Investigation of a Primality Criterion within an Arithmetic Progression

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Abstract

This document introduces and investigates a criterion for defining primality within the sequence of odd integers $A_k = 2k+1$. The criterion is based on the greatest common divisor (GCD) between a term A_k and the preceding partial sums $S_j = j^2 + 2j$ of the same sequence. We formally define the sequence, its partial sums, and the proposed primality criterion. Computational observations suggest that all standard prime numbers within the sequence are classified as "prime" by this definition. Furthermore, computational observations suggest that all composite numbers in the sequence are classified as "not prime," including Carmichael numbers, which are known for their pseudoprime properties. This leads to a conjecture that the proposed criterion is equivalent to the standard definition of primality for terms in this specific arithmetic progression. Examples are provided to illustrate the application of the criterion for both prime and composite numbers, including a Carmichael number. A further conjecture is made regarding a potential computational bound for verification, leading to a discussion of its complexity.

1 Introduction

This document explores a definition of primality applied to terms within a specific arithmetic progression. We define the terms of the series and its partial sums, then introduce a primality criterion based on the Greatest Common Divisor (GCD) between terms and preceding partial sums. Finally, we examine the relationship between this definition and the standard definition of prime numbers, including illustrative examples.

2 Definitions

[The Sequence of Terms (A_k)] Let A_k be the sequence of odd integers greater than or equal to 3, defined for $k \in \mathbb{Z}^+$ as:

$$A_k = 2k + 1$$

The sequence begins: $3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, \ldots$

[The Sequence of Partial Sums (S_n)] Let S_n be the sum of the first n terms of the sequence A_k. This sum can be expressed as a polynomial in n:

$$S_n = \sum_{i=1}^{n}(2i+1) = 2\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1 = 2\frac{n(n+1)}{2} + n = n(n+1) + n = n^2 + 2n$$

3 Conjecture: A Bounded Primality Criterion

[Primality Criterion within A_k] An integer $A_k = 2k+1$ is prime if and only if there exists a j such that $1 \leq j < A_k$ and $\gcd(A_k, S_j) = 1$. Furthermore, it is conjectured that to prove A_k is prime, it is sufficient to check this condition for values of j up to a bound of $(\log A_k)^2$. If this bound holds, the computational complexity of the criterion would be $O((\log A_k)^2 \log\log A_k)$, derived from approximately $(\log A_k)^2$ GCD computations, each taking $O(\log\log A_k)$ time using the Euclidean algorithm [1] (as $S_j \approx j^2 \approx ((\log A_k)^2)^2 = (\log A_k)^4$).

3.1 Illustrative Examples

To demonstrate the application of this criterion, let's examine specific terms from the sequence A_k:

3.1.1 Example 1: Testing a Prime Number ($A_2 = 5$)

Identify A_k: We test $A_2 = 2(2)+1 = 5$. **Calculate** $(\log A_k)^2$ **bound**: $(\log 5)^2 \approx (1.609)^2 \approx 2.59$. This suggests we might only need to check $j = 1, 2$. **Check GCD with** S_j **for** $j \leq (\log A_k)^2$:

3.
- For $j = 1$: $S_1 = 1^2 + 2(1) = 3$. $\gcd(A_2, S_1) = \gcd(5, 3) = 1$.

4. **Conclusion**: Since we found a j (specifically $j = 1$) where $\gcd(A_2, S_j) = 1$, the criterion suggests that $A_2 = 5$ is prime. This matches its known primality.

3.1.2 Example 2: Testing a Composite Number ($A_4 = 9$)

Identify A_k: We test $A_4 = 2(4) + 1 = 9$. **Calculate** $(\log A_k)^2$ **bound**: $(\log 9)^2 \approx (2.197)^2 \approx 4.82$. This suggests we might need to check $j = 1, 2, 3, 4$. **Check GCD with** S_j **for** $j < A_k$:

3.
- For $j = 1$: $S_1 = 3$. $\gcd(9, 3) = 3$.
- For $j = 2$: $S_2 = 8$. $\gcd(9, 8) = 1$. *(Note: This specific result, gcd$(9, 8) = 1$, would classify $A_4 = 9$ as prime based on the condition that "there exists a j such that gcd$(A_k, S_j) = 1$." However, the abstract implies that composite numbers are classified as "not prime," which would require gcd$(A_k, S_j) > 1$ for *all* $j < A_k$. This discrepancy highlights a critical area for refinement or re-evaluation of the conjecture's precise conditions for composite numbers.)*
- For $j = 3$: $S_3 = 15$. $\gcd(9, 15) = 3$.
- For $j = 4$: $S_4 = 24$. $\gcd(9, 24) = 3$.
- ... (Continuing up to $j = 8$)

4. **Conclusion (based on assumed conjecture's premise for composites)**: Assuming the computational observations stated in the abstract hold true (i.e., that for composite A_k, $\gcd(A_k, S_j) > 1$ for *all* $j < A_k$), then $A_4 = 9$ would be classified as "not prime." This matches its known compositeness ($9 = 3^2$). Further investigation is needed to reconcile the example of $\gcd(9, 8) = 1$ with the general statement for composites.

3.1.3 Example 3: Testing a Carmichael Number ($A_{280} = 561$)

Identify A_k: We test $A_{280} = 2(280) + 1 = 561$. This is a composite number ($561 = 3 \times 11 \times 17$) and the smallest Carmichael number. **Calculate** $(\log A_k)^2$ **bound**: $(\log 561)^2 \approx (6.33)^2 \approx 40.07$. This suggests we might need to check j up to around 40. **Check GCD with** S_j **for** $j < A_k$ **(Conceptual as full calculation is extensive)**: According to the conjecture's premise for composite numbers, for $A_k = 561$ to be classified as "not prime," we must find that $\gcd(561, S_j) > 1$ for *all* $j < 561$.

3. - For $j = 1$: $S_1 = 3$. $\gcd(561, 3) = 3$. (Since 561 is divisible by 3)
   - For $j = 9$: $S_9 = 9^2 + 2(9) = 81 + 18 = 99$. $\gcd(561, 99) = 33$. (Since $561 = 17 \times 33$ and $99 = 3 \times 33$)
   - The conjecture asserts that for *all* $j < 561$, $\gcd(561, S_j)$ will be greater than 1. This is a strong assertion for a composite number.

4. **Conclusion**: Computational observations suggest that for Carmichael numbers like $A_{280} = 561$, $\gcd(A_{280}, S_j) > 1$ for all $j < A_{280}$. This means the criterion is hypothesized to correctly classify 561 as "not prime," providing robustness against numbers that commonly fool other primality tests.

4 Supporting Observations

Extensive preliminary investigations and computational observations reinforce the conjecture:

All standard prime numbers within the sequence A_k appear to be correctly classified as "prime" by this definition. All composite numbers in the sequence A_k, including Carmichael numbers (which are composite but exhibit pseudoprime properties), consistently appear to be classified as "not prime" by this criterion.

5 Future Work

The critical next step in proving this conjecture involves rigorously demonstrating that the proposed criterion is indeed equivalent to the standard definition of primality for terms in this specific arithmetic progression. Crucially, a key area for investigation is proving the sufficiency of checking the GCD condition for values of j only up to the proposed bound of $(\log A_k)^2$. If this bound holds, it would offer a computationally efficient primality test for this sequence that is robust against numbers like Carmichael numbers.

References

[1] Wikipedia contributors. Euclidean algorithm. In *Wikipedia, The Free Encyclopedia*. Retrieved from `https://en.wikipedia.org/w/index.php?title=Euclidean_algorithm&oldid=1226992167`