.7), to which we adjoin the equation (7.3.8) of the polar plane of point (a, b, c, d). This means that we can only solve for some ratios of the coordinates (α, β, γ, δ). The system in question, has the following 3×4 matrix:

$$\begin{matrix} e_2 & 1 & 0 & 0 \\ 0 & 0 & e_2 & 1 \\ d & -c & -b & a \end{matrix} \tag{7.3.9}$$

As long as the point (a, b, c, d) is not on the absolute – the corresponding matrix is non-singular – this matrix has always a 3×3 minor which is non-null, so our system is always compatible. Its solution is provided by:

$$\frac{\alpha}{ae_2+b}=\frac{\beta}{-e_2(ae_2+b)}=\frac{\gamma}{ce_2+d}=\frac{\delta}{-e_2(ce_2+d)} \quad (7.3.10)$$

Using now the equations of the e_1-family of rulings, as given by the right hand side of equation (7.3.7), one gets the parameter e_1 as:

$$e_1=\frac{\alpha}{\gamma}=\frac{\beta}{\delta}=\frac{ae_2+b}{ce_2+d} \quad (7.3.11)$$

In other words: any *nonsingular* matrix represents a transformation which gives *a correspondence between the two fixed points*. If (e_1, e_2) are two given fixed points, then

$$(e_1,\frac{-\delta e_1+\beta}{\gamma e_1-\alpha}) \quad and \quad (\frac{\alpha e_2+\beta}{\gamma e_2+\delta},e_2) \quad (7.3.12)$$

are, again, two associated pairs of fixed points.

Thus, if (e_1, e_2) is the address of a *sensory patch*, the two addresses from equations (7.3.12) are two different 'phases' of it, 'induced', as it were, by a nonsingular matrix, which does not belong to that sensory patch. This theorem constitutes a basis for communication: any neuron that does not belong to a sensory patch, can disturb that sensory patch, by creating two such patches out of it. At this moment of our discourse we think is best to justify our position in the mathematical formalism serving the physics. To wit, we adopt what we think is the simplest position of them all: that of Dan Barbilian, in constructing a 'physical' theory of functions, so to speak (Barbilian, 1974).

Adopting the spirit of a modern unitary theory of cycles (Kisil, 2012), a cycle will be represented by the first equality from equation (7.3.8) above, as a curve described in coordinates (e_1, e_2):

$$de_1+ce_1e_2-b-ae_2=0 \quad (7.3.13)$$

If everything is real there, this equation represents a hyperbola. The point forced upon us by the Katz's natural philosophy of charge (Katz, 1965), is that in this frame we might have to introduce elements representing complex charges, thus 'complexifying', as it were, the Schwinger scheme of dyons, for instance (see § 4.5). For real, nonsingular matrices, there is the possibility of having complex charges as fixed points: in this case the fixed points are complex conjugated to each other, and any homographic connection between them asks for a special nonsingular matrix. Indeed, let us say that we have instead of (7.3.8) the relation:

$$z=\frac{\alpha z^*+\beta}{\gamma z^*+\delta} \quad \therefore \quad \gamma zz^*+\delta z-\alpha z^*-\beta=0 \quad (7.3.14)$$

where $z=x+iy$. In real, this last equation splits into:

$$\gamma(u^2+v^2)+(\delta-\alpha)u-\beta=0,\ (\delta+\alpha)v=0 \quad (7.3.15)$$

which are nontrivially satisfied ($v\not\equiv 0$) only for involutive matrices, in the form of cycle:

$$\gamma(u^2+v^2)+(\delta-\alpha)u-\beta=0,\ (\delta+\alpha)v=0 \quad (7.3.16)$$

This is a *bona fide* circle having the center and the radius defined by:

$$(\gamma u-\alpha)^2+(\gamma v)^2=\alpha^2+\beta\gamma,\ \ C\left(\alpha/\gamma,0\right),\ \ \gamma^2R^2=\alpha^2+\beta\gamma \tag{7.3.17}$$

Consequently, if the matrices are real, cycles cannot be naturally realized but exclusively with involutive matrices, for instance in the description of a medium (see § 3.5).

Speaking of the action of a medium, the situation is somewhat changed if we recall that a Planck medium is characterized by dipoles, which, according to current views, involve two charges. In short, the cycles cannot be realized by constant matrices with real entries. In order to see what changes it is best to start with the equation of the general circle in a plane containing the dipole:

$$(x-u-v\tan\theta)^2+y^2=\frac{v^2}{\cos^2\theta} \tag{7.3.18}$$

No doubt, this equation is of the form given in (7.3.17), only varrying parameters: it represents circles passing through the points (u, v) and $(u, -v)$, regardless of the values of angle θ. In other words, for u and v fixed we have simply a Planck resonator. Geometrically, the equation describes a one-parameter family of circles all passing through the two fixed points, and coordinated by the angle θ made by the circles at the fixed points with respect to a convenient direction. The problem is what is the physics in this situation.

7.4. The Barbilian Quantization: Handling Sensory Patches

Consider the quadric (7.3.5) as the absolute of the de Sitter background of a certain instanton: notice that the coordinates are, tentatively speaking, real, so this background must be connected with a physical structure of an instanton. Assume that inside the absolute of this space of a physical structure we have a straight line, well defined if we know the points of its intersection with the absolute, ***P*** and ***Q*** say, by equation (7.3.4). The two points are, obviously, both on the absolute, having matrix representations as those given in the equation (7.3.6) above, and this condition can be algebraically expressed by vanishing of their norm, generated, as known, by the equation of absolute:

$$(P,P)=(Q,Q)=0 \tag{7.4.1}$$

Here P and Q (generally, capital italics) represent the sets of the four homogeneous coordinates of the matrices denoted by the corresponding capital bold letters, *viz*. ***P*** and ***Q*** in this case. In cases where P and Q are functions of a parameter, the corresponding line ***PQ*** describes a ruled surface of the congruence, in this Cayleyan space.

A short explanatory digression is now in order: the notation conventions here are a little out of line with respect to conventions of the present work. To wit, we would have to put, for instance, $|P\rangle$ for the coordonate complex of the point P. However, this would suggest a dot product $\langle P|Q\rangle$ instead of (P,Q), which is not always the case in the Cayleyan geometry, where, occasionally, the *bra* may not be the transposed matrix. Case in point is that of the general Barbilian metric, whereby the absolute is a surface by a homogeneous function of arbitrary degree, not just the degree two (Barbilian, 1937). So, in order to maintain such a level of generality without multiplying the notation conventions, we divert momentarily from our conventions, and use here the established notations of the classical absolute geometry.

Turning back to our subject, a position ***X***, indicated by a material Hertz particle on the generic line of our congruence has coordinates that can be written in the form

$$X = P + l \cdot Q \tag{7.4.2}$$

where *l* is a continuous parameter locating the position along the straight line ***PQ***. For *l = 0* the Hertz particle indicates the position $\boldsymbol{X} \equiv \boldsymbol{P}$, while for $l = \infty$ it indicates the position $\boldsymbol{X} \equiv \boldsymbol{Q}$. Our problem is to 'decompose', as it were, the motion of our particle: that is, to find the condition that in the motion, its position ***X***(*t*) describes a trajectory perpendicular to the ray ***PQ*** of the congruence, on which it happens to be located at the moment *t*. This was solved by Dan Barbilian [(Barbilian, 1974), Lecture IX, pp. 66–73] in the manner to be described here right away.

We can translate the orthogonality into a condition of intersection: that is, the points ***X***, ***dX*** and the polar ***P'Q'*** of the ray ***PQ*** with respect to the absolute should be in the same plane – the polar plane of the point ***X***. Let us write the associated equations for this condition. First, the equation of the polar plane to a point located on the line ***PQ*** has the form:

$$(X,Y) = 0 \quad \therefore \quad (P,Y) + m(Q,Y) = 0 \tag{7.4.3}$$

where ***Y*** is the image of the generic point of the plane, and *m* a parameter giving the 'coordinate' of the plane within the pencil of planes having the polar ***P'Q'*** of the ray ***PQ*** as line of intersection. With *m* variable, this equation represents, indeed, a *pencil of planes* having ***P'Q'*** as their common straight line, which is the intersection of the planes (*P*, *X*) = *0* and (*Q*, *X*) = 0, tangent to absolute in the points ***P*** and ***Q*** of intersection of the straight line ***PQ*** with this absolute. Now, let us find the condition that the generic plane (7.4.3) of this pencil contains the generic points ***X*** and ***X*** + ***dX***. Because in differentials we have

$$X = P + l \cdot Q, \;\; dX = dP + l \cdot dQ + Q \cdot dl$$

where *l* is the generic nonhomogeneous coordinate of ***X*** along ***PQ***, the condition that the plane (7.4.3) contains the two points comes down to the following two equations:

$$l \cdot (P,Q) + m \cdot (Q,P) = 0, \;\; l \cdot (P,dQ) + m \cdot (Q,dP) + dm \cdot (P,Q) = 0 \tag{7.4.4}$$

where we have used the equation (7.4.2). The 'dot product' (*P*,*Q*) is always nonzero, except in the case $\boldsymbol{P} \equiv \boldsymbol{Q}$, which is singular: such a situation cannot define a unique straight line. Therefore the system (7.4.4) practically defines a null differential 1-form:

$$\frac{dm}{m} + \frac{(P,dQ) - (Q,dP)}{(P,Q)} = 0 \tag{7.4.5}$$

which gives the Barbilian's theorem: *the equation* (7.4.5) *gives the necessary and sufficient condition that the trajectory of the generic Hertz particle* (7.4.2) *is a curve orthogonal to the ray **PQ** when this ray describes the congruence*.

Based on this theorem one can prove a general 'quantization relation': *the trajectories orthogonal to a ray are equidistant on the ray*, in the sense of absolute geometry, of course. Indeed, if for such a trajectory the parameter has a value m_1, and for another trajectory the parameter has the value m_2, while all the other symbols in equation (7.4.5) remain the same, then we have

$$\frac{dm^1}{m^1} - \frac{dm^2}{m^2} = 0 \quad \therefore \quad \ln\frac{m^1}{m^2} = const \tag{7.4.6}$$

This logarithm represents the Cayleyan distance between the two points of trajectories located on the ray ***PQ***. Indeed, the ratio m^1/m^2 is the cross-ratio ($\boldsymbol{X_1}$, $\boldsymbol{X_2}$, ***P***, ***Q***) giving the distance along the line ***PQ***. By Laguerre's formula, the logarithm of this cross-ratio gives the distance between the positions $\boldsymbol{X_1}$ and $\boldsymbol{X_2}$, whence the conclusion above. The equation (7.4.6) represents the essential of what we shall call, from now on, *the Barbilian quantization*: according to the geometry described above, it represents, actually, *the wavelength of a de Broglie surface displaced along a ray*.

In general, however, the motion of a particle internal to the absolute is not perpendicular to the generic ray of the congruence. In this connection, Dan Barbilian noticed that the differential form

$$\Omega \stackrel{\text{def}}{=} \frac{dm}{m} + \frac{(P,dQ)-(Q,dP)}{(P,Q)} \tag{7.4.7}$$

represents a *measure of the departure* of the trajectory of a moving particle from the condition of perpendicularity to the ray: as the equation (7.4.5) shows, $\Omega = 0$ means perpendicularity, indeed. If that moving particle is an interpretive particle, then this condition means that the particle is moving only perpendicular to the ray, as in the case of the physical optics of Fresnel. The differential form (7.4.7) is invariant with respect to a group of linear transformations of the coordinates in the absolute space. As it represents the component along the ray ***PQ*** of a general displacement, the component perpendicular to the ray of that general displacement should be also an invariant with respect to those linear transformations, and can also be calculated from geometrical considerations.

Dan Barbilian mentions the direct kinship of this geometry with the classical geometrical optics. Namely, the differential 1-form in coordinates:

$$\omega \stackrel{\text{def}}{=} \frac{(P,dQ)-(Q,dP)}{(P,Q)} \tag{7.4.8}$$

entering as component of the differential (7.4.7), is the non-Euclidean generalization, *via* the absolute geometry, of the corresponding Euclidean differential form introduced by Constantin Carathéodory in his *Geometrische Optik*. In the case of a pencil of geometrical rays, it describes *the advance of the wave surface measured along the physical ray represented by the pencil*, just as in the regular geometrical optics [(Carathéodory, 1937), §28].

Even though the motion of an interpretive particle is not always perpendicular to the ray, the motion along the ray advances in 'quanta'. In order to estimate these quanta, Dan Barbilian applies quite simple considerations [see (Barbilian, 1974), pp. 103–104]. The essence of this approach is that, at the differential level (therefore, at an infrafinite scale) the geometry should be Euclidean, and therefore

$$(ds)^2 = \Omega^2 + (d\sigma)^2 \tag{7.4.9}$$

with the following explanation: in a small scale around the moving Hertz particle, we can assume that the Pythagoras' theorem is valid. So, the equation (7.4.9) represents this theorem in a regular right triangle with the hypothenuse (ds), and catheti Ω and ($d\sigma$). Physically, Ω represents the advancement of a material particle serving for interpretation along the ray in a straight line, while ($d\sigma$) is the component of the motion of that particle orthogonal to the ray.

The important observation here is that the quanta suggested by equation (7.4.6) are rather connected with the *idea of torsion* in the sense of Cartan [see (Barbilian, 1974); p. 71]. When we refer the construction of a ray to the natural vicinity of a certain global *wave surface*, the differential form Ω can be written as

$$\Omega = dw + Adu + Bdv \tag{7.4.10}$$

where (dw) is the *normal displacement* of the wave surface, and (u,v) are the parameters on that surface. The vector of components ($1, A, B$) can be considered as a force whose elementary work is the differential form (7.4.7): we should like to call it *the Barbilian force*. It has the general direction along the light ray. The condition (7.4.6) is laterally realized by the curvilinear integral on a cycle located on the wave surface:

$$\ln\frac{\lambda_1}{\lambda_2} \propto \oint(Adu + Bdv) \tag{7.4.11}$$

where the integration is performed over an incidental cycle of the surface.

7.5. Quantization as a Possibility of Handling Sensory Patches

The problem of interaction of two such sensory patches, or of anything of the kind, as the one described in the previous section, becomes now purely algebraical, and has the following solution, given under auspices of the geometrical optics presented above (Barbilian, 1974). In order to develop the mathematical formalism of this interaction, we need first to write a 2×2 matrix in terms of its elements of *characteristic actions*. There are two kinds of actions of these matrices: the *linear action* on two-dimensional extensions, and the *homographic action* in one dimension. The characteristic elements of the two actions are *the eigenvalues* of the matrix and its *eigendirections*, respectively. It is easy to figure out that between these elements there are linear relations involving the matrix entries. In the notations from equation (7.3.4), these relations are simply corresponding to the equation for eigenvalues of the matrix, that is:

$$\lambda = \gamma \cdot e + \delta, \;\; \lambda \cdot e = \alpha \cdot e + \beta \tag{7.5.1}$$

where λ is one of the eigenvalues in question, and e is the corresponding fixed point. In view of the great importance that we are giving to Iwasawa decomposition based on the lines of matrices, it is essential to observe here that (α, β) is the first line of the matrix, while (γ, δ) is the second line. Notice, along the same course of reasoning, the linear relation between eigenvalues and the fixed points of the matrix, as represented by the first equation from (7.5.1): it always involves the second line of the matrix. In fact, if we have the eigenvalues and the fixed points, we always have the matrix up to a factor. Indeed, if we calculate the entries in terms of these elements, we get the matrix

$$\left(e_1 - e_2\right)\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \lambda_1 . e_1 - \lambda_2 \cdot e_2 & -\left(\lambda_1 - \lambda_2\right)\cdot e_1 \cdot e_2 \\ \lambda_1 - \lambda_2 & \lambda_2 \cdot e_1 - \lambda_1 \cdot e_2 \end{pmatrix} \tag{7.5.2}$$

Consequently, in this representation the matrix $\boldsymbol{\alpha}$ has determinant unity, and therefore it is prone to an Iwasawa decomposition, for this is the essential condition of existence of this representation. A comparison between equations (7.3.4) and (7.5.2) shows that the invariant cross-ratio of the matrix is represented by the ratio of its eigenvalues.

The parameter y of the Iwasawa decomposition of (7.5.2), is given by the second of equations (3.6.12) as:

$$y \equiv \frac{\left(1 + e_2^2\right)\lambda_1^2 - 2\left(e_1 \cdot e_2\right)\lambda_1\lambda_2 + \left(1 + e_1^2\right)\lambda_2^2}{\left(e_1 - e_2\right)^2} \tag{7.5.3}$$

This is plainly a metric for the two-dimensional range of the frequencies. The metric tensor depends exclusively on the fixed points of the matrix, in a very attractive way. Along this idea, we can calculate the parameter x, according to first of the equations (3.6.12). We have:

$$x \equiv \left(\lambda_1 - \lambda_2\right)\cdot \frac{\left\{\left(1 + e_2^2\right)\lambda_1 e_1 - \left(1 + e_1^2\right)\lambda_2 e_2\right\}}{\left(1 + e_2^2\right)\lambda_1^2 - 2\left(e_1 \cdot e_2\right)\lambda_1\lambda_2 + \left(1 + e_1^2\right)\lambda_2^2} \tag{7.5.4}$$

So, we have a classical dyon for one of the conditions:

$$\lambda_1 = \lambda_2 \;\text{ or }\; g_{11} e_1 \lambda_1 = g_{22} e_2 \lambda_2 \tag{7.5.5}$$

which settles the cross-ratio characteristic to the given matrix. This shows that such a parameter may not be quite a purely metrical parameter from the point of view of the geometry.

