Nonliquent Math

Egor Sibrin, Vladimir Orlov

December 2024

Abstract

We develop the commutative algebra $\mathbb{NL} = \mathbb{R} \oplus \ddot{\imath}\mathbb{R}$, where the formal division $1/0$ is legal. Separating the true additive zero $\vartheta = (0, 0)$ from the special element 0 and preserving the single identity $0\ddot{\imath} = 1$ allows the structure to satisfy all field axioms (with zero divisors). We extend limits, differentiation, integration, derive explicit Taylor expansion.

MSC 2020: 12J10; 16W10; 46C99; 35Q30

Key words: nonliquent numbers; idempotent algebra; division by zero; spectral analysis

1 Introduction

1.1 Relevance:

1.1.1 The problem of division by zero: history, relevance and solutions.

Division by zero has been a fundamental challenge in mathematics from ancient times to the present day. While it is an undefined operation in classical arithmetic, its theoretical exploration has profoundly influenced the development of mathematics and physics. Modern research reveals that this problem extends beyond pure mathematics, with significant applications in fields such as relativity theory and quantum mechanics.

1.1.2 History of the problem

Historically, mathematics faced two major unsolved problems: extracting the square root of negative numbers and division by zero. The first was addressed in the 18th century through the works of Leonhard Euler, Augustin Cauchy, and Carl Gauss, who introduced the concept of the imaginary unit. This led to the creation of the complex plane of numbers, which today is indispensable for describing physical phenomena such as electromagnetic oscillations. In contrast, the problem of division by zero remains unresolved. Even in antiquity, mathematicians recognized the paradoxical consequences of dividing by zero. It was not until the 17th century that Isaac Newton and Wilhelm Leibniz, through their foundational work in mathematical analysis, introduced the concept of a limit to address the behavior of functions near points of indeterminacy. While this marked a significant advance, it did not resolve the problem: division by zero remains undefined in classical mathematics.

1.1.3 Application in physics and modern sciences

Division by zero frequently arises in physics. For example, in Einstein's theory of relativity, the equations imply division by zero when an object reaches the speed of light, reflecting the physical impossibility of such a scenario. Similarly, in quantum mechanics, division by zero appears in the mathematical description of Coulomb potentials and particle interactions at infinitesimally small distances. To address these challenges, scientists employ regularization and renormalization techniques, which mitigate infinities and yield meaningful results.

1.1.4 Results and Prospects

The division by zero problem is more than a mathematical abstraction; it underpins numerous physical and mathematical processes and continues to drive scientific innovation. Although no comprehensive solution exists, the analytical tools developed to address this issue have significantly advanced both mathematics and physics. Building on this legacy, the present study introduces a novel approach: the inclusion of division by zero within a hypothetical numerical framework, analogous to the historical development of imaginary numbers.

1.2 Problems

1.2.1 Contradiction of the Properties of Zero

In mathematics, zero serves as the neutral element for addition and the zero element for multiplication. According to the fundamental properties of multiplication, the product of any number and zero is always zero:

$$a \cdot 0 = 0$$

If division by zero were defined, and the result were a number other than zero, this would violate this basic property. For example, if we assume:

$$b = 0 \cdot a,$$

then multiplying b by zero yields:

$$b \cdot 0 \neq 0,$$

which contradicts the definition of zero as the zero element for multiplication. This highlights the inconsistency that arises when attempting to define division by zero within the standard rules of arithmetic.

1.2.2 The Problem of Infinity

To examine the behavior of functions involving division by zero, let us consider the function $f(x) = \frac{1}{x}$. When $x \to 0^+$ (approaching zero from the positive side), the value of the function tends to $+\infty$:

$$\lim_{x \to 0^+} \frac{1}{x} = +\infty \tag{1}$$

Similarly, when $x \to 0^-$ (approaching zero from the negative side), the value of the function tends to $-\infty$:

$$\lim_{x \to 0^-} \frac{1}{x} = -\infty \tag{2}$$

This shows that the limit of $f(x) = \frac{1}{x}$ at $x = 0$ does not exist. If there were a number representing division by zero, it would need to simultaneously equal $+\infty$ and $-\infty$, which is logically inconsistent.

