Pi as Alternating Continued Fraction

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Abstract: An infinite alternating continued fraction for π is introduced .
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1 The main result

Theorem 1.1

$$\pi = 4 - \cfrac{2}{1 + \cfrac{1}{1 - \cfrac{1}{1 + \cfrac{2}{1 - \cfrac{2}{1 + \cfrac{3}{1 - \cfrac{3}{\ddots}}}}}}}$$

Proof.
Let $f_n(x) = 1 + \frac{n}{1-\frac{n}{x}} = \frac{(1+\frac{1}{n})x-1}{\frac{1}{n}x-1}$. We want to show that

$$f_1 \circ \cdots \circ f_n(1) \to \frac{2}{4-\pi}.$$

Define the matrix

$$A_n = \begin{pmatrix} 1+\frac{1}{n} & -1 \\ \frac{1}{n} & -1 \end{pmatrix}.$$

Let $X_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and define

$$X_n = \begin{pmatrix} u_n \\ v_n \end{pmatrix} = A_1 \cdots A_n X_0.$$

We want to show that

$$\frac{u_n}{v_n} \to \frac{2}{4-\pi}.$$

We have

$$X_n = A_1 \dots A_{n-1} \begin{pmatrix} \frac{1}{n} \\ \frac{1}{n} - 1 \end{pmatrix} = \frac{1}{n} X_{n-1} + A_1 \dots A_{n-2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{n} X_{n-1} + X_{n-2}.$$

Thus,

$$u_n = \frac{1}{n} u_{n-1} + u_{n-2} \quad (*).$$

Similarly, v_n satisfies the same recurrence relation $(*)$. By induction, we establish that

$$u_{2n} = u_{2n+1} = \frac{1 \times 3 \times \cdots \times (2n+1)}{2 \times 4 \times \cdots \times (2n)}.$$

Similarly, the sequence

$$w_n = 2u_n - v_n$$

satisfies

$$w_{2n} = w_{2n-1} = \frac{2 \times 4 \times \cdots \times (2n)}{1 \times 3 \times \cdots \times (2n-1)}.$$

Using Stirling's approximation, we obtain

$$u_n \sim \sqrt{\frac{2n}{\pi}}, \quad w_n \sim \sqrt{\frac{n\pi}{2}}.$$

From this, we deduce

$$\frac{u_n}{v_n} = \frac{u_n}{2u_n - w_n} \to \frac{2}{4-\pi}.$$