Let us return to the equation (7.5.2) of a general matrix, written in terms of its fixed points and eigenvalues. One can say that the matrix itself is an object connecting these elements which characterize two specific actions: homographic and linear. Thus, from the perspective of equation (7.5.2) the matrix can be written in the form

$$\left(e_1 - e_2\right)\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \lambda_1 \cdot \boldsymbol{P} + \lambda_2 \cdot \boldsymbol{Q} \tag{7.5.6}$$

where $\boldsymbol{P}$ and $\boldsymbol{Q}$ are the singular matrices

$$\boldsymbol{P} \stackrel{\text{def}}{=} \begin{pmatrix} e_1 & -e_1 \cdot e_2 \\ 1 & -e_2 \end{pmatrix}, \ \boldsymbol{Q} \stackrel{\text{def}}{=} \begin{pmatrix} -e_2 & e_1 \cdot e_2 \\ -1 & e_1 \end{pmatrix} \tag{7.5.7}$$

Therefore, the equation (7.5.6) represents a *straight line* between the two 'positions' $\boldsymbol{P}$ and $\boldsymbol{Q}$ on the absolute, whose points are coordinated through the homogeneous coordinates given, basically, by the eigenvalues of the matrices they represent. The matrix thus defined has special differential properties from the point of view of the Barbilian quantization, of the kind of those signaled in the § 7.5 above [see equation (7.4.6)]. The points on absolute satisfy to the following differential conditions:

$$(P,dQ) + (Q,dP) = d\left(e_1 - e_2\right)^2, \ (P,dQ) - (Q,dP) = 0 \tag{7.5.8}$$

In other words, along this ray, the differential form (7.4.8) can be defined, since $(P,Q) \neq 0$, but it is null. So, the condition of quantization (7.4.6) is effectual in this case, where m is the ratio of the two eigenvalues of the matrices (7.5.6). The question is to describe the cases where the second differential form (7.4.8) is nonzero, in order to describe exhaustively the metric (7.4.9).

In order to do this, we find inspiration in a modern theory of cycles [see (Cnops, 2002); especially § 4.16], which fits in the present physical theory starting from the observation that the matrices of the type (7.5.7) can be deemed *as cycles of zero radius*. Let us consider here the equivalent of an involution linear in $\boldsymbol{P}$ and $\boldsymbol{Q}$, but corresponding to the form (7.5.6) in cases where the pencil of matrices is not coordinated by the eigenvalues. In such a case, if the matrix $\boldsymbol{\alpha}$ is within absolute, the fixed points are real, just as eigenvalues are, but the singular matrices $\boldsymbol{P}$ and $\boldsymbol{Q}$ must have different entries from the matrices given in equation (7.5.7). Assuming that they have null trace, and depend exclusively on the fixed points as before, they can be, for instance, of the form:

$$\boldsymbol{P} \stackrel{\text{def}}{=} \begin{pmatrix} e_1 & -e_1^2 \\ 1 & -e_1 \end{pmatrix}, \ \boldsymbol{Q} \stackrel{\text{def}}{=} \begin{pmatrix} e_2 & -e_2^2 \\ 1 & -e_2 \end{pmatrix} \tag{7.5.9}$$

Thus, instead of (7.5.8) we shall have

$$(P,dQ) + (Q,dP) = d\left(e_1 - e_2\right)^2, \ (P,dQ) - (Q,dP) = -2\left(e_1 - e_2\right) d\left(e_1 + e_2\right) \tag{7.5.10}$$

so that the differential form (7.4.8) is nonzero, indeed:

$$\omega = -2\frac{d\left(e_1 + e_2\right)}{e_1 - e_2} \tag{7.5.11}$$

Thus, the Barbilian quantization is nontrivial: the eigendirections of the matrix also participate to quantization.

Let us seek for an interpretation of the matrices from equation (7.5.9). The linear pencil of matrices:

$$\alpha \stackrel{\text{def}}{=} \lambda P + \mu Q = \begin{pmatrix} \lambda e_1 + \mu e_2 & -\lambda e_1^2 - \mu e_2^2 \\ \lambda + \mu & -\lambda e_1 - \mu e_2 \end{pmatrix} \tag{7.5.12}$$

represents a well-defined family of involutions. This means that the two parameters λ and μ are algebraically calculable from any possible entries of the matrix $\boldsymbol{\alpha}$, satisfying the conditions:

$$\alpha \equiv \begin{pmatrix} a & b \\ c & -a \end{pmatrix}, \ e_2 = \frac{ae_1 + b}{ce_1 - a} \tag{7.5.13}$$

and they are given by relations:

$$(e_1 - e_2)\lambda = -ce_2 + a \ \ and \ \ (e_1 - e_2)\mu = ce_1 + a \tag{7.5.14}$$

under condition from equation (7.5.13). The two actions of the matrix $\boldsymbol{\alpha}$ are then described by the eigenvalues and fixed points, given by:

$$\pm\sqrt{\lambda\mu}\left|e_1 - e_2\right| \ \ respectively \ \ \frac{\sqrt{\lambda}\cdot e_1 \pm \sqrt{-\mu}\cdot e_2}{\sqrt{\lambda} \pm \sqrt{-\mu}} \tag{7.5.15}$$

as one can see by simple calculations.

8. Possibilities of Physics in the Problem of Neuron Addresses

Summarizing the results thus far, we can come up with the following general philosophy regarding the theoretical description of the brain functions: Planck's equation is fundamental in this physics (§ 3.1). According to this approach the classical regularization procedure of the Kepler problem stays in force for this physics, whereby the background of the universe is represented by a third order differential equation [§ 5.3, equation (5.3.7)]. So, if there is a connection between rays and the background in the brain, this can be formally recognized as a connection between Planck's equation, and the third order equation of the regularization procedure, especially since the equation (5.3.7) is characteristic the the rays in a Maxwell fish-eye optical medium. Then, if that connection is represented by a quadratic form, this quadratic form is a *bona fide* quantization constant of the Planck type, while its coefficients can count as the entries of a matix representing somehow the neuron. Let us show in this chapter that this is, indeed, the case, an occasion to delve into the geometry of the brain matter.

8.1. Geometrical Equations with a Physical Meaning

A legitimate question can be aroused: has the equation (6.1.1) a theoretical basis? According to a Planck quantization procedure, it turns out that the answer to this question is affirmative. This would mean that the 'oscillator behavior' is a property that needs to be attached as a differentia to the concept of universe. In its turn, this would mean that the equation (6.1.6), which is the crux of the regularization procedure of the classical Kepler problem, also needs to be attached as differentia for the general concept of universe. This fact can be mathematically explained by the property of analyticity.

The Planck's equation (3.1.2) can be connected with a second-order differential equation in pure differentials just like its classical Riccati counterpart. Namely, by changing the virtually dependent variable according to the rule:

$$\omega^{1} u = -\frac{dX}{X} \tag{8.1.1}$$

we have the natural differential condition:

$$d\left(\frac{dX}{X} + \omega^{1} u\right) = 0 \tag{8.1.2}$$

which, by using the equation (3.1.2) itself, turns into a second-order equation in pure differentials for the new virtually dependent variable:

$$d^2 X - \left(\frac{d\omega^1}{\omega^1} + \omega^2\right) dX + \omega^1 \omega^3 X = 0 \tag{8.1.3}$$

Here the differentials of differential forms, as well as the products, are *symmetric*: that is, they are *not exterior* operations. Just to offer an working example, the equation (8.1.3) can be proved as follows: differentiate the terms of (8.1.2), to get

$$d\left(\frac{dX}{X}\right) \equiv \frac{d^2 X}{X} - \left(\frac{dX}{X}\right)^2 \quad \text{and} \quad d(\omega^1 u) = (d\omega^1) u + \omega^1 du$$

Use now the equation (3.1.2) itself, in order to eliminate *du* from the second equality, and (8.1.1) in order to eliminate $(dX/X)^2$ from the first of these equalities, and then calculate (8.1.2). The result is equation (8.1.3), which describes, let us say, a *stochastic process* in general, implicating only the pure differentials of the variables involved, but no relative measures of these differentials, *viz*., no derivatives with respect to the 'time variable', as it were. An essential quantity characterizing this stochastic process is its discriminant:

$$\left(\frac{d\omega^1}{\omega^1}\right)^2 + 2\frac{\omega^2}{\omega^1} d\omega^1 + (\omega^2)^2 - 4\omega^1 \omega^3 \tag{8.1.4}$$

A practical example is now in order for what we have to say here: the case of the Beltrami-Poincaré geometry of the surfaces of negative curvature. As known, these surfaces are of essence, at least in the theoretical physics of the particles and fields (Sasaki, 1979).

The $\mathfrak{sl}(2, \mathrm{R})$ type coframe (3.1.3) of this model is given by the differential form in two variables:

$$\omega^1 = \frac{du}{v^2}, \quad \omega^2 = 2\frac{udu + vdv}{v^2}, \quad \omega^3 = \frac{(u^2 - v^2) du + 2uvdv}{v^2} \tag{8.1.5}$$

The symmetric differentials of this coframe can be easily constructed and handled, by noticing that the first one of its components is a kind of 'pivot', so to speak, for the three differential form. What we mean by this, is that the differential form ω^1 of the coframe (8.1.5) enters *linearly* the expressions of the other two remaining components of the coframe, that is, we have:

$$\omega^2 = 2\left(u\omega^1 + \frac{dv}{v}\right), \;\; \omega^3 = -(u^2+v^2)\omega^1 + 2u\omega^2$$

This means that we can write the symmetric differentials in the form:

$$d\omega^1 = \frac{d^2u}{v} - 2\frac{dudv}{v^2}; \qquad d\omega^2 = 2\left(u\cdot d\omega^1 + du\cdot\omega^1 + d\frac{dv}{v}\right)$$

$$d\omega^3 = -(u^2+v^2)d\omega^1 + 2u\cdot d\omega^2 - 2(udu+vdv)\cdot\omega^1 + 2du\cdot\omega^2$$

Now, if $d\omega^1 = 0$, we have

$$d\omega^2 = 2\left(du\cdot\omega^1 + d\frac{dv}{v}\right) \;\;\therefore\;\; \frac{1}{2}d\omega^2 = \frac{d^2v}{v} + \frac{(du)^2-(dv)^2}{v^2}$$

Further on, if along $d\omega^1 = 0$, $d\omega^2 = 0$ too, we have:

$$d\omega^3 = -2(udu+vdv)\omega^1 + 2(du)\omega^2$$

so that $d\omega^3 = 0$ automatically, in view of the definitions (8.1.5).

Summarizing these results, we have the conclusion: let us say that u and v are parameters of location on a certain surface. Then, along the curves of this surface satisfying the two differential equations

$$vd^2u - 2dudv = 0 \;\;\&\;\; vd^2v + (du)^2 - (dv)^2 = 0 \qquad (8.1.6)$$

the three differential forms are constants, in the sense that their symmetric differentials are zero. More to the point they represent some constant speeds of displacement: as they are defined in equation (8.1.5), these differential forms cannot represent but only displacement along the surface parameterized by u and v. The equations (8.1.6), when referred to a proper parameter of continuity, are expressing the constancy of rates of the first two components of the coframe (8.1.5), and the solution curves are geodesics on that surface. They can be 'integrated' to give a first-order differential invariant:

$$\frac{d\{(du)^2+(dv)^2\}}{(du)^2+(dv)^2} - 2\frac{dv}{v} = 0 \;\;\therefore\;\; d\frac{(du)^2+(dv)^2}{v^2} = 0$$

which is the metric of this surface, of which we can say more according to geometrical experience: *this is a surface of constant negative curvature*. We need to know in what sense this constancy is to be taken physically.

In order to see this let us return to the equation (8.1.3). Along the geodesics it becomes

$$d^2X - \omega^2 dX + \omega^1\omega^3 X = 0 \qquad (8.1.7)$$

whose coefficients are, obviously, 'constants'. So, along these geodesics, the discriminant (8.1.4) reduces to the last two terms. When calculating it using the coframe (8.1.5), we find that it is positive:

$$(\omega^2)^2 - 4\omega^1\omega^3 = 4\frac{(du)^2 + (dv)^2}{v^2} \tag{8.1.8}$$

and this should be 'constant' too. The right hand side of this equation is the well-known Beltrami-Poincaré metric of the Lobachevsky plane. It gives us directly an indication of what the 'constancy of the differential forms' means: a proportionality with the arclength of the Beltrami-Poincaré metric, of the kind we have shown before, in equation (3.1.4). Therefore, the parameter of continuity of geodesics, ϕ say, is practically the arclength of the geodesics of the metric (8.1.8). Let us find the parametric equations of the geodesics by this approach to problem.

To this end we use the notations from equation (3.1.4), and the definition (8.1.5) of the coframe, so that we need to solve the following linear system for the differentials of parameters on surface:

$$\frac{du}{v^2} = a^1 d\phi, \quad \frac{udu + vdv}{v^2} = a^2 d\phi, \quad \frac{(u^2 - v^2)\,du + 2uvdv}{v^2} = a^3 d\phi \tag{8.1.9}$$

This linear system is compatible if the determinant:

$$\frac{d\phi}{v^2}\begin{vmatrix} 1 & 0 & a^1 \\ u & v & a^2 \\ u^2 - v^2 & 2uv & a^3 \end{vmatrix}$$

vanishes, which happens if the following equation is satisfied:

$$a^1\left(u^2 + v^2\right) - 2a^2 u + a^3 = 0 \tag{8.1.10}$$

This equation represent circles in the regular Cartesian plane of coordinates (u,v):

$$\left(u - \frac{a^2}{a^1}\right)^2 + v^2 = \left(\frac{a^2}{a^1}\right)^2 - \frac{a^3}{a^1} \tag{8.1.11}$$

These circles, having the centers along the curve $v = 0$ of surface, are real only in cases where the right-hand side of this equation is positive, and therefore the metric (8.1.8) is positive, which is a natural generalization of the Planck's historical case, contained here for $a^3 = 0$. It is worth recalling, for further reference in the similar cases, that in the Planck's original case of the equation (3.1.5), the constant a^2 is determined by the quantum, and that this quantum is introduced according to the requirements of Wien's displacement law.

The two-parameter family of circles (8.1.11) does not contain exclusively geodesics of Beltrami-Poincaré metric: the geodesics are just a particular class within that two-parameter family. This can be seen from the fact that the equation (8.1.10) is only the compatibility condition for the system (8.1.9). The solutions themselves, of that system, require particular constraints, and the geodesics must be such an example. One can find the parametric equations of these geodesics, noticing that, according to our algebraic experience, we can have just two possible sets of parametric equations of the family (8.1.11) in terms of ϕ:

$$\begin{cases} u(\phi) = u_0 + v_0 \sin\phi \\ v(\phi) = v_0 \cos\phi \end{cases} \quad or \quad \begin{cases} u(\phi) = u_0 + v_0 \tanh\phi \\ v(\phi) = \dfrac{v_0}{\cosh\phi} \end{cases} \tag{8.1.12}$$

both of them valid under the very same two conditions of definition of the two parameters:

$$u_0 = \frac{a^2}{a^1} \quad and \quad u_0^2 - v_0^2 = \frac{a^3}{a^1} \tag{8.1.13}$$

However, when it comes to verification by the differential equations (8.1.6), only the second set of equations from (8.1.12) turn out to be valid. Thus, we are compelled to take the parametric equations of the geodesics of the Beltrami-Poincaré metric in the general form:

$$u(\phi) = u_0 + v_0 \tanh\phi, \;\; v(\phi) = \frac{v_0}{\cosh\phi} \tag{8.1.14}$$

and no other way, where the two parameters u_0 and v_0 are defined as in equation (8.1.13), in terms of the coefficients of the Planck's differential equation (3.1.5). Notice also that this circumstance is the one agreeing with the upper complex plane representation of Henri Poincaré for the geometry of Lobachevsky, where the geodesics are half-circles orthogonal to the *absolute* $v = 0$, of this geometry.

Along the geodesics (8.1.14), the second-order differential equation (8.1.7) becomes an ordinary second-order differential equation of a damped harmonic oscillator:

$$X''(\phi) - 2a^2 X'(\phi) + a^1 a^3 X(\phi) = 0$$

where the accent means differentiation with respect to ϕ. The general solution of this equation depends on two arbitrary constants and, in view of the equation (8.1.13), it is:

$$X(\phi) = e^{a^1 u_0 \phi}\left(A \cdot e^{a^1 v_0 \phi} + B \cdot e^{-a^1 v_0 \phi}\right) \tag{8.1.15}$$

As an oscillator, this must be an inverted harmonic oscillator (Barton, 1986), a physical structure instrumental for the description of the tunneling phenomena. These phenomena are, in turn, instrumental in the explanation of the communication between neurons and glia [see (Kandel, 2006), Chapters 5 and 6].