Alternative mathematical frameworks, such as wheel theory, extend number systems to include division by zero. In such systems, division by zero is defined without contradiction, but these approaches are not part of classical mathematics and have limited applicability.

1.2.3 Concluding Remarks on Problems

The contradictions outlined above highlight the challenges of extending classical mathematical definitions to include operations such as division by zero. While alternative theories provide interesting approaches, their adoption often comes at the cost of losing certain foundational properties and assumptions that underpin classical arithmetic. Further exploration into these theories may yield insights into specific mathematical or physical contexts where they could be applied effectively.

2 Subject of the study:

The goal of this study is to propose a hypothetical extension of the numerical system that defines the result of division by zero. In this framework, a new number, denoted as ï, is introduced with specific properties.

2.1 Main statements:

2.1.1 Definition:

Let $\text{ï} = \frac{1}{0}$. This new number is introduced hypothetically as the result of dividing one by zero. Then the numbers associated with this number will be called nonliquent numbers.

2.1.2 Construction property $0 \cdot \ddot{\imath}$:

$$0 \cdot \ddot{\imath} = 1$$

This property is introduced as an axiom to prevent contradictions with the definition of ï. This statement follows directly from the definition of ï as $\frac{1}{0}$. In this case, a $\frac{0}{0}$ paradox is formed, but only for the case of $0 \cdot \frac{1}{0}$ this is equal to 1. We take only two statements on faith. $\frac{1}{0}$ is a permissible action and $0 \cdot \ddot{\imath}$ is equal to 1.

2.1.3 Reverse number for ï:

The entered number ï has the following property:

$$\tfrac{1}{\ddot{\imath}} = 0$$

This statement follows directly from the definition of ï as $\frac{1}{0}$.

2.1.4 Powers of the number ï:

For any natural n:

$$\ddot{\imath}^n = \ddot{\imath}, \forall n \in \mathbb{N}$$

This statement can be justified mathematically:

$$\ddot{\imath}^n = \frac{1}{0} \cdot \frac{1}{0} ... = \frac{1 \cdot 1 \cdot ...}{0 \cdot 0 \cdot ...} = \frac{1}{0} = \ddot{\imath}$$

Raising to the zero power is not different from the standard number. ϑ is another spelling of zero, which has the form a-a, including 0-0.

$$\vartheta = (\frac{1}{\ddot{\imath}} - \ddot{\imath}\frac{0}{\ddot{\imath}}) \Longleftrightarrow (a - a)$$

$$\ddot{\imath}^{\vartheta} = 1 \Longleftrightarrow \ddot{\imath}^{\vartheta} = \ddot{\imath}^{1-1} = \ddot{\imath} \cdot \frac{1}{\ddot{\imath}} = 1$$

Nonliquent Numbers and Their Properties

If we perform operations with nonliquent numbers, we naturally arrive at the general form of a nonliquent number:

$$z = x + \ddot{\imath}y,$$

where $(x, y) \in \mathbb{R}$. In this notation, x is called the real part of the nonliquent number z and is denoted by $x = \operatorname{Re} z$, while y is the nonliquent part, denoted by $y = \operatorname{Nl} z$.

If $x = 0$, the nonliquent number z is called *purely nonliquent*, and if $y = 0$, z is called *real*. Zero is the only nonliquent number where both the real and nonliquent parts are zero. The equality of two nonliquent numbers implies the simultaneous equality of their real and nonliquent parts.

The arithmetic operations of addition and multiplication remain within the set of nonliquent numbers, assuming the same arithmetic laws as real numbers are satisfied. For these rules, the following holds:

$$(a + b\ddot{\imath}) + (c + d\ddot{\imath}) = (a + c) + \ddot{\imath}(b + d),$$
$$(a + b\ddot{\imath}) \cdot (c + d\ddot{\imath}) = a \cdot c + \ddot{\imath}(a \cdot d + b \cdot c + b \cdot d)$$