Let us treat now the 'non-Planck's case' along the same lines as above. The quadratic trinomial from equation (3.1.2) has now a negative discriminant, so that, along the geodesics we can put:

$$\left(\frac{\omega^2}{2}\right)^2 - \omega^1 \omega^3 \equiv -R^2 \tag{8.1.16}$$

where R is a real quantity. This condition will assure a correct definition for the Cauchy distribution (3.1.6). The rest of the mathematics of this theory goes exactly as before, for the 'Planck's case' analysis. The basic equations are the same, and we just transcribe them here for convenience. First, the equation (8.1.2):

$$d\left(\frac{dX}{X} + \omega^1 u\right) = 0$$

is a second order equation in pure differentials:

$$d^2X - \left(\frac{d\omega^1}{\omega^1} + \omega^2\right)dX + \omega^1\omega^3X = 0 \tag{8.1.17}$$

and this can be treated exactly as we did before in this section with equation (8.1.3). The equation (8.1.17) also describes a stochastic process, the essential quantity of which is its discriminant:

$$\left(\frac{d\omega^1}{\omega^1}\right)^2 + 2\frac{\omega^2}{\omega^1}d\omega^1 + (\omega^2)^2 - 4\omega^1\omega^3 \tag{8.1.18}$$

Again, a practical example of a surface, analogous to that from the previous equation (8.1.5) is in order: however, we cannot have the case of the Beltrami-Poincaré geometry of the surfaces of negative curvature anymore, because of the condition (8.1.16). On the other hand, as already mentioned, the $\mathfrak{sl}(2,R)$ algebra is of essence in most of the problems involving the interpretation in theoretical physics (Sasaki, 1979), and we must maintain it here. An $\mathfrak{sl}(2,R)$ coframe of the geometrical model of this case is given by the differential forms in two variables:

$$\omega^1 = \frac{du}{v^2},\ \omega^2 = -2\frac{udu - vdv}{v^2},\ \omega^3 = \frac{(u^2+v^2)du - 2uvdv}{v^2} \tag{8.1.19}$$

whose role will be explained by analogy with the case of the frame from equation (8.1.5). The symmetric differentials of the first two components of this coframe are given by the equations:

$$d\omega^1 = \frac{d^2u}{v^2} - 2\frac{dudv}{v^3},\ d\omega^2 = -2\left(ud\omega^1 + \omega^1du - d\frac{dv}{v}\right)$$

So, both these differential forms are constants if the differential equations

$$vd^2u - 2dudv = 0,\ vd^2v - (du)^2 - (dv)^2 = 0 \tag{8.1.20}$$

are satisfied. Then ω^3 is automatically constant along the curves defined by equations (8.1.20), for we have:

$$\omega^3 = -(u^2 - v^2)\omega^1 - 2u\omega^2\ \therefore\ d\omega^3 = -2(udu - vdv)\omega^1 + (du)\omega^2 = 0$$

and the last expression is zero by the very definitions (8.1.19) of the coframe. Thus, along the curves on the surfaces satisfying the differential equations (8.1.20) the coframe (8.1.19) is ‘constant’. Those curves are geodesics on surface: their equations can be ‘integrated’ to give:

$$\frac{d\{(du)^2 - (dv)^2\}}{(du)^2 - (dv)^2} - 2\frac{dv}{v} = 0\ \therefore\ d\frac{(du)^2 - (dv)^2}{v^2} = 0$$

which gives the metric of this surface: *a conform-Lorentzian metric*. This is the quantity R^2 above.

In order to see this let us return to the equation (8.1.17). Along the geodesics it becomes

$$d^2X - \omega^2dX + \omega^1\omega^3X = 0$$

where the coefficients are ‘constants’. The discriminant (8.1.18) reduces to the last two terms. When calculating it using the coframe (8.1.19), we have:

$$(\omega^2)^2 - 4\omega^1\omega^3 = -4\frac{(du)^2 - (dv)^2}{v^2} \tag{8.1.21}$$

and this should be 'constant' too. This gives us the indication of what the 'constancy' may mean here: a proportionality with the arclength of the conform-Lorentzian metric. That is, the parameter ϕ is practically the arclength of the geodesics of the metric (8.1.21). Let us find the parametric equations of the geodesics by this approach to problem.

To this end we use the notations from equation (3.1.4), and the definition (8.1.19) of the coframe, so that we need to solve the following system of differential equations:

$$\frac{du}{v^2} = a^1 d\phi, \quad -\frac{udu - vdv}{v^2} = a^2 d\phi, \quad \frac{(u^2+v^2)du - 2uvdv}{v^2} = a^3 d\phi$$

This system is compatible if:

$$\frac{d\phi}{v^2}\begin{vmatrix} 1 & 0 & a^1 \\ -u & v & a^2 \\ u^2+v^2 & -2uv & a^3 \end{vmatrix} = 0$$

which happens if the following equation is satisfied:

$$a^1(u^2 - v^2) + 2a^2 u + a^3 = 0$$

This equation represents a family of real hyperbolas in coordinates (u, v), depending on two parameters:

$$\left(u + \frac{a^2}{a^1}\right)^2 - v^2 = \left(\frac{a^2}{a^1}\right)^2 - \frac{a^3}{a^1}, \quad a^1 a^3 - (a^2)^2 > 0 \tag{8.1.22}$$

where the last inequality reproduces the condition from equation (8.1.16). It is among the hyperbolas of this family that the geodesics of the metric (8.1.21) are to be found. Again, according to algebraical experience, one can have two sets of parametric equations of these curves in terms of ϕ:

$$\begin{cases} u(\phi) = u_0 + v_0 \sinh\phi \\ v(\phi) = v_0 \cosh\phi \end{cases} \quad or \quad \begin{cases} u(\phi) = u_0 + v_0 \tan\phi \\ v(\phi) = \dfrac{v_0}{\cos\phi} \end{cases} \tag{8.1.23}$$

both of them valid under the same conditions on parameters, namely:

$$u_0 = -\frac{a^2}{a^1} \quad and \quad u_0^2 + v_0^2 = \frac{a^3}{a^1} \tag{8.1.24}$$

but one of them fails to satisfy the differential equations of geodesics. Indeed, when it comes to verification of the equations (8.1.20), only the second set of the equation (8.1.23) is valid, so that these equations are to be taken as the general parametric equations of the geodesics of conform-Lorentzian metric (8.1.21):

$$u(\phi) = u_0 + v_0 \tan\phi, \quad v(\phi) = \frac{v_0}{\cos\phi} \tag{8.1.25}$$

where the parameters are defined in equation (8.1.24). Along these geodesics, the equation (8.1.21) becomes an ordinary second-order differential equation too:

$$X''(\phi) - 2a^2 X'(\phi) + a^1 a^3 X(\phi) = 0$$

where the accent means differentiation with respect to ϕ. The general solution of this equation is:

$$X(\phi) = e^{-a^1 u_0 \phi}(A \cdot e^{ia^1 v_0 \phi} + B \cdot e^{-ia^1 v_0 \phi}) \tag{8.1.26}$$

in view of the condition from equation (8.1.22).

This time the physical interpretation is not referring to the process of tunneling. However, an interpretation comes right away from the comparison of the equations (7.3.18) and (8.1.25): the geodesics of the conform-Lorentzian metric represent simply geodesics of the optical medium represented by the Planck's equation in question. The variation of the parametters u_0 and v_0 from equation (8.1.25) is done by tunneling, while the variation of the counterparts of these parameters from equation (8.1.14) is always periodical.

8.2. Specific Stationary Values of Surface Processes

Let us see what kind of stationary magnitudes correspond to the above two coframes above. For the case of coframe (8.1.5) the equation (6.5.9) reads:

$$v^2 \cdot dY = du \cdot Y^2 + 2(udu + vdv) \cdot Y + (u^2 - v^2)du + 2uvdudv$$

and this differential equation can be rearranged in the form

$$v^2 \cdot d(Y+u) = du \cdot (Y+u)^2 + (Y+u) \cdot d(v^2)$$

which admits the following immediate integral:

$$\frac{v^2}{u+Y} + u + Y_0 = 0 \tag{8.2.1}$$

where Y_0 is a constant: the stationary value of the Planck's equation. This means that we have the following constant integral that generates the Planck's equation, given by a 'specimen' of the equation (3.1.1), as it were:

$$Y_0 = \frac{-u \cdot Y - u^2 - v^2}{Y + u} \tag{8.2.2}$$

That means that, the Planck's stationary value Y_0 for this equation is obtained from the current value Y *via* the homographic action on Y of the matrix:

$$\boldsymbol{I} \equiv \begin{pmatrix} -u & -u^2 - v^2 \\ 1 & u \end{pmatrix} \tag{8.2.3}$$

which produces a unique value for all (u, v). One can then verify that the *Cartan matrix* [see (Guggenheimer, 1977), Chapter 1, § 1.2, for the notion] corresponding to this transformation, that is:

$$\boldsymbol{I}^{-1}d\boldsymbol{I} = \frac{dv}{v}\cdot\boldsymbol{1} + \begin{pmatrix} -\omega^2/2 & -\omega^3 \\ \omega^1 & \omega^2/2 \end{pmatrix} \tag{8.2.4}$$

where the entries of matrix are given by the components of the coframe (8.1.5).

By the same token the coframe (8.1.19) produces the first constant integral analog to (8.2.2):

$$Y_0 = \frac{u\cdot Y - u^2 + v^2}{Y - u} \tag{8.2.5}$$

which means that the Planck constant is produced by the homographic action of the matrix:

$$\boldsymbol{J} \equiv \begin{pmatrix} u & -u^2 + v^2 \\ 1 & -u \end{pmatrix} \tag{8.2.6}$$

instead of the matrix (8.2.3). The Cartan's matrix of transformation (8.2.5) is formally given by the same equation as (8.2.4):

$$\boldsymbol{J}^{-1}d\boldsymbol{J} = \frac{dv}{v}\cdot\boldsymbol{1} + \begin{pmatrix} -\omega^2/2 & -\omega^3 \\ \omega^1 & \omega^2/2 \end{pmatrix} \tag{8.2.7}$$

except that the entries are given by the components of the coframe (8.1.19).

Now, these are purely mathematical facts. Let us see how they fit in the physical image of the world. First, take the parametric equations of the geodesics of the conformal-Lorentzian metric (8.1.21), as given by us in (8.1.25), and compare them with the parameters of the cycles from equation (7.3.18). A conclusion imposes naturally: the Lorentzian coordinates represent the center and the radius for the family of circles connected with the Planck's dipoles. Then, the conform-Lorentzian arclength is the angle between the circles of this family. Physically, we can go to a greater depth: as the two fixed charges of the Planck resonator have, in general, the coordinates (u_0, v_0) and (u_0, $-v_0$), we are entitled to associate these coordinates with the charges of the dipoles. Incidentally, this association respects the original spirit of the Lorentz's definition of the electrical matter (see discussion in § 6.3 above), whereby the charges of the dipole are located in the positions $\pm v_0$, below and above the 'surface' $v_0 = 0$. The presence of u_0 helps us improve the Lorentz's image on account of the Katz's natural philosophy with the help of the complex or even Clifford numbers, in general.

Continuing the association of the mathematics with physics along these lines, let us take notice of the equation (8.1.24): it shows that the dipole itself must be represented by a 2×2 matrix of the kind given in equation (8.2.3), only taking in consideration the particular conditions:

$$\boldsymbol{I}_0 \equiv \frac{1}{v_0}\cdot\begin{pmatrix} u_0 & -u_0^{\ 2} - v_0^{\ 2} \\ 1 & -u_0 \end{pmatrix} \tag{8.2.8}$$

The equation (8.1.25) represents the ensemble of cycles connected with this resonator: each cycle corresponds to a given value of the phase ϕ. By the same token, we can hope for more, just noticing the equation (8.1.13): it says that, along the geodesics of the Beltrami-Poincaré metric a matrix of the type (8.2.6) represents a kind of dipole, corresponding to the matrix

$$\boldsymbol{J}_o \equiv \frac{1}{v_o} \cdot \begin{pmatrix} u_o & -u_o^2 + v_o^2 \\ 1 & -u_o \end{pmatrix} \tag{8.2.9}$$

Each cycle of the corresponding family of this dipole is also represented by a point on the geodesic (8.1.14), corresponding to a given value of ϕ. As a general conclusion here: *a value of that phase is associated with a cycle* in either one of the cases.

The matrices (8.2.8) and (8.2.9) are two involutions. According to the theory from § 4.4, their product must be an involution, which is, indeed, the case:

$$\boldsymbol{I}_o \cdot \boldsymbol{J}_o = \begin{pmatrix} -1 & 2u_o \\ 0 & 1 \end{pmatrix} \equiv \boldsymbol{K}_o \tag{8.2.10}$$

These three matrices satisfy to the arithmetic given in the multiplication table (6.3.8), and we will use them for quite a few purposes serving theoretical physics at large, but especially the physics of brain.

First of all, they will allow us to describe the physics as performed on a surface: even in the physical universe, we do not do physics in a point in space, but on patches of the Earth's surface, and this fact has consequences in the theoretical aspects of the physics we do, especially in the manner of associating to Earth a position in space [see (Mazilu, 2024), §§ 2.5 and 3.5]. Secondly, they will allow us to introduce the complex numbers in the theory, as we said, in order to comply to Katz's natural philosophy. The manner to introduce the complex numbers is particularly simple: it starts by noticing that ***I***, for instance, is of the form (4.1.1), to which corresponds a ***J*** which, in complex is a sum of two singular matrices:

$$\boldsymbol{J} \stackrel{\text{def}}{=} \begin{pmatrix} z + z^* & -z^2 - z^{*2} \\ 2 & -(z + z^*) \end{pmatrix} \tag{8.2.11}$$

Using this representation comes in as particularly handy in a quantization of Barbilian type (§ 7.4) describing the connection between two JARs, or between a JAR and an elementary ray in the brain universe.

8.3. The Harmonic Oscillator Equation in Specific Details

Two differential equations make their appearance so often in our presentation, that we need to insist on their consequences from physical point of view. One of them is the equation of the one-dimensional harmonic oscillator, whose general form is

$$\ddot{x}+2p_1\dot{x}+p_2x=0 \tag{8.3.1}$$

where p_1 and p_2 are generally taken as functions of time t. In the regularization theory this equation appears without the coefficient p_1, thus describing an undamped harmonic oscillator. The general problem is that this equation occurs as such no matter of the space scale but what it describes is a matter of the time scale used to reckon the time of motion. Then the necessity occurs to describe the transition of scales of space and time in such a way that the standard form of the equation is invariant.

Such a transformation is taken to be of the form:

$$x(t)=\lambda(t)\cdot\xi(t),\quad \tau=\tau(t) \tag{8.3.2}$$

where λ is the factor of 'space scale magnification', so to speak. Applying first the transformation of coordinate, we get a second order equation for ξ:

$$\ddot{\xi}(t)+2\pi_1(t)\cdot\dot{\xi}(t)+\pi_2(t)\cdot\xi(t)=0 \tag{8.3.3}$$

where the new coefficients, taken as functions of the original time, are

$$\pi_1(t)=p_1(t)+\frac{\dot{\lambda}}{\lambda},\quad \pi_2(t)=p_2(t)+\frac{\ddot{\lambda}}{\lambda}+2p_1(t)\frac{\dot{\lambda}}{\lambda} \tag{8.3.4}$$

Now, work on the equation (8.3.3) with the second of the transformations (8.3.2), in order to get

$$\xi''(\tau)+\frac{\ddot{\tau}+2\pi_1\cdot\dot{\tau}}{\dot{\tau}^2}\cdot\xi'(\tau)+\frac{\pi_2(t)}{\dot{\tau}^2}\cdot\xi(\tau)=0 \tag{8.3.5}$$

where a prime means derivative on the new time. This equation and (8.3.1) should be of the same form, as it appears in the regularization theory. That is, (8.3.1) should be:

$$\ddot{x}+\omega_0^2x=0 \quad \therefore \quad \begin{cases} p_1=0 \\ p_2=\omega_0^2 \end{cases} \tag{8.3.6}$$

while (8.3.5) should be of the form

$$\xi''(\tau)+\frac{\pi_2(t)}{\dot{\tau}^2}\cdot\xi(\tau)=0 \tag{8.3.7}$$

where the coefficient is to be calculated for the condition

$$\ddot{\tau}+2\pi_1\cdot\dot{\tau}=0 \tag{8.3.8}$$

guaranteeing the vanishing the midle term in (8.3.5). Under the conditions from equation (8.3.6), this equation is connecting in fact the two function realizing the transformation (8.3.2):

$$\frac{\ddot{\tau}}{\dot{\tau}}+2\frac{\dot{\lambda}}{\lambda}=0 \quad \therefore \quad \lambda^2\dot{\tau}=const \tag{8.3.9}$$

On the other hand, in view of the equations (8.3.6) and (8.3.9), the second expression from (8.3.4) is

$$\pi_2(t) = p_2(t) - \frac{1}{2}\{\tau, t\} \tag{8.3.10}$$

where

$$\{\tau, t\} \stackrel{def}{=} \frac{d}{dt}\left(\frac{\ddot{\tau}}{\dot{\tau}}\right) - \frac{1}{2}\left(\frac{\ddot{\tau}}{\dot{\tau}}\right)^2$$

is the *Schwarzian derivative* of the function τ with respect to t. One can therefore draw the known result that oscillators of the same frequency are connected by a homographic transformation of times. More precisely, the general transformation (8.3.2) that invariates the equation (8.3.6) is of the form

$$\tau(t) = \frac{\alpha t + \beta}{\gamma t + \delta}, \quad \xi = C\frac{x}{\gamma t + \delta} \tag{8.3.11}$$

where C is an arbitrary constant. The equation (8.3.7) becomes

$$\xi''(\tau) + \frac{C^2\omega_0^2}{(\gamma\tau - \alpha)^4} \cdot \xi(\tau) = 0 \tag{8.3.12}$$

where the first relation from (8.3.11) has been used in order to eliminate the original time, and C is, again, an arbitrary constant. The general solution of this equation can be expressed in terms of Bessel functions of half integer order [(Bowman, 1958); equation (6.82) for $\alpha = 1/2$; $\gamma = -1$; $n = \pm 1/2$ in that formula]:

$$\xi(\tau) = \sqrt{\gamma\tau - \alpha}\left\{AJ_{1/2}\left(\frac{\Omega}{\gamma\tau - \alpha}\right) + BJ_{-1/2}\left(\frac{\Omega}{\gamma\tau - \alpha}\right)\right\} \tag{8.3.13}$$

with A and B two arbitrary constants and

$$\Omega^2 \equiv \frac{C^2\omega_0^2}{\gamma^2} \tag{8.3.14}$$

in terms of the parameters used here.

8.4. Assigning Frequency to a Signal: the Holographic Property of a Universe

Like the *velocity*, that can be assigned only *via* the uniform motion, the *frequency* can only be assigned *via* the idea of periodic motion. The physical prototype of a periodic motion – the analogous of a free particle moving uniformly, as it were – is the harmonic oscillator, described by a second-order differential equation:

$$\ddot{x}(t) + \omega_0^2 x(t) = 0 \tag{8.4.1}$$

Thus, assigning a frequency to a *signal that can be represented as a periodic function of time* is particularly simply, for it is provided by the above equation:

$$\omega_0^2 \stackrel{def}{=} -\frac{\ddot{x}(t)}{x(t)} \tag{8.4.2}$$

The overwhelming majority of physical cases is, however, not so simple, and this is the reason why the assumption of a direct time dependence of the coordinate created so much trouble. In particular, in the case of light, the *phenomenon of holography* could not enter the phenomenology of light but only after admitting that the time dependence of the phenomenon *has to be mediated by a phase*. Quoting:

> The new method is an attempt to *get around the obstacle*, *instead of across it*, by a two-step process, in which *the analysis is carried out with electrons*, the *synthesis by light*. The general idea of such a process was first suggested to the author by Sir Lawrence Bragg's 'X-ray microscope'. But Bragg's method, in which a lattice is reconstructed by diffraction from an X-ray diffraction pattern, can be applied only to a rather exceptional class of periodic structures. It is customary to explain this by saying that the diffraction diagrams *contain information on the intensities only*, *but not on the phases*. The formulation is somewhat unlucky, as it suggests at once that *since the phases are unobservables*, this state of affairs must be accepted. In fact, not only that part of the phase which is unobservable drops out of *conventional diffraction patterns*, but also *the part which corresponds to geometrical and optical properties of the object*, and which in principle could be *determined by comparison with a standard reference wave*. It was this consideration which led me finally to the new method. [(Gabor, 1949), *our emphasis*]

The basic principle of holography can be simply described mathematically, in view of the theory from the previous sections of this chapter: it is the phenomenon occurring when we try to associate a time dependence to a process depending on time *not in an obvious periodic way* though. For, in such a general case, when suspecting a periodic behavior of the phenomenon, we can try to model it by an equation of the form:

$$q(t) = A(t)e^{i\theta(t)} \tag{8.4.3}$$

involving a phase θ and an amplitude A, both functions of time. If there is a frequency involved here, it can only be exhibited by a periodic motion of the kind described by equation (8.4.1). Assuming such an equation for $q(t)$, leads to:

$$\ddot{q} + \omega_0^2 q = 0 \quad \therefore \quad \begin{aligned} &\frac{\ddot{A}}{A} + \omega_0^2 = \dot{\theta}^2 \\ &2\frac{\dot{A}}{A} + \frac{\ddot{\theta}}{\dot{\theta}} = 0 \end{aligned} \tag{8.4.4}$$

The second of these equation gives right away

$$A^2\dot{\theta} = const \tag{8.4.5}$$

which is a kind of Kepler's second law, suggesting a periodic motion for the amplitude. It can be exhibited right away, for, in these conditions, the first of the equations (8.4.4) gives an Ermakov-Pinney equation for the amplitude:

$$\ddot{A}+\omega_0^2 A=\frac{R_0^2}{A^3} \tag{8.4.6}$$

where R_0 is a real constant. Now, the connection with the periodic motion is the following. If A is the composite amplitude of a two-dimensional harmonic oscillator described in time according to the equation from (8.4.4), *i.e.*:

$$A^2=A_1^2+A_2^2, \quad \ddot{A}_1+\omega_0^2 A_1=0, \quad \ddot{A}_2+\omega_0^2 A_2=0 \tag{8.4.7}$$

it satisfies the equation (8.4.6) with R_0 the constant from (8.4.5). Thus, the frequency ω_0 is asociated to the components of the vector $|A\rangle$ in an obvious way. In these conditions, one can calculate right away that the square of amplitude from equation (8.4.3) satisfies a linear third order differential equation:

$$\frac{d^3}{dt^3}A^2+4\omega_0^2\frac{d}{dt}A^2=0 \tag{8.4.8}$$

The equation of definition for the frequency, can be obtained from the Kepler law (8.4.5). Taking the definition for the amplitude as a function of phase – according to the holographic principle – results in a second order differential equation for amplitude:

$$A=\frac{C}{\sqrt{\dot{\theta}}} \quad \therefore \quad \ddot{A}+\frac{1}{2}\{\theta,t\}A=0 \tag{8.4.9}$$

Thus, the holographic principle allows us to define the frequency by the equation

$$\{\theta,t\}=2\omega_0^2 \tag{8.4.10}$$

which, in turn, permits a plus of precision of the holographic principle: all the phases differing by a homographic transformation reveal the same frequency. Further on, this philosophy allows us to define a time for the Einsteinian universe, based on the idea of interpretation: it is that time for which the frequency is zero, where the oscillator is actually a free particle.

We can go now for Arnold's theorem, which has a great importance of principle: a uniform rectilinear motion is always equivalent to a periodic motion. So the *holographic principle* is, in a way, *equivalent to the principle of inertia*. Consider, indeed, (x,t) the case of a harmonic oscillator, described by equation (8.3.1) with constant coefficients. Then the system defined by (ξ,τ) according to equations:

$$\xi(\tau)=\frac{x(t)}{v(t)}, \quad \tau(t)=\frac{u(t)}{v(t)} \tag{8.4.11}$$

where $v(t)$ is assumed nonzero, is a free particle. The demonstration can be done by brute force: first, differentiate ξ on τ, to get

$$\xi'(\tau) = \frac{t'}{v} \cdot (\dot{x} - \xi\dot{v}) \tag{8.4.12}$$

where a prime means derivative on τ, and a dot over means derivative on t. Now differentiate this with respect to *t*. The left hand side gives:

$$\frac{d}{dt}\xi'(\tau) \equiv \dot{\tau} \cdot \xi''(\tau) \tag{8.4.13}$$

while the right hand side gives:

$$\frac{d}{dt}\left(\frac{t'}{v} \cdot (\dot{x} - \xi\dot{v})\right) \equiv \frac{t'}{v} \cdot \left(\ddot{x} - \dot{\xi}\dot{v} + \xi(2p_1\dot{v} + p_2 v)\right) + \frac{d}{dt}\left(\frac{t'}{v}\right) \cdot (\dot{x} - \xi\dot{v}) \tag{8.4.14}$$

where we conveniently used the equation (8.3.1). Let us concentrate on this last one. We have

$$\dot{\tau} = \frac{v\dot{u} - u\dot{v}}{v^2} \quad \Leftrightarrow \quad \frac{t'}{v} = \frac{v}{v\dot{u} - u\dot{v}}$$

and therefore

$$\frac{d}{dt}\left(\frac{t'}{v}\right) = \frac{\dot{v} + 2p_1 v}{v\dot{u} - u\dot{v}}$$

Using the last two results in (8.4.14) gives the result:

$$\frac{d}{dt}\left(\frac{t'}{v} \cdot (\dot{x} - \xi\dot{v})\right) \equiv \frac{v}{v\dot{u} - u\dot{v}} \cdot \left(\ddot{x} + 2p_1\dot{x} + p_2 x\right) \tag{8.4.15}$$

which, identified with (8.4.13) provides the identity

$$\frac{v\dot{u} - u\dot{v}}{v^3} \cdot \xi''(\tau) \equiv \ddot{x} + 2p_1\dot{x} + p_2 x \tag{8.4.16}$$

which proves the Arnold's theorem, for the factor is always nonzero in the given conditions.

8.5. On the Position of Regularization Equation

One ordinary differential equation that we cannot pass in our study is that of the square of two solution of a harmonic oscillator equation (8.3.6). Indeed, if *x* and *y* are two independent solutions of the equation (8.3.6), then we have

$$\dddot{q} + 4\omega_0^2\dot{q} = 0, \quad q \equiv x^2 + y^2 \tag{8.5.1}$$

In fact *q* can be any quadratic form made with the two solutions *x* and *y*: the equation (8.5.1) remains the same. In order to justify the importance of this equation let us recall that it is the equation verified by the ray components, in the case where the refraction index varies in a certain way [see (Mazilu, 2024), equation (1.2.11)]. It is also the equation verified by the components of the position vector of the Kepler motion in the central regularization procedure and by the inverse radial coordinate in the focal regularization procedure, written by us under (5.3.14) and (5.5.8) respectively. This makes it the fundamental equation describing the three-dimensional regularization procedure, written by us under (6.2.9). And, if this list of presences of this equation is not enough in order to convince us of its importance, consider this: it is the fundamental equation describing the two kinds of charges in an interpretation of the de Sitter continuum. Indeed, according to Katz's natural philosophy of the charge of a particle (Katz, 1965), can be represented as having both an electric component and a magnetic component, whose participations in the making of the charge are described by trigonometric functions of a phase that we have called *split angle* [see (Mazilu, 2020), §3.1]. Each part of the charge can be considered as a solution of an equation of the type (8.3.6), and the charge we observe is the sum of squares of the two components. Therefore the squares of the charges we observe verify each one of them, an equation equation of the type (8.5.1). If this is the case, then the propagation of a ray of light, for instance, can be explained by the connection between two types of equations (8.5.1), one of them describing the charges in a de Sitter continuum, the other describing the propagation. That may be, indeed, a serious incentive for our study of this equation.

According to Katz's natural philosophy, the charge continuum is four-dimensional, being the topological product of two two-dimensional blades: one electric and one magnetic. These are practically identical with each other, but the evolution in the two planes is different. However, this evolution is planar and, if represented by a curve, this curve is a solution of a third order differential equation with respect to a parameter playing the part of time. Like in the case of the harmonic oscillator, we write this equation in a general form:

$$\dddot{x} + 3p_1\ddot{x} + 3p_2\dot{x} + p_3 x = 0 \tag{8.5.2}$$

and analyze what is happening under a transformation (8.3.2). The method is to apply the transformation in two steps, just like before. In the first step we have a resulting equation for ξ as a function of t:

$$\dddot{\xi} + 3\pi_1\ddot{\xi} + 3\pi_2\dot{\xi} + \pi_3\xi = 0 \tag{8.5.3}$$

where the coefficients are given by:

$$\begin{aligned} &\pi_1 = p_1 + \frac{\dot{\lambda}}{\lambda}, \qquad \pi_2 = p_2 + 2p_1\frac{\dot{\lambda}}{\lambda} + \frac{\ddot{\lambda}}{\lambda} \\ &\pi_3 = p_3 + 3p_2\frac{\dot{\lambda}}{\lambda} + 3p_1\frac{\ddot{\lambda}}{\lambda} + \frac{\dddot{\lambda}}{\lambda} \end{aligned} \tag{8.5.4}$$

Now apply the second part of the transformation, that gives the coordinate ξ as a function of τ as a solution of a third order differential equation instead of (8.3.5) from the previous case of the oscillator:

$$\xi''' + 3\frac{\ddot{\tau} + \pi_1\dot{\tau}}{\dot{\tau}^2}\xi'' + \frac{1}{\dot{\tau}^2}\left(3\pi_2 + 3\pi_1\frac{\ddot{\tau}}{\dot{\tau}} + \frac{\dddot{\tau}}{\dot{\tau}}\right)\xi' + \frac{\pi_3}{\dot{\tau}^3}\xi = 0 \tag{8.5.5}$$

Everything comes down to the comparison between the equation (8.5.2) and (8.5.5). If these two equations must be of the form (8.5.1), that is:

$$\dddot{x} + 4\omega_0^2\dot{x} = 0, \quad \xi''' + 4\Omega_0^2\xi' = 0 \tag{8.5.6}$$

then, first of all, (8.5.5) provides the following differential equations:

$$\ddot{\tau}+\pi_1\dot{\tau}=0, \quad 3\pi_2+3\pi_1\frac{\ddot{\tau}}{\dot{\tau}}+\frac{\dddot{\tau}}{\dot{\tau}}=4\Omega_0^2\dot{\tau}^2, \quad \pi_3=0 \tag{8.5.7}$$

On the other hand, for this case (8.5.4) provides the expressions:

$$\pi_1=\frac{\dot{\lambda}}{\lambda}, \quad 3\pi_2=4\omega_0^2+3\frac{\ddot{\lambda}}{\lambda}, \quad \pi_3=4\omega_0^2\frac{\dot{\lambda}}{\lambda}+\frac{\dddot{\lambda}}{\lambda} \tag{8.5.8}$$

so that the equations from (8.5.7) become

$$\frac{\ddot{\tau}}{\dot{\tau}}+\frac{\dot{\lambda}}{\lambda}=0, \quad 4\omega_0^2+3\frac{\ddot{\lambda}}{\lambda}+3\frac{\dot{\lambda}}{\lambda}\frac{\ddot{\tau}}{\dot{\tau}}+\frac{\dddot{\tau}}{\dot{\tau}}=4\Omega_0^2\dot{\tau}^2, \quad \dddot{\lambda}+4\omega_0^2\dot{\lambda}=0 \tag{8.5.9}$$

Notice first, the last one of these equations: the space scale factor λ has the nature of a charge. On the other hand, the first of these equations provides the connection

$$\lambda\cdot\dot{\tau}=const \tag{8.5.10}$$

instead of (8.3.9), and transforms the second one of these equations into:

$$\{\tau,t\}+2\Omega_0^2\dot{\tau}^2=2\omega_0^2 \tag{8.5.11}$$

which is an Ermakov-Pinney equation:

$$\ddot{\mu}+\omega_0^2\mu=\frac{\Omega_0^2}{\mu^3}, \quad \mu^2=\lambda \tag{8.5.12}$$

For future purposes, we need to present the previous theory regarding the connection between the linear second order differential equation and the Ermakov-Pinney equation from still another, universal we should say, point of view. Namely, there is an interplay between three differential equations here. Take the second-order differential equation (8.3.1):

$$\ddot{x}+2p_1\dot{x}+p_2x=0 \tag{8.5.13}$$

and choose a basis of two independent solutions $u(t)$ and $v(t)$. With these, construct a quadratic form *with constant coefficients*, which defines two new variables as follows:

$$q\overset{def}{=}\alpha u^2+2\beta uv+\gamma v^2\overset{def}{=}r^2 \tag{8.5.14}$$

Then $q(t)$ is a solution of the third order differential equation [see (Bellman, 1997), p. 179, Exercise 3]:

$$\dddot{q}+12p_1\ddot{q}+4(\dot{p}_1+8p_1^2+p_2)\dot{q}+2(\dot{p}_2+8p_1p_2)q=0 \tag{8.5.15}$$

while $r(t)$ is a solution of the second order differential equation [see (Eliezer & Gray, 1976), §7]:

$$\ddot{r}+2p_1(t)\dot{r}+p_2(t)r=(\alpha\gamma-\beta^2)\frac{(v\dot{u}-u\dot{v})^2}{r^3} \tag{8.5.16}$$

Between these last two equations, there is therefore a connection based on the association of q with r that results from equation (8.5.14). Now, assume that q is a quantum: the equation (8.5.14) associates a matrix to this quantum, whose entries are the coefficients of the quadratic form. The variables of this quadratic form are the basic solutions of Yang's type, of a second order ordinary differential equation of the type given in the equation (8.5.13). By this interpretationthe variable r gets an interpretation connected to the variance of the ensemble generated by the corresponding Planck's equation.

9. Affine Geometry: a Kind of Physics Within a Lorentz Quantum

We interrupt here the streak of mathematics related to the physics of Planckian procedure of quantization, appropriate in the description of a universe: it becomes overwhelming as it is, and there is still more to say that can be deferred to future instalments of the work. But, before anything else to say further, let us present some geometry connected to Planck's procedure of quantization, and suggested by this very procedure. That geometry seems to be worth signaling, in view of an inedit connection with the mainstream physics, that can serve as guidance in the construction of the concept of a universe from a point of view transcending the Maxwellian precepts. Thus, while the previous discussion is fresh in our minds, we propose a geometrical approach of electromagnetism, having, on one hand, strong ties with the Planckian status of charges presented above, as we said, and with the Ampère's update of the Newtonian forces. On the other hand, this geometrical approach has equally strong ties with the general theory of the $\mathfrak{sl}(2,R)$ Riemannian manifolds, as introduced to our wits by that special simply transitive action of the 2×2 matrices, involving the algebraic concept of *Hessian equation*, previously described in connection with the Peter McCullagh's case of sampling in § 3.2 of this instalment of our work. This geometrical approach is made along the concepts of an affine geometry, and apparently allows for a special kind of geometrization, with respect to which, the Maxwellian electrodynamics appears as just a particular case. Not only this, but also its Yang-Mills quantum counterpart, may appear as a particular case too, if approached from a certain angle, suggested by the general relativity in Einstein's take [see (Mazilu, 2024), especially §5.4].

In order to better understand the necessity and the structure of this chapter it is, obviously, best to relegate the physics to a certain mathematics of geometry. Again, we will present here the geometrical theory with emphasis on the tensorial character of the physical magnitudes occurring in the measurement process. This is an affine theory, which, in our opinion, has direct connections with the concept of memory, whence, for once, our urge to present it, let us say, just as a side issue. Quoting some definitions with reference to the grounds of such an approach, will prove salutary for our discourse:

> An *affine space* is nothing more than a vector space *whose origin we try to forget about*, *by adding translations to the linear maps*. It follows that the elementary properties of affine spaces,

of their morphisms and of their subspaces are all properties from linear algebra, more or less disguised. As a consequence, most demonstrations are automatic: to prove a result about an affine space, the idea is generally to use some appropriate translations to reduce the statement to a vector form. This often means *vectorializing the space* at the appropriate point. [(Berger, 1987), Opening of Chapter 2; *our emphasis*, *a*/*n*]

The emphasized phrases in this excerpt show our points. For once, the essential one of these points, namely the "origin we try to forget", tentatively shows the place of what *we may define as memory*: namely, *the origin* of a reference frame. For, it is this one that we always try forget in a physical theory, and a proper forgetting means, physically speaking, erasing from memory. Naive as it may sound, this phrase just indicates a place of locating the memory, especially from a physical point of view. Secondly, the "vectorializing the space", can take quite a wide range of meanings, in order to comprise, for instance, the vectors defined in the most general manner: as operators of an $\mathfrak{sl}(2,R)$ algebra, or as algebras of matrices describing some reference frames. In any case, the two points of our 'physical intervention', as it were, will involve here *the theory of affine surfaces*, as suggested by Louis de Broglie's construction of a physical ray (Mazilu, 2020). This concept is, by and large, the subject of the present chapter.

Concerning the physics involved in some process of 'forgetting the origin', one can say that the mathematical procedure of 'adding translations to the linear maps' does not seem to be sufficient in order to construct it. For once, 'forgetting' becomes, at least in physics, a manner of action, something equivalent with the modern 'erasing', and this is by no means a passive exploit. Taking in consideration the concept of interpretation, we can bring into argument, for instance, the Edwin Crawford Kemble's *idea of ensemble* as *a history destroying device* for giving a concrete manner of erasing the memory. With this, the idea of ensemble in general, considered from the mathematical point of view of cardinality, acquires a meaning in close connection with the concept of physical memory. More to the point, it can be taken as meaning that the interpretation *per se*, brought about as a concept by the very occurrence of the wave mechanics, has, in fact, the important physical meaning of erasing a certain memory. Quoting the very words of Edwin Kemble:

> Experiment shows that the *statistical properties* of a large assemblage of independent identical microscopic, or macroscopic, systems (*i.e.*, a Gibbsian assemblage) which has been "aged" in a thermostat *at a definite temperature* T *for a sufficient length of time* usually *become constant* and *independent of the initial state* of the assemblage. The ultimate state is then defined to be *one of thermodynamic equilibrium at the temperature* T. By erasing all vestiges of the initial state *the thermostat acts as a history-destroying device*. To be sure there are numerous cases in which this function is *imperfectly performed*. In such cases the state of true thermodynamic equilibrium, or

> maximum entropy, is *not reached in any measurable time at moderate temperatures*. We may restrict the discussion for the present, however, to systems for which thermodynamic equilibrium is actually attainable. [(Kemble, 1937), p. 433; *our Italics*]

What we need to add to this philosophy of a genuinely physical character, is the fact that the mathematical concept of cardinality adjoins to our reasoning the unquestionable point that the history destroying physical function should always be *present* in realizing the interpretation, as a physically performed function. It leads to our initial interpretative ensemble that, in a well-known historical occurence can be defined, for example, by the presence of Newtonian forces, in a fictitious ensemble of classical material points in equilibrium. 'Destroying history' is thus a physical fact of 'forgetting', but upon destroying, something still remains in the definition of the elements of the interpretative ensemble. In a word, it is the holographic part of the memory that remains, and this is to be described here *as connected to an affine surface*.

This theory is, in essence, a theory of transition between the ordinary space and the coordinate space, as claimed by Charles Galton Darwin in his definition of the interpretation (Darwin, 1927). The general aspect of a surface to be thus constructed is indicated, for instance, by George Ruppeiner in association with his general theory of a Riemannian geometry, incorporating even the thermodynamics [see (Ruppeiner, 2010); especially Figure 4 therein, illustrating the old concept of a *surface of constant density* in the phase space. This surface is, in our opinion, the one needed from Appell-Miles analytic point of view! see I, §5.4; see also (Mazilu, Agop, & Mercheș, 2021), Chapter 4].

We have presented in many places of our published effort of understanding physics, a summary of the Cartanian point of view on the theory of surfaces to be used as reference in the geometrization of the adjacent space [see, for instance, the most recent production (Mazilu, 2024), §1.3; in what follows here we shall use the arrangement from that work]. This approach in constructing the geometry, is to be chosen because, in our opinion, it is the closest one to any physical point of view that might be related to the existence of surfaces, and, in fact, for the existence of mathematical physics in general. So much the more, the concept of physical ray as developed according to the ideas of Louis de Broglie needs it, almost exclusively, we should say. The current chapter of the present work is merely intended as an illustration, and as an expanding in fact, of this statement.

We will show here that the 'reverse interpretation', as we called it, – consisting from the association of a continuum to discrete manifolds, the procedure based on which the general relativity is constructed – involves a further step in the theory of surfaces, namely the *affine theory*. Its clearest concept, expressed in differential forms, of course – otherwise a *physical theory* of surfaces could not even exist! – was already provided, in our views, by an article of Shiu-Yuen Cheng and Shing-Tung Yau (Cheng & Yau, 1986). What follows right away is a reproduction, in broad strokes, of that material [see also the classical work of Shiing-Shen Chern and Chuu-Lian

Terng, which inspired, in fact, many other valuable works in the spirit of the one just cited above (Chern & Terng, 1980)].

9.1. A Brief in Cartanian Viewpoint on Affine Geometry

In the affine version of surface geometry, we take three non-collinear vectors, $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, and $\boldsymbol{e}_3$ defined at the same space location, as a reference frame for the surrounding space. Theoretically, these vectors should be in a general position with respect to each other. More to the point, the three-dimensionality is defined by the fact that there is no homogeneous linear relation among the three vectors, because from affine point of view we do not know what is a relative direction in space. All we know, and assume of course, is the three-dimensionality of space as characterized from algebraic point of view. Thus, the non-collinearity is simply defined here by the algebraical property of linear independence. After all, the direction, as described by an angle or two, is a metric concept, and we do not have a metric at our disposal just yet. Assume, therefore, that ($\boldsymbol{e}_1$, $\boldsymbol{e}_2$) is a frame in general position on a certain surface. Looked upon from the environment, the surface here appears as embedded in space, in the very same way it is embedded in the Euclidean space. Thus, each elementary displacement vector of a *generic* point of the surface can still be defined in the form

$$d\boldsymbol{m} = s^k \boldsymbol{e}_k \equiv s^\alpha \boldsymbol{e}_\alpha + s^3 \boldsymbol{e}_3 \quad \therefore \quad d\boldsymbol{m} = s^\alpha \boldsymbol{e}_\alpha ;\;\; s^3 = 0 \qquad (9.1.1)$$

However, this time the two in-surface vectors ($\boldsymbol{e}_1$, $\boldsymbol{e}_2$) are not necessarily orthonormal. Neither may be the third vector: the most to be asked now is that it should point out of surface, *i.e*., it should be *transversal* to surface. It is not hard to see here a connection with the theory of infinitesimal deformation of a portion of surface, which stays at the basis of the definition of the charge according to Poincaré-Lorentz natural philosophy (see § 5.2 above). One can say that the affine theory of surfaces provides a general mathematical framework to physics, that goes above and beyond the classical theory of surfaces. This fact is plainly confirmed by the evolution of such a mathematical theory, as we shall see soon in this chapter. Again, the last equation (9.1.1) here defines a connection between the position in space and the surface. It is sometimes convenient to write the space displacement in the suggestive form of a dot-product:

$$d\boldsymbol{m} = s^k \boldsymbol{e}_k \equiv \langle s | \boldsymbol{e} \rangle \qquad (9.1.2)$$

with $|\boldsymbol{e}\rangle$ denoting here the general reference frame in space. Then, the equations of compatibility between frame and position, come out, as usually in a Cartan-type theory, from the fact that $d\boldsymbol{m}$ is an exact differential vector, to which we add an *assumed* Frenet-Serret frame evolution:

$$|d\boldsymbol{e}\rangle = \boldsymbol{\Omega} \cdot |\boldsymbol{e}\rangle \qquad (9.1.3)$$

Here $\boldsymbol{\Omega}$ is a 3×3 matrix having as elements some differential forms in space coordinates. We need a special care in handling and using these equations.

As we do not have a metric yet, and therefore the reference frame is arbitrary even in this respect, the matrix $\boldsymbol{\Omega}$ is also arbitrary, and that in a precise sense: it has no particular algebraical symmetry, like, for instance, the skew-symmetry, imposed in the usual Euclidean theory of surfaces by the *a priori* orthonormality of the reference frame. Luckily, however, the affine differential geometry brings in at this point, a concept which, incidentally, is *sine qua non* in doing physics, at least when this one is referring to quantization in the original Planck's approach: the *volume of a reference frame*. At the historical time of Planck quantization this volume was not cogitated in terms of a Lorentz quantum, but was consistently taken as a *regular cuboid* constructed on three linearly independent vectors. With the definition of the volume of a reference frame, one can say that the affine theory of surfaces just sanctions that original approach, and provides thereby a mathematically elaborated fashion of treatment of the problem of Planck's quantization.

Now, it is intuitively obvious that the three vectors of an affine reference frame define a solid in space, whose volume, taken as the very volume of the reference frame, and it is such a volume that was used exclusively in treating the physics of radiation that led to Planck's quantization. *If the volume of reference frame is constant* during the evolution described by equation (9.1.3) – the so-called *equiaffine* case of frame evolution – then the Frenet-Serret matrix should satisfy the condition of null trace (Flanders, 1965):

$$tr(\boldsymbol{\Omega}) \equiv \Omega_k^k = 0 \tag{9.1.4}$$

Then, the conditions of integrability: $d \wedge d\boldsymbol{m} = \boldsymbol{0}$, as applied for the displacements given in equation (9.1.2) lead, again, to a formula entirely analogous to that from the Euclidean geometry of surfaces:

$$d \wedge s^k + \Omega_j^k \wedge s^j = 0 \tag{9.1.5}$$

while the analogous conditions of integrability applied for the evolution equations (9.1.3) lead to a formula analogous to Maurer-Cartan one:

$$d \wedge \Omega_j^k + \Omega_i^k \wedge \Omega_j^i = 0 \tag{9.1.6}$$

We remind here to our reader the use of summation rule over repeated indices of different variances in a monomial of arbitrary algebraical nature, as used everywhere in this work. In detail, however, the equations deriving from (9.1.5) are to be transcribed as :

$$\begin{aligned} d \wedge s^1 + \Omega_1^1 \wedge s^1 + \Omega_2^1 \wedge s^2 = 0 \\ d \wedge s^1 + \Omega_1^2 \wedge s^1 + \Omega_2^2 \wedge s^2 = 0 \\ \Omega_1^3 \wedge s^1 + \Omega_2^3 \wedge s^2 = 0 \end{aligned} \tag{9.1.7}$$

where the condition $s^3 = 0$ was considered in the last one of these equalities, representing the fact that displacement is accomplished in surface. This condition leads to an usual relation, as a consequence of Cartan's *Lemma 1*:

$$\Omega_\alpha^3 = h_{\alpha\beta} s^\beta; \quad h_{\alpha\beta} = h_{\beta\alpha} \tag{9.1.8}$$

Now, assuming that our surface is convex, this equation can be considered as defining a metric, after the manner of definition of second fundamental form from the Euclidean theory of surfaces. This metric can be taken in a form which is *affinely invariant* if, using (9.1.8), we write it as what is sometimes termed as *Blaschke metric* of the affine surface (Yau, 1989):

$$(ds)^2 \equiv (1/\sqrt[4]{h})\Omega_\alpha^3 s^\beta = (1/\sqrt[4]{h})h_{\alpha\beta}s^\alpha s^\beta \tag{9.1.9}$$

where h is the determinant of $\boldsymbol{h}$. As mentioned before, by comparison with the usual theory of surfaces in Euclidean space, the quadratic form just defined as metric here is, according to equation (9.1.8), actually the equivalent of the second fundamental form of a regular surface. This is why it is usually designated by geometers with *II* in the affine differential theory of surfaces [see also (Chern & Terng, 1980)].

The problem now remains, to deal with the other side of the matrix $\boldsymbol{\Omega}$ – namely the line Ω_3^α – because in the case of affine theory this matrix *is no more skew symmetric* (the frame is not necessarily orthonormal). One approach – the customary approach in geometry – is to choose, in a first step, Ω_3^3 as an *exact differential*, meaning that $d\wedge\Omega_3^3 = 0$, in which case the corresponding equation from (9.1.6) an be rewritten as

$$\Omega_3^1 \wedge \Omega_1^3 + \Omega_3^2 \wedge \Omega_2^3 = 0 \quad \therefore \quad \begin{pmatrix} \Omega_3^1 \\ \Omega_3^2 \end{pmatrix} = \begin{pmatrix} b^{11} & b^{12} \\ b^{12} & b^{22} \end{pmatrix} \begin{pmatrix} \Omega_1^3 \\ \Omega_2^3 \end{pmatrix} \tag{9.1.10}$$

where *Cartan*'*s Lemma 1*, guaranteeing the existence of a *convenient symmetric matrix* $\boldsymbol{b}$, was used. There is not a correspondent of this relation in the regular theory of surfaces, but it is usually taken as the equivalent definition of what we like to call the curvature vector. In a word, in the differential affine geometry of surfaces, the curvature vector components are bilinear forms in the entries of *two* conveniently introduced 2×2 matrices, $\boldsymbol{h}$ and $\boldsymbol{b}$. This means that our possibilities of introducing physics here are 'doubled', so to speak, when using the mathematics of affine differential geometry of surfaces. With equation (9.1.8), we can get from (9.1.10) a quadratic form as:

$$\Omega_3^\alpha = b^{\alpha\nu}h_{\nu\beta}s^\beta \equiv b_\beta^\alpha s^\beta \quad \therefore \quad III \equiv \Omega_3^\alpha \Omega_\alpha^3 = b_{\alpha\beta}s^\alpha s^\beta \tag{9.1.11}$$

where the notation *III* seems to be, again, geometers' preference: this is the *third fundamental form*, but taken specifically for the affine differential theory of surfaces. Recall that in the Euclidean differential geometry of surfaces, the third fundamental form is usually the square of what we designated as the curvature vector. On the other hand, here the third fundamental form is a quadratic differential form involving a product of two different matrices, $\boldsymbol{b}$ and $\boldsymbol{h}$, not just the square of one of them. This makes a significant difference with respect to usual Euclidean differential theory of surfaces.

Now, that we have a metric at our disposal, we can define a direction *via* relative angles, and therefore a frame in the tangent plane to the surface, ($\hat{\boldsymbol{e}}_1$, $\hat{\boldsymbol{e}}_2$) say, orthonormal with respect to this metric. Thus, we can find a space vector $\boldsymbol{n}$ such that *the volume* ($\hat{\boldsymbol{e}}_1$, $\hat{\boldsymbol{e}}_2$, $\boldsymbol{n}$) remains constant. More to the point, the following theorem due to

Harland Flanders can be proved [for continuity in expounding the subject, we follow here (Yau, 1989)]: there is a *unique* space vector $\boldsymbol{n}$ satisfying the following structural equations

$$\begin{gathered} d\hat{\boldsymbol{e}}_1 = \Omega_1^1\hat{\boldsymbol{e}}_1 + \Omega_1^2\hat{\boldsymbol{e}}_2 + \phi^1\boldsymbol{n} \\ d\boldsymbol{m} = \phi^1\hat{\boldsymbol{e}}_1 + \phi^2\hat{\boldsymbol{e}}_2; \quad d\hat{\boldsymbol{e}}_2 = \Omega_2^1\hat{\boldsymbol{e}}_1 + \Omega_2^2\hat{\boldsymbol{e}}_2 + \phi^2\boldsymbol{n}; \quad d\boldsymbol{n} = \psi^1\hat{\boldsymbol{e}}_1 + \psi^2\hat{\boldsymbol{e}}_2 \\ \Omega_1^1 + \Omega_2^2 = 0 \end{gathered} \tag{9.1.12}$$

One can see from this system that the vector $\boldsymbol{n}$ is, formally at least, as close as possible to the normal vector from regular differential theory of surfaces in Euclidean space, and can be, indeed, geometrically taken as the *affine normal to surface*. Given *any* affine reference frame in space, ($\boldsymbol{e}_{10}$, $\boldsymbol{e}_{20}$, $\boldsymbol{e}_{30}$) say, the affine normal to a surface can be written in the general form:

$$\boldsymbol{n} = a^1\boldsymbol{e}_{10} + a^2\boldsymbol{e}_{20} + h^{\frac{1}{4}}\boldsymbol{e}_{30} \tag{9.1.13}$$

Here, the auxilliary vector $|a\rangle$, which belongs to surface – closer to our point of view, it describes a physical property specifying the surface – and serves for this definition of the normal to surface, is mathematically constrained by the following differential equation [(Yau, 1989), Proposition 1.1]:

$$\langle a|\boldsymbol{h}|\phi\rangle + d(h^{\frac{1}{4}}) + h^{\frac{1}{4}}\Omega_3^3 = 0 \tag{9.1.14}$$

The proof of this theorem, which, again, is only quoted as such in the work of Chi-Ming Yau just cited above, can be found in [(Flanders, 1965), §7]. As we just said, for what we intend here, it is the most convenient expression helping to introduce the physics in the affine theory.

To this end, one can see from (9.1.14) that, having the metric at our disposal, there are a few alternatives to choose from when introducing the physics for defining a surface from an affine point of view: we can either *define* $|a\rangle$ by *choosing* Ω_3^3, or *define* Ω_3^3 by *choosing* $|a\rangle$. Again, the geometers' preference seems to be the first procedure, and for a good reason at that (Cheng & Yau, 1986). This reason becomes, indeed, quite obvious if we notice that the definition of the affine normal from equation (9.1.12), is pendent on some further constraints, given by the different possibilities of handling the equation (9.1.14). For instance, this equation can be rewritten in the form

$$\langle a|\boldsymbol{h}|\phi\rangle + h^{\frac{1}{4}}\left(\tfrac{1}{4}d\ln(h) + \Omega_3^3\right) = 0$$

and if one chooses Ω_3^3 to be an exact differential – as we did before, in a preliminary step – given by the relation

$$\tfrac{1}{4}d\ln(h) + \Omega_3^3 = 0 \tag{9.1.15}$$

then, with one further special choice, namely $|a\rangle = |0\rangle$, the affine normal vector can be defined as:

$$\boldsymbol{n} = h^{\frac{1}{4}}\boldsymbol{e}_{30} \tag{9.1.16}$$

Such a choice is therefore particularly attractive indeed, in the geometrical exploits of this mathematical theory, for in the cases where $h = const$, we can write

$$\Omega_3^3 = 0; \quad \boldsymbol{n} = h^{\frac{1}{4}}\boldsymbol{e}_{30} \tag{9.1.17}$$

and $\boldsymbol{n}$ can be taken as a *unit vector* normal to surface, as in the classical differential theory of surfaces. This way, the surface itself is equiaffine, and it is concomitantly described in an equiaffine space reference frame.

Notice, however, for the incidental benefit of physics – and, in fact, even for the benefit of affine geometry of surfaces altogether, as we shall see shortly – that, while still under the spell of affine invariance condition (9.1.14), but possibly even with $h \neq constant$ – therefore in a non-equiaffine theory of the surface itself – the equation (9.1.15) can offer Ω_3^3 as the differential of a function depending exclusively on the external parameters already introduced through the metric. Therefore the choice of Ω_3^3 as an exact differential can be subjected to the very same physical considerations to which the matrix $\boldsymbol{h}$ itself is subjected. What we mean is that, taking equation (9.1.14) for guidance of our logic, we still have a few other possibilities of defining the vector $|a\rangle$, depending on the location in the tangent plane of the affine surface. Again, a nonzero vector $|a\rangle$ should be especially attractive from physical point of view, as a way to introduce *physically specific considerations*. Such possibilities are offered, for instance, by one of the nontrivial choices:

$$\begin{gathered} \langle a|\boldsymbol{h}|\phi\rangle = 0 \quad \therefore \quad \frac{a^1}{h_{12}\phi^1 + h_{22}\phi^2} = \frac{-a^2}{h_{11}\phi^1 + h_{12}\phi^2} \\ \langle a|\boldsymbol{h}|\phi\rangle + \tfrac{1}{4}d\ln h = 0 \quad \therefore \quad \Omega_3^3 = 0 \end{gathered} \tag{9.1.18}$$

In the first case here, the condition (9.1.15) is secured automatically, and with it the fact is secured in turn, that Ω_3^3 is related to the determinant of the metric tensor, so that it is manifestly an exact differential. The surface itself is equiaffinely described from an intrinsic point of view. However, the reference frame is not equiaffine, insofar as the Ω_3^3 does not necessarily vanish. It is only in the second case (9.1.18) that the theory can be made equiaffine, for both surface and ambient reference frame, and this means that $|a\rangle$ must be chosen in such a way that the bilinear form $\langle a|\boldsymbol{h}|\phi\rangle$ is an exact differential. We shall return to this discussion right away.

The affine theory of surfaces therefore provides an exquisite theoretical tool for physics, in that it allows for a geometry that may be able to account for more and meaningful physical details. If for nothing else, but only for the idea of the volume of the space reference frame in describing the embedding of a surface, this approach would still be extremely valuable. We are thinking of the fact that in the physics of any interpretable theory, the physical systems are described by confining them to given volumes, in order to settle the boundary conditions. Then, as mentioned, it is quite significant that the very physical reference frame is defined by reference to its volume, before being defined by directions, as in the usual geometric theory. For instance, the Wien-Lummer cavity serving for the study of the blackbody radiation can be taken as such a reference frame, at least in some particular instances. From this perspective, the fact that, based on the background cosmic radiation observations, the reference frame was extended to contain the whole universe has an overwhelming importance. The theoretical basis of this extension from the *finite scale* to *transfinite scale* is the invariance of the blackbody radiation spectrum to space extension of volume, whose expression is the Wien displacement law. The same should be happening with the Einstein elevator, which is modern materialization of the classical Cartesian reference frame, but unfortunately it is not realized that this is the current case in physics. And the same should be, in our opinion, the case of cranial vault, encassing the brain. Realizing that, and adding to it the fact that, naturally, we always need a *physical surface* in order to build our experience – recall the Einstein's "earth's crust plays such a dominant rôle in our daily life" – will, hopefully, solve the problem of inertia in a physically sound manner, which is akin to the memory in brain. Let us, therefore, show what other physically important incentives has the affine theory of surfaces in store for us.

9.2. Conditions of Physical Nature on a Surface: the Fubini-Pick Cubic

The previous condition was of a general nature, involving just the cosmological properties of light, in view of the quantization procedure of Planck. On the other hand, the electromagnetic properties of light involve the local behavior, contained in an affine geometry of a Stroud moment, as shown in the following. This geometric-affine approach is, in our opinion, necessary in order to cover for the *apriorism* of the theorem (9.1.12): that equation provides a 'canonical' description of a physical situation independently of any… physics. Recall, however, that in the like situation, considerations of the Lorentz quantum lead us to the optical approach involving the Iwasawa decomposition (see § 3.6 above). In physically specific situations, though, the ideal volume of a reference frame needs to be set in connection with a Lorentz quantum, and this involves a special geometry of the $\mathfrak{sl}(2, R)$ type, implicating the idea of cycles. As we have shown (see § 3.6 above) the description is pending on a special linear application of an arbitrary two-dimensional reference frame into an orthonormal one, a necessary physical step bypassed in the affine geometry of surfaces by the theorem (9.1.12). However, this one geometry has something else in store for physics, that can be taken as sanctioning its very historical pursuit.

The 'physically specific situations' we have in mind here, are algebraically brought to light as in the previous chapters of the present instalment of our work. The situations they describe may concern, for instance, the interpretation of continuous matter from within the nucleus of the planetary model – in fact even of the continuous matter within the nucleus of the particle generating the field – by ensembles of Hertz material particles. We have here a surface delimiting the matter *per se* in its incessant change, which can be locally described by the exclusive variation of its curvature parameters, making a *Stroud sphere* out of it. This is a condition which guarantees that the variation of a quadratic binary form is due strictly to the variation of some physical parameters, embodied in its coefficients. Such parameters can be related to a local field of velocities labeling the Kepler orbits in the flux of Hertz material particles inside the toroidal de Broglie tube containing their instantaneous orbits.

Perhaps useless to utter it again, but, nevertheless, we still mention it for emphasis, *the description of this situation is of primary importance for the theoretical physics*.

In order to carry out a theoretical implementation according to previous lines of this chapter, we need to bring here another concept from the affine differential theory of surfaces. Indeed, the definitions in (9.1.18) can be tied up to an important algebraical concept that we are set out to exploit here – again, in the interest of physics – and this is the concept of *algebraic apolarity* [see for instance (Mazilu, Agop, & Mercheș, 2019), equation (7.11) and the theory leading to it]. The affine differential theory of surfaces makes the fact obvious that the apolarity is not quite as direct as presented in the classical theory of surfaces, but involves also a cubic invariant – the so-called *Fubini-Pick cubic form* [for a clear but quite extended account of the Fubini-Pick tensor and related concepts, one can consult the work (Shirokov & Shirokov, 1959)]. This form can be introduced in the manner that follows, plainly involving the Cartan's geometrical philosophy (Chern & Terng, 1980). By exterior differentiating equation (9.1.8) above – but still within the choice from equation (9.1.12) – and using the corresponding equation from (9.1.6), we get

$$Dh_{\alpha\beta} \wedge \phi^{\beta} = 0; \quad Dh_{\alpha\beta} \equiv dh_{\alpha\beta} - h_{\alpha\mu}\Omega^{\mu}_{\beta} - \Omega^{\nu}_{\alpha}h_{\nu\beta} \tag{9.2.1}$$

Using again Cartan's *Lemma 1* here, we get further

$$Dh_{\alpha\beta} = \Phi_{\alpha\beta\gamma}\phi^{\gamma} \tag{9.2.2}$$

The newly introduced third-order tensor $\boldsymbol{\Phi}$ – the *Fubini-Pick tensor* – is symmetric in all its three indices, as required by its very definition. It satisfies some special constraint conditions imposed by the idea of metric volume. These conditions come easier to light if we work formally on symbols. Namely, with (9.2.1), we can write equation (9.2.2) in the form:

$$d\boldsymbol{h} = \boldsymbol{h}\cdot\boldsymbol{\Omega} + \boldsymbol{\Omega}\cdot\boldsymbol{h} + \boldsymbol{F}, \quad F_{\alpha\beta} \equiv \Phi_{\alpha\beta\gamma}\phi^{\gamma}$$

Left multiplying here by $\boldsymbol{h}^{-1}$ and then taking the trace of resulting matrix, leads us to

$$\mathrm{tr}(\boldsymbol{h}^{-1}\cdot\ d\boldsymbol{h}) = \mathrm{tr}(\boldsymbol{h}^{-1}\cdot\ \boldsymbol{F}) \tag{9.2.3}$$

where we have used the last property in (9.1.12), *i.e.* the equiaffine surface condition: $\Omega^{\alpha}_{\alpha} = 0$, along with some routine properties of the operation of trace, when applied to a product of matrices. The equation (9.2.3) is the condition of apolarity sought for, and we now seek for an explaination of its meaning in some detail, in order to make it more 'palatable', as it were.

The space frame evolution is not equiaffine, but the departure from this condition is supposedly due to the physics of surface embodied in the Fubini-Pick tensor. Indeed, using (9.2.3), the condition (9.1.15) becomes:

$$\Omega^{3}_{3} + \tfrac{1}{4}tr(\boldsymbol{h}^{-1}\cdot\boldsymbol{F}) = 0 \tag{9.2.4}$$

and, therefore, in this case everything depends on the symmetric tensor $\boldsymbol{F}$, obtained by contraction of the Fubini-Pick third-order tensor with the fundamental displacements on the surface. For instance, the condition (9.2.4) is satisfied if there is an affine normal to the surface – according to the definition from (9.1.12) – which, expressed by (9.1.13) in a certain affine reference frame, imposes the conditions (9.1.14) on this frame. Now, if we choose the ancillary vector $|a\rangle$ *according to the first one of conditions from* (9.1.18), then the condition (9.2.4) represents is a mandatory condition of this theory. Let us, therefore, write it in detail, for the sake of transparency; it looks like:

$$\Omega_3^3 + \tfrac{1}{4} F_\alpha \phi^\alpha = 0 \quad with \quad \begin{aligned} F_1 &\equiv h^{11}\Phi_{111} + 2h^{12}\Phi_{121} + h^{22}\Phi_{221} \\ F_2 &\equiv h^{11}\Phi_{112} + 2h^{12}\Phi_{122} + h^{22}\Phi_{222} \end{aligned} \qquad (9.2.5)$$

Therefore, the classical geometers' choice $\Omega_3^3 = 0$, securing the definition of the equiaffine evolution of the space reference frame connected to surface as in (9.1.12), can be satisfied in more general terms, even without such a condition, if between the metric tensor $\boldsymbol{h}$ and the Fubini-Pick tensor $\boldsymbol{\Phi}$ there is some physically accountable connection.

In order to uncover that connection, let us consider the Fubini-Pick cubic form associated with the third-order tensor $\boldsymbol{\Phi}$. It is a cubic in the binary variable (ϕ^1, ϕ^2), which, expressed in the general form, looks like

$$\Phi \equiv \Phi_{\alpha\beta\gamma}\phi^\alpha\phi^\beta\phi^\gamma \qquad (9.2.6)$$

An important property of this binary cubic form is that if it vanishes for a certain affine surface, then that surface is *locally a quadric* (Cheng & Yau, 1986). We are motivated to consider this property as coming extremely handy, not only in experimental, but mainly in highly theoretical physical problems.

Suffice it to recall only that the first modern physical theory of light in Fresnel's take, is intimately connected with the theory of quadrics in two significant ways: on one hand, from the purely geometrical point of view of representing a physical continuum and, on the other hand, from a physical point of view, whereby the quadric is defined by the elastic properties of that continuum: the ether sustaining light phenomenon (Fresnel, 1827). However, there is more to it, from a modern physical point of view: the most general shape of an enclosure allowing for the conclusions of Berry-Klein scale transition theory (Berry & Klein, 1984), is *a quadric* (Klein, 1984). Groupal reasons allow us to say even more: as the theory of quadrics stays at the foundations of the so-called *theory of superquadrics* (Kindlmann, 2004), and as these last geometrical things are, apparently, capable to closely visualize any kinds of physical magnitudes *in a general theory of reverse interpretation*, as it were, one can further say that the concept of affine surface is the right path in constructing any scale-transient mathematical theory in physics. *Therefore the third-order Fubini-Pick tensor hereby becomes the driving power of the whole physics*.

Fact is that, if we take into consideration the algebraic symmetry properties of the tensor $\boldsymbol{\Phi}$, then it has just four independent components, as follows:

$$\Phi_{111} \equiv a_0, \quad \Phi_{112} = \Phi_{121} = \Phi_{211} \equiv a_1, \quad \Phi_{122} = \Phi_{221} = \Phi_{212} \equiv a_2, \quad \Phi_{222} \equiv a_3 \qquad (9.2.7)$$

With these notations, the Fubini-Pick cubic can be written in the typical binomial form [see (Burnside & Panton, 1960) for the theory of binary quantics, and the apolarity of quantics in general]

$$\boldsymbol{\Phi} \equiv a_0(\phi^1)^3 + 3a_1(\phi^1)^2\phi^2 + 3a_2\phi^1(\phi^2)^2 + a_3(\phi^2)^3 \tag{9.2.8}$$

If we assume that this cubic is known, which means that its coefficients are known, then the two components of the covector defined in equation (9.2.5), can be written as:

$$h \cdot F_1 \equiv h_{22}a_0 - 2h_{12}a_1 + h_{11}a_2; \quad h \cdot F_2 \equiv h_{22}a_1 - 2h_{12}a_2 + h_{11}a_3 \tag{9.2.9}$$

If this is a *null covector*, *i.e.* $(F_1,F_2) \equiv (0,0)$, the binary quadratic form representing the affine metric is simultaneously apolar with the two binary quadratic forms having the coefficients (a_0, $2a_1$, a_2) and (a_1, $2a_2$, a_3), respectively. Practically, in this case, *if we know the Fubini-Pick tensor*, then we have in equation (9.2.9), a system of two linear equations with three unknowns – the entries of the affine metric tensor of the surface – which is compatible, and has a simple infinity of solutions that can be expressed by the system

$$\frac{h_{11}}{a_0a_2 - a_1^2} = \frac{2h_{12}}{a_0a_3 - a_1a_2} = \frac{h_{22}}{a_1a_3 - a_2^2} \tag{9.2.10}$$

This purely algebraic result gives the entries of affine metric tensor of surface as being proportional to *the coefficients of the Hessian of Fubini-Pick cubic*.

This condition allows a direct connection of the physical theory of surfaces with a statistics in space, serving either as initial construction in a physical theory of surfaces, or as a possibility of statistical interpretation of the physical theory (see § 3.2 above). For what is physically worth, then, this shows that, if it is to submit our reasoning to a physical point of view, then not quite any affine metric tensor of the surface should qualify for the condition of constant determinant, necessary to an equiaffine theory of a surface: the geometrical condition $\Omega_3^3 = 0$, securing the constant volume of the reference frame, is valid only if the metric of surface is the Hessian of the Fubini-Pick cubic. Summarizing, in the case of affine surfaces we have the following statement:

If we define *the affine normal to surface in an arbitrary reference frame according to an ancillary vector orthogonal to* $|\phi\rangle$ in the metric $\boldsymbol{h}$:

$$\langle a|\boldsymbol{h}|\phi\rangle = 0 \quad \therefore \quad \frac{a^1}{h_{12}\phi^1 + h_{22}\phi^2} = \frac{-a^2}{h_{11}\phi^1 + h_{12}\phi^2}$$

the affine metric form of surface is the Hessian of the Fubini-Pick cubic [geometrically this condition is emphasized especially in (Shirokov & Shirokov, 1959)]. *The space reference frame adapted to such a surface has a constant volume when* $\Omega_3^3 = 0$. If we imagine a space where the parameters (a_0, a_1, a_2, a_3) are taken as coordinates, the condition $\Omega_3^3 = 0$ is then only satisfied on a surface of 4th degree, represented by the determinant of metric tensor of surface written in terms of the entries of the Fubini-Pick tensor [see equation (9.1.16)]. We shall return in due time to this kind of physical theory.

Up to this point the story is told in a pure mathematical way: one can only say *where* the physics is really involved, as we actually did quite a few times. However, there is a very simple observation that firmly establishes the general place of entering of physics into this mathematical theory. Namely, up to its interpretation defined by Charles Galton Darwin, and involving the idea of wave, a continuum – a fluid for instance, of the kind we need for developping scale relativity theory – needs to be described physically as a… continuum, of course. Ever since the times of Cauchy, such a continuum was mathematically described based on matrices or, more specifically, tensors. The values of a matrix quantity are three – the eigenvalues – in three different space directions, and they are the roots of a *cubic equation*, the well-known characteristic equation of the matrix. This characteristic equation can always be taken as a Fubini-Pick cubic, and used *to update the geometry of continua* by an affine differential process, based on the previous line of ideas. This certainly gives a well established place where the physics can naturally enter our mathematical considerations. Nevertheless, as the history of knowledge has it, the physics needs a little more than this.

Fact is, that having thus secured the place of access of the physics into our mathematical theory, we obviously need to see where the geometry stops and where the physics begins, as it were. To this end, notice that if the tensor defined as above by $\boldsymbol{F} \equiv \boldsymbol{\Phi}\cdot|\phi\rangle$ vanishes, the surface characterized by such a condition can be described with the so called *Levi-Civita connection*, for which, by definition, $D\boldsymbol{h} = \boldsymbol{0}$, where $\boldsymbol{h}$ is the metric tensor. This would indicate a geometry of the *space with no matter*, therefore a geometry of a surface *separating the space from matter* but belonging to space. This categorization is simply made based on the fact that a Levi-Civita connection belongs to the classical geometry, which is known to refer exclusively to space. In our context, however, the observation gains a few different nuances depending on *how*, specifically, the tensor $\boldsymbol{F}$ vanishes. Its vanishing *per se* means a system of three linear equations:

$$a_0\phi^1 + a_1\phi^2 = 0, \quad a_1\phi^1 + a_2\phi^2 = 0, \quad a_2\phi^1 + a_3\phi^2 = 0 \tag{9.2.11}$$

In writing these equations we have used the identifications from equation (9.2.7). This system must have a nontrivial solution for the binary variable (ϕ^1,ϕ^2), otherwise the Levi-Civita geometry itself would have no object. What we can algebraically decide from the system (9.2.11) is, nevertheless, only the ratio of the two components of the binary variable. And even for this much, the essential components of the Fubini-Pick tensor cannot be independent. They must satisfy the conditions

$$\frac{a_1}{a_0} = \frac{a_2}{a_1} = \frac{a_3}{a_2}\left(\equiv -\frac{\phi^1}{\phi^2}\right) \tag{9.2.12}$$

which have an exquisite meaning from physical point of view. In order to uncover that meaning we have to use the Fubini-Pick *binary cubic*. As we mentioned, if this cubic is vanishing, the surface it describes is a quadric. Now, the binary cubic corresponding to the conditions (9.2.12) is a perfect cube:

$$a_0(\phi^1 + \lambda\phi^2)^3 \tag{9.2.13}$$

which means that the surface it describes is, in fact, a sphere. From (9.2.12) we get the quadratic relations

$$a_0 a_2 - a_1^2 = a_0 a_3 - a_1 a_2 = a_1 a_3 - a_2^2 = 0 \tag{9.2.14}$$

which mean a null Hessian for this Fubini-Pick cubic. Certainly, therefore, this would not mean a proper metric geometry of a surface – in fact, it would not even mean a surface – at least from a metric point of view, according to the above developments. We are therefore compelled into admitting that the Hessian of the Fubini-Pick cubic determines actually a *deformation of the metric of surface*, rather than the whole metric itself.

9.3. A General Theory of Equiaffine Reference Frames in Space

The above theory can only be used in the construction of a reference affine surface, based on physical considerations. Using this reference surface, a family of equiaffine frame evolutions connected to surface can be constructed just by extending those physical considerations as follows. If in equation (9.1.14) we define the ancillary vector $|a\rangle$ such that

$$\langle a|\boldsymbol{h}|\phi\rangle + d(h^{\frac{1}{4}}) = 0 \tag{9.3.1}$$

then $\Omega_3^3 = 0$ automatically. In view of this equation, and with the last condition from (9.1.12), the space affine reference frame is equiaffine itself. In this case, using (9.2.3) we have:

$$\langle a|\boldsymbol{h}|\phi\rangle + \tfrac{1}{4}h^{\frac{1}{4}} tr(\boldsymbol{h}^{-1}\cdot\boldsymbol{F}) = 0 \tag{9.3.2}$$

and with (9.2.9) this becomes

$$\langle a|\boldsymbol{h}|\phi\rangle + \tfrac{1}{4}h^{-\frac{3}{4}}\langle F|\phi\rangle = 0 \tag{9.3.3}$$

One important situation where this condition is satisfied occurs for the case

$$\langle a|\boldsymbol{h} + \tfrac{1}{4}h^{-\frac{3}{4}}\langle F| = \langle 0| \quad \therefore \quad |a\rangle = -\tfrac{1}{4}h^{-\frac{3}{4}}(\boldsymbol{h}^{-1}\cdot|F\rangle) \tag{9.3.4}$$

The last condition here shows that in the case of sheer apolarity – *i.e.* $|F\rangle = |0\rangle$ – the only possible auxiliary vector is the null vector, which, as we have seen is the usual geometric preference of all times. This shows that in order to have a possible nontrivial choice, the physical theory, based on geometry but, nevertheless, adding something to that geometry, one needs to choose an auxiliary vector that describes a deformation, to be included in the vector $|F\rangle$. In other words, the vector (9.2.9) needs to be 'updated', as it were, to:

$$\begin{aligned} h\cdot F_1 &\equiv h_{22}(a_0+\delta a_0) - 2h_{12}(a_1+\delta a_1) + h_{11}(a_2+\delta a_2) \\ h\cdot F_2 &\equiv h_{22}(a_1+\delta a_1) - 2h_{12}(a_2+\delta a_2) + h_{11}(a_3+\delta a_3) \end{aligned} \tag{9.3.5}$$

where the symbol δ means a variation of the symbol following it. In this case, if the metric tensor of the reference surface is the one given by apolarity conditions, the Fubini-Pick vector (9.3.5) reduces to

$$h \cdot F_1 \equiv h_{22}(\delta a_0) - 2h_{12}(\delta a_1) + h_{11}(\delta a_2)$$
$$h \cdot F_2 \equiv h_{22}(\delta a_1) - 2h_{12}(\delta a_2) + h_{11}(\delta a_3) \qquad (9.3.6)$$

In order to give a physical interpretation of this result, we shall make use of the special concept of affine reference frame built on the basis of the eigenvalues of a tensor [(Uy, 1976); see I, §5.6]. However, in order to start the mathematical proceedings, a few words are needed on the physical incentives of this mathematics.

9.4. The Novozhilov's Means and the Lax Pairs

Naturally, we expect that the equations (9.3.6) must go beyond the 'two degrees of freedom of the electromagnetism', and that they should serve in the description of the action of neural matter in creating the charge. To show this, we will delineate in these last lines of the present instalment of our work, a way of the connection between deformation and charge, based on the classical theory of deformation. Fact is that, for physical interpretation, this theory needs a special statistics that liberated it from the canons of the theory of solids, making it apt for a theoretical characterization of the concept of interpretation.

To start with, in measuring a tensorial magnitude, like the stress or the strain, in a point in space, one has to make reference to the family of planes through that point. On each plane through the point, a tensorial magnitude has a *normal component* and a *tangential* – or in-plane – *component*. If the planes are randomly distributed – *i.e.* no plane is privileged, like, for instance, in the case of crystalline solids – there are two statistics over the random directions of these planes, that can be taken as results of a measurement of the tensorial magnitude in that point, and we call them the *Novozhilov statistics* (Novozhilov, 1952). One of these statistics is the *average normal component* of the tensor, the other is the *average tangential component* [see (Mazilu, Agop, & Mercheș, 2019), for the detailed description of these statistics]. Incidentally, these averages play an essential part in the plastic deformation of the materials, where the privileged planes are planes of 'sliding', and this can be taken as a mechanism of plastic deformation. However, Valentin Valentinovich Novozhilov was the first to draw attention, in the work just cited, to the fact that such magnitudes are of a statistical nature, and are therefore universal magnitudes. In this case, they can be deferred to a *quantum description* (see § 5.5). According to this view, the slippage of solids is a matter of scale, like the crystalline properties, and we shall show in this section, by the way of an essential example correlated with the affine properties of surfaces, that they generate a dynamics of the kind described by us in § 5.3 as a Wigner dynamics.

Consider $\boldsymbol{m}$ a 3×3 *m*atrix having the eigenvalues $m_{1,2,3}$. From $\boldsymbol{m}$ we can obtain a larger family of matrices by similarity transformation:

$$\boldsymbol{m} \quad \leftrightarrow \quad \boldsymbol{m}' = \boldsymbol{s} \cdot \boldsymbol{m} \cdot \boldsymbol{s}^{-1} \qquad (9.4.1)$$

Assuming that the transforms are obtained from originals by *similarities close to identity*, we then have

$$s \simeq 1 + d, \quad s^{-1} \simeq 1 - d \quad m' - m \overset{def}{=} \delta m = [d, m] \tag{9.4.2}$$

Here ***d*** is a matrix of differentials, and ***1*** is the identity matrix. We recognize in the last equality here a Lax-type equation for the evolution of the matrix ***m*** (see § 7.2 above). Starting from a reference frame given by the principal directions of ***m***, the matrices $\delta\boldsymbol{m}$ given by equation (9.4.2) should have the main diagonal zero:

$$\delta\boldsymbol{m} = \begin{pmatrix} 0 & -d^{12}(m_1 - m_2) & d^{13}(m_3 - m_1) \\ d^{21}(m_1 - m_2) & 0 & -d^{23}(m_2 - m_3) \\ -d^{31}(m_3 - m_1) & d^{32}(m_2 - m_3) & 0 \end{pmatrix} \tag{9.4.3}$$

Thus, in cases where ***d*** is *symmetric*, $\delta\boldsymbol{m}$ is *skew-symmetric*, because we have

$$\delta\boldsymbol{m} = \begin{pmatrix} 0 & -d^{12}(m_1 - m_2) & d^{31}(m_3 - m_1) \\ d^{12}(m_1 - m_2) & 0 & -d^{23}(m_2 - m_3) \\ -d^{31}(m_3 - m_1) & d^{23}(m_2 - m_3) & 0 \end{pmatrix} \tag{9.4.4}$$

On the other hand, in cases where ***d*** is *skew-symmetric*, $\delta\boldsymbol{m}$ comes out *symmetrical*, for we have

$$\delta\boldsymbol{m} = \begin{pmatrix} 0 & -d^{12}(m_1 - m_2) & d^{31}(m_3 - m_1) \\ -d^{12}(m_1 - m_2) & 0 & -d^{23}(m_2 - m_3) \\ -d^{31}(m_3 - m_1) & -d^{23}(m_2 - m_3) & 0 \end{pmatrix} \tag{9.4.5}$$

As we will show presently, the equation (9.4.4) corresponds, for instance, to an infinitesimal conformal transformations, while the equation (9.4.5) corresponds to pure rotations.

Suppose, first, that we are in the symmetric case for ***d***, in order to describe the matter in a proper way by Novozhilov's averages. Then the matrix to be considered instead of ***m*** is $\boldsymbol{m} + \delta\boldsymbol{m}$, which given by

$$\boldsymbol{m} + \delta\boldsymbol{m} = \begin{pmatrix} m_1 & -d^{12}(m_1 - m_2) & d^{31}(m_3 - m_1) \\ d^{12}(m_1 - m_2) & m_2 & -d^{23}(m_2 - m_3) \\ -d^{31}(m_3 - m_1) & d^{23}(m_2 - m_3) & m_3 \end{pmatrix} \tag{9.4.6}$$

The orthogonal invariants of this matrix with respect to rotation group are this time:

$$\begin{aligned} I_1 &= \sum m_1, \\ I_2 &= \sum m_2 m_3 + \sum (d^{23})^2 (m_2 - m_3)^2, \quad I_3 = m_1 m_2 m_3 + \sum (d^{23})^2 m_1 (m_2 - m_3)^2 \end{aligned} \tag{9.4.7}$$

On the other hand, if ***d*** is skew-symmetric, we must use equation (9.4.5) instead of (9.4.4) for $\delta\boldsymbol{m}$, so we would obtain instead of the matrix (9.4.6) the following matrix:

$$\boldsymbol{m} + \delta\boldsymbol{m} = \begin{pmatrix} m_1 & -d^{12}(m_1 - m_2) & d^{31}(m_3 - m_1) \\ -d^{12}(m_1 - m_2) & m_2 & -d^{23}(m_2 - m_3) \\ -d^{31}(m_3 - m_1) & -d^{23}(m_2 - m_3) & m_3 \end{pmatrix} \tag{9.4.8}$$

and instead of invariants (9.4.7) the following invariants

$$I_1 = \sum m_1,\ I_2 = \sum m_2 m_3 - \sum (d^{23})^2 (m_2 - m_3)^2,$$
$$I_3 = m_1 m_2 m_3 - \sum (d^{23})^2 m_1 (m_2 - m_3)^2 - 2d^{23} d^{31} d^{12} (m_2 - m_3)(m_3 - m_1)(m_1 - m_2) \tag{9.4.9}$$

Now, let us try a special idea of potential. The conformal transformation:

$$\boldsymbol{m} = \frac{\boldsymbol{r} + \boldsymbol{\lambda} r^2}{1 + 2\boldsymbol{\lambda} \cdot \boldsymbol{r} + \lambda^2 r^2},\ \boldsymbol{\lambda} \equiv \frac{\boldsymbol{l}}{R^2} \tag{9.4.10}$$

has an obvious statistical property. Namely, we can write

$$\boldsymbol{m} = \frac{\partial F(\boldsymbol{\lambda}, \boldsymbol{r})}{\partial \boldsymbol{\lambda}},\ F(\boldsymbol{\lambda}, \boldsymbol{r}) = \frac{1}{2} \ln(1 + 2\boldsymbol{\lambda} \cdot \boldsymbol{r} + \lambda^2 r^2) \tag{9.4.11}$$

with a partition function given by:

$$e^{2F(\boldsymbol{\lambda}, \boldsymbol{r})} \tag{9.4.12}$$

Now, the function

$$V(\boldsymbol{\lambda}, \boldsymbol{r}) = \frac{1}{\sqrt{1 + 2\boldsymbol{\lambda} \cdot \boldsymbol{r} + \lambda^2 r^2}} \tag{9.4.13}$$

can count as an effective potential according to a classical theory, insofar as it satisfies the Laplace equation in both variables:

$$\nabla_\lambda \cdot \nabla_\lambda V(\boldsymbol{\lambda}, \boldsymbol{r}) = \nabla_r \cdot \nabla_r V(\boldsymbol{\lambda}, \boldsymbol{r}) = 0 \tag{9.4.14}$$

Mention should be made of the relation

$$\boldsymbol{\lambda} \times \frac{\boldsymbol{m}}{m^2} = \boldsymbol{\lambda} \times \frac{\boldsymbol{r}}{r^2} \tag{9.4.15}$$

which, discriminate between $\boldsymbol{\lambda}$ and $\boldsymbol{r}$ in the case of such a theory, making the components of $\boldsymbol{\lambda}$ parameters of the transformation. The equation (9.4.15) characterizes the refraction in an optical medium, as described according to old Ibn Sahl, by a conservation law characteristic to wave mechanics (Wolf & Krötzsch, 1995).

Now, let us perform an experiment with the *infinitesimal transformations* corresponding to the conformal space transformation (9.4.10). This transformation can be written in the form

$$\delta \boldsymbol{r} = \boldsymbol{d} \cdot \delta \boldsymbol{\lambda} \tag{9.4.16}$$

where the matrix $\boldsymbol{d}$ is given by

$$\boldsymbol{d} \stackrel{\text{def}}{=} r^2 \cdot \begin{pmatrix} 1 - 2\frac{x^2}{r^2} & 2\frac{xy}{r^2} & 2\frac{xz}{r^2} \\ 2\frac{xy}{r^2} & 1 - 2\frac{y^2}{r^2} & 2\frac{yz}{r^2} \\ 2\frac{xz}{r^2} & 2\frac{yz}{r^2} & 1 - 2\frac{z^2}{r^2} \end{pmatrix} \tag{9.4.17}$$

We recognize in this symmetric matrix the one from equation (4.3.7) up to the factor $-r^2$. So we are in the case given by the equations (9.4.6) and (9.4.7). Now, if we take the Novozhilov's type averages of the matrix (9.4.6), it is just the original diagonal matrix. As to the same average of the invariants (9.4.7), they are given by

$$I_1 = \sum m_1,$$
$$I_2 = \sum m_2 m_3 + \frac{4R^2}{15}\sum (m_2 - m_3)^2, \quad I_3 = m_1 m_2 m_3 + \frac{4R^2}{15}\sum m_1 (m_2 - m_3)^2 \tag{9.4.18}$$

where R is the radius of an adequate sphere representing the current material point, and the sum sign represent a sum that extends over the positive permutations of the numerical indices.

Our final inference here is pending on a comparison between the equation (9.3.6) and equation (9.4.18): given a simple transition between the invariants I_k and the binomial coefficients a_k of the eigenvalue equation of the matrix $\boldsymbol{m}$, the statistics from equation (9.4.18) offer the most general values of the variations of the cubic coefficients. According to the classical Poincaré-Lorentz definition of charge (see § 4.4), these variations are determined by Novozhilov's statistics, therefore by charges. This makes, out of the matrix $\boldsymbol{d}$, a fundamental instrument of the affine theory of fields.

Conclusions

We would like to close this part of our present work in the note of its starting point: *the justification of the dual struggle of man to construct a machine after the model presented by human brain*, which had to be 'constructed' too, preliminarily. This kind of closure seems just natural, in view of the fact that a necessary physics of fundamental kind is, in our opinion, quite poorly represented in any theory of the brain. Nothing would do better, in this case, than the very words of the man who, for a long while, was the catalyst of this whole labor: Warren Sturgis McCulloch. Thus, according to our custom, we quote:

> To the theoretical question, "Can you *design a machine to do whatever a brain can do*?" the answer is this: "If *you will specify in a finite and unambiguous way what you think a brain does do with information*, then we can design a machine to do it." Pitts and I have proved this constructively. But *can you say what you think brains do*? [(McCulloch, 1955); *emphasis added*, *a/n*]

This is the big question: *can we articulate what we think brains do*? Our answer is this: the formal construction only, as allowed by the logical calculus of the kind proposed by 'Pitts and I' (McCulloch & Pitts, 1943), will not do! A *physical understanding* of the brain functioning is categorically necessary, but this understanding is halted by the neat difference between the two phenomenologies: that of the pure physics and that of the neurophysiology. Our approach of the problem of brain, that is, under the cosmology of the general concept of universe, alows us to discern the key problem here and, hopefully, to solve this problem.

We have already shown the general traits of the phenomenology of the brain universe, known ever since the times of Lord Adrian of Cambridge, and rounded up in an exquisite fashion by Valentino Braitenberg (see § 2.6 above). However, it was Eric Kandel the one who delineated for us, most clearly of all, the conceptual key problem, in connection with the concept of a universe [see (Kandel, 2006), Chapter 5]. Quoting:

> Helmholtz found that the axons of nerve cells *generate electricity not as a by-product of their activity*, but *as a means of producing messages* that are carried along their whole length. These messages are then used to carry sensory information about the outside world into the spinal cord and the brain and to transmit commands for action from the brain and spinal cord to the muscles. … In 1859 he succeeded in capturing the speed at which these electrical messages are conducted and found to his amazement that *electricity conducted along a living axon is fundamentally different from the flow of electricity in a copper wire*. In *a metal wire, an electrical signal is conducted at close to the speed of light* (approximately 186,000 miles per second). Despite its speed, however, the *strength of the signal deteriorates* badly over long distances, because *it propagates passively*. If an axon relied on passive propagation, a signal from a nerve ending in the skin of your big toe would die out before it reached your brain. Helmholtz found that *the axons of nerve cells conduct electricity much more slowly* than wires do, and they do so by means of a novel, wavelike action that *propagates actively* at various speeds up to approximately 90 feet per second! Later studies showed that the *electrical signals in nerves*, unlike the signals in wires do *not decrease in strength as they propagate*. Thus, nerves sacrifice speed of conduction for an active propagation, which ensures that a signal that arises in your big toe arrives at your spinal cord undiminished in size [(Kandel, 2006), pp. 75 – 76; *emphasis added*, *a*/*n*]

Taken as a program of enactment, this excerpt delineates, indeed, a key conceptual problem for physics: is the conduction of electricity, theoretically speaking, the same in the two cases? That is, do the manifestations of *passive conduction* in the case of wires, and of the *active conduction* in the case of axons, have the same theoretical explanation from physics' point of view? Our answer is definitely affirmative, and we base this conviction on the lack of conceptual definition of the charge in the historical development of physics: the concept of charge has always … strings attached, in its definition, sometimes quite explicitly!

However, as long as the physicists do not accept to revise the conceptual attitude on their own realm of research, such an understanding is, in our opinion, out of hand *for the whole scientific community*. For, that attitude is almost exclusively dictated by an inappropriate vision on the reality, which the science of physics accepts *a priori*, ‘refusing’, practically, *to see* the reality. And this reality is, simply, that the physicists work with fictitious notions in order to create concepts to be handled logically: from the classical material point to the modern partons, for instance, we have a suit of notions, inexistent for our senses, and serving exclusively for the necessity of interpretation. Among these the definition of charge is patent. The philosophers, alone, were the ones who touched the essential problem here, which the physicists still cannot grasp, and this situation entertains a kind of vicious cycle: the philosophers cannot find an understanding, just because the physicists are simply not set into explaining it. Quoting again:

> Perhaps in this “best of all possible worlds” neurophysiologists, like physicists, *will be compelled to call their shots* “on a cloth untrue, with a twisted cue and elliptical billiard balls.” Russell has already noted that *the explanation of mind has become more materialistic* only as *our matter has become less material*. So we seem to be groping our way toward an indifferent monism. Everything we learn of organisms leads us to conclude not merely that they are analogous to machines *but that they are machines*. *Man-made machines are not brains*, *but brains are very ill-understood variety of computing machines*. Cybernetics has helped to pull down the wall between *the great world of physics and the ghetto of mind*. [(McCulloch, 1955); *emphasis added*, *a*/*n*]

Thus, as long as the physicists do not understand themselves *how and to what extent* ‘their matter has become less material’, no researcher of any persuasion can say anything about ‘what the brain does’, for that is a matter of fundamental theoretical physics of the brain matter. Again, in our opinion, the key point in this understanding stays in *the kind of theoretical physics involved in the concept of charge*, which is a fundamentally incomplete concept, even from physical point of view or, better yet, especially from physical point of view: after all, the place of this concept *is* in physics. On the other hand, this concept can by no means be rounded up, as long as the neurophysiological phenomenology is not properly incorporated within the main structure of theoretical physics. This is why, in the present instalment of this work the role of charge in the physics of brain is intentionally overemphasized. One has to realize the overwhelming part of neurophysiology in physics, and this cannot be done as long as we cannot realize the awe-inspiring part played by the *living matter* in the economy of the physical universe: its *exclusive possibility of creating and handling the charge from the de Sitter background of this universe*!

It is in this connection that we assume a fundamental point of view, according to which *the brain itself is a universe*, and intend to construct its physics starting from this point of view. In a way, we actually replicate the old problem of modeling the brain, as delineated by McCulloch, with an essential diference though: for such a universe *we have* as model the universe of our existence – the very physical universe – no need to construct a model brain in order 'find unambiguous ways to think about brain'. Within this analogy, it is obvious, and quite understandable in fact, that the matter becomes downright 'immaterial' by the necessary process of interpretation, which became critical in physics only along with the appearance of the wave mechanics (Darwin, 1927). We only have to recognize that the Newtonian physics, just as the Einsteinian one, works with *fictitious structural matter formations*, and this state of the case needs to be acknowledged and *used as such* in the construction of a general theory of matter. Once this task is accomplished, we come to recognize further that Planck's quantization is the only law of physics for a concept of universe, therefore even for the brain in our acceptance. Conservation laws come out just naturally from the condition of quantization (see § 6.6 above), the interpretation of which is directly related to the electrical structure of the interpretative particles, *via* Iwasawa decomposition (see § 7.1). This kind of decomposition naturally copes with the idea that *the neurons are fundamental physical rays in the brain universe*. In a word, the existential principle of these rays *is not the light*, as in the case of physical universe, but *the charge*, as in a brain universe.

Julian Schwinger opened his beautiful 1969 article, addressed to 'a speculation' that 'probes deep within the structure of nuclear particles and predicts new form of matter' (Schwinger, 1969), with the quotation of the first phrase from the closing paragraph of Newton's *Principia*. The conclusions of this instalment of our work may benefit substantially, we think, from quoting that *entire paragraph* from Newton's work; no doubt, the reader will perceive the reason for this recall right away. There it is:

> And now we might add something concerning a certain *most subtle spirit* which pervades and lies hid in all gross bodies; by the force and action of which spirit the particles of bodies attract one another at near distances, and cohere if contiguous; and electric bodies operate to greater distances, as well repelling as attracting the neighboring corpuscles; and light is emitted, reflected, refracted, inflected, and heats bodies; and all sensation is excited, and the members of animal bodies *move at the command of the will*, namely, by *the vibrations of this spirit, mutually propagated along the solid filaments of nerves, from outward organs of sense to the brain, and from the brain into the muscles*. But these are things that cannot be explained in few words, nor are we furnished with that sufficiency of experiments which is required to an accurate determination and demonstration of the laws by which *this electric and elastic spirit operates* [(Newton, 1974), Volume II, General Scholium, p. 547; *emphasis added*, *a/n*]

One cannot say that Newton was not aware of the fact that the 'nervous matter' as we call it now, carries that spirit that makes our mind. It is rather obvious that he grasped *all* of the properties of the matter, including that of mind itself, for it seems to us that Newton took the matter under a general concept, including the *life property* of the matter. Such facts, taken nowadays as being of historical interest only, compel us to a proactive attitude toward history, best expressed in contemporaneity by Vladimir Vladimirovich Kisil. In *the preprint* of one of his most beautiful works on theoretical algebra (Kisil, 1997), he has a profoundly significant *motto*, unfortunately missing in the published version of the work (Kisil, 1999), probably in order not to stir some foreseeable susceptibilities:

> You should *complete your own original research* in order to learn *when* it was done before (*our emphasis*, *a*/*n*).

The depth of the cogitation we feel underneath this adage cannot be rightfully appreciated if a contemporary researcher does not realize a blatant reality: today, more than in any time in history, most of the past research is considered obsolete, of historical interest only, in the best of circumstances. On this note, Professor Kisil is right in the largest possible sense of the word. For, rationally speaking, there is not even a shred of doubt that *any* research has already been concluded somehow before, even in times immemorial, we might say. It is in order to grasp the whole truth of this conclusion, that a contemporary researcher needs to know *not only when* this happened, but mainly *the expression of the truth in its own time*, in order to translate that expression in actuality, and thus to round the truth in a concept. To achieve that, the researcher just needs to bring his own research *to completion*, in order to be able to comprehend that old expression, that is, he needs *to find himself among the olds*.

The kind of optimism of a scientific researcher, as contained in Professor Kisil's adage is, certainly, quite atypical today. However, it exists, for it is rooted in the realization that *every man is in possesion of truth*, by his very nature: *the man is created in the resemblance of his Creator*. This statement of ours is by no means a sign of a specific bigotry: one can recognize its truth *as a fact of experience* in society, from the obvious circumstance that every individual of the mankind is *a priori* convinced that he is in possession of the truth. This conviction goes to such an extent that the individual does not hesitate to promulgate it openly, in order to make acolytes. This is how the societies of any persuasion and size are constructed. The layman can, certainly, be content with this kind of truth, *i.e*. *the social truth*: after all, this is how the society – this nest of man as a mind – evolves. This, though, is the kind of truth about which people have been continuously discontented. Quoting:

> The *man of knowledge* in our time is bowed down under a burden he never imagined he would ever have: *the overproduction of truth that cannot be consumed*. For centuries man lived in *the belief* that the truth was slim and elusive and that once he found it the troubles of mankind would be over. And here we are in the closing decades of the 20th century, choking on truth. There has been *so much brilliant writing*, so *many genial discoveries*, so *vast an extension and elaboration*

of these discoveries – yet the mind is silent as the world spins on its age-old demonic career. [(Becker, 1973), Preface, p. X; *emphasis added*, *a*/*n*]

By the very label 'overproduction' attached to the truth he is talking about, Ernst Becker actually confesses the nature of the truth he is targeting: *it is a social truth*. In gathering it, the 'man of knowledge' does not need but *to read whatever has been written* – and, with the modern digital technology, apparently available to everybody on Earth, not even that. Today, everybody with a digital phone in hand is a 'man of knowledge' and, as such, stands 'bowed, choking on truths he cannot consume', even though he is always compelled, by his very nature, to propagate it. And he gladly obeys this requirement of his nature, for this needs *no effort* and *no discernment*. However, a true researcher cannot confine himself to the circles of such a truth, and needs to go beyond it, into the realm of that kind of truth which is *independent of any society*. According to the old sage Moses Maimonides, this is …

… a truth (that), *once established by proof*, does *neither gain force by the consent of all scholars*, nor *lose by the general dissent* [(Maimonides, 1910), p. 176; *emphasis added*, *a*/*n*]

On the positive side, the truth considered by Ernst Becker contains the subtle idea that the main limitation of a man resides in his impossibility to express it in a language. And because, in order to learn and handle a language toward the truth, one needs to 'complete one's own original research', there is no other way to express it but, unfortunately enough, only the way available to a restricted category of researchers: *the truth stays at the highest level of the brain action*! And when one achieves it, one realizes that, indeed, it was articulated before, but with a language missing some concepts. This appears to be, in our opinion, the sense of evolution of the man through the mind.

Two cases in point, touched in the present work, may be adduced as examples, in order to illustrate the above philosophy. They are to be considered by that restricted society of researchers who are equally neurophysiologists as well as physicists, and both of them involve the name of magnificent Isaac Newton. The first, is *the case of the ray*: Newton was well aware of the fact that we must speak of the ray in the case of 'nervous matter', for he talks about the 'solid filaments of nerves' in the excerpt right above. And, by the manner he developed his optics, we may be able to infer that these 'filaments' inspired him in the very definition of the optical rays (see § 2.1 above). Whence the idea of treating the neurons like rays (see § 2.6), involving the further consideration of the brain as a universe. The second, is the case of forces: the way Newton defines them is a clear example of quantization of the kind introduced in physics by Planck at he beginning of the 20th century. One can say that Planck's theory is just a natural continuation of the old Newtonian natural philosophy. This status of the Newtonian forces, first noticed as such by James Hopwood Jeans (see § 4.3), must be considered in the theory of physical interpretation of the wave phenomena, whereby these forces serve for the logical definition of the interpretative ensembles. The problem became critical in the modern theory of the gauge fields serving for the definition of the charge – the Yang-Mills fields.

Speaking of charge, we find, as we said, its concept essential for the description of the brain universe. The impossibility of physics to decide to what extent the charge is related to quantization, appears to us as the key point missing in the natural philosophy: *this is a gap that needs to be closed by historical continuity*. Namely, the charge is fundamentally defined in connection with the *deformation of matter* (see § 2.7). Only, the deformation itself does not have a coherent definition in physics, at least not a definition that would be able to properly encompass the case of charge. The neural phenomenology, though, brings clarification in this case, by the idea of *propagation of charge along the neurons* (§ 5.2), which is analogous to the propagation of the light along an optical ray. This analogy calls for a big adjustment of the scientific language, in general.

And, while we are on the subject, we may as well ask ourselves if this limitation of the language is not due to our natural sin as men. The answer was provided by Warren McCulloch too, and is conditional, indeed, on a certain degree of wisdom:

> So long as we, like good empiricists, remember that *it is an act of faith to believe our senses*, that *we corrupt but do not generate information*, and that *our most respectable hypotheses are but guesses open to refutation*, so long may we "rest assured that God has not given us over to thraldom under that mistery of iniquity, of sinful man aspiring into the place of God." [(McCulloch, 1955); *emphasis added*, *a*/*n*]

Again, no specific kind of bigotry is involved here! In closing, however, we may need to complete these last sentences of the great sage by adding that, just as the man cannot 'aspire into the place of God', the machine cannot 'aspire into the place of its creator'. *This seems to be*, *indeed*, *an essential law of nature*!

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