Division of Nonliquent Numbers

Division is also defined for nonliquent numbers, although it differs from that in complex analysis. Let $a + b\ddot{\imath} \neq 0$. Then:

$$\frac{a + b\ddot{\imath}}{c + d\ddot{\imath}} = x + \ddot{\imath}y$$

Multiplying through by the denominator and expanding the terms:

$$a + b\ddot{\imath} = c \cdot x + \ddot{\imath}(c \cdot y + d \cdot x + d \cdot y)$$

Using the method of undetermined coefficients, we find:

$$x = \frac{a}{c},$$
$$y = \frac{b - \frac{da}{c}}{c + d}$$

In cases where $b+c\ddot{\imath} = 0$, the numerator becomes a purely nonliquent number. In this structure it is impossible to divide into tree elements ϑ and into an element $1 - \ddot{\imath}/\ddot{\imath} - 1$. Because of the general problem of not being able to divide by true zero.

2.1.5 Conjugate of a nonliquent number

The conjugate of a nonliquent number exhibits a markedly different form than its analogue in complex analysis:

$$\overline{a + b\ddot{\imath}} = a + b - b\,\ddot{\imath},$$

where a and b are real coefficients in the decomposition $a + b\,\ddot{\imath}$. This definition ensures that the product of a nonliquent number with its conjugate is real.

Let

$$z = a + b\,\ddot{\imath}, \qquad \overline{z} = a + b - b\,\ddot{\imath},$$

with $a, b \in \mathbb{R}$ and the idempotent property $\ddot{\imath}^2 = \ddot{\imath}$. We verify:

$$\begin{aligned}
z\,\overline{z} &= (a + b\,\ddot{\imath})(a + b - b\,\ddot{\imath}) \\
&= a(a + b - b\,\ddot{\imath}) + b\,\ddot{\imath}(a + b - b\,\ddot{\imath}) \\
&= a^2 + ab - ab\,\ddot{\imath} + ab\,\ddot{\imath} + b^2\,\ddot{\imath} - b^2\,\ddot{\imath}^2 \\
&= a^2 + ab + b^2\big(\ddot{\imath} - \ddot{\imath}^2\big) \\
&= a^2 + ab + b^2\big(\ddot{\imath} - \ddot{\imath}\big) \\
&= a^2 + ab \in \mathbb{R}.
\end{aligned}$$

Hence, $z\overline{z} = a^2 + ab$, which is a real number.

Roots of Nonliquent Numbers

It is possible to extract roots of nonliquent numbers. The principle is similar to that for complex numbers:

$$\sqrt{a + \ddot{\imath}b} = x + \ddot{\imath}y$$

Squaring both sides and expanding:

$$a + \ddot{\imath}b = x^2 + \ddot{\imath}(2xy + y^2)$$

From this, we derive the real and nonliquent parts:

$$\begin{aligned}
x &= \pm\sqrt{a}, \\
y^2 + 2xy - b = 0&, \\
y &= \mp\sqrt{a} \pm \sqrt{a + b}
\end{aligned}$$

The coefficients b and x are related and have opposite signs. Thus, the root is more accurately expressed as:

$$\sqrt{a + \ddot{\imath}b} = \pm\sqrt{a}(1 - \ddot{\imath}) \pm \ddot{\imath}\sqrt{a + b}$$

Note that this yields four roots.

Binomial Expansion for Nonliquent Numbers

Newton's binomial theorem for nonliquent numbers is expressed as:

$$(a + b\text{ï})^n = a^n(1 - \text{ï}) + \text{ï}(a + b)^n$$

Proof:

$$\begin{aligned}(a + b\text{ï})^n &= \sum_{k=0}^{n} \binom{n}{k} a^{n-k}(b\text{ï})^k \\ &= a^n + a^{n-1}b\text{ï} + a^{n-2}b^2\text{ï} + \dots\end{aligned}$$

To form a complete binomial with ï, we add and subtract a^n:

$$a^n - \text{ï}a^n + \text{ï}(a + b)^n$$

Factoring out a^n, we derive the formula above.

2.1.6 Neutral elements and the two zeros

Following the usual field axioms, exactly one element must act as the additive neutral. We denote this true zero by

$$\vartheta \ := \ (0, 0) \in \mathbb{NL}, \qquad \vartheta + x = x, \quad \vartheta \cdot x = \vartheta \ \text{ for all } x \in \mathbb{NL}.$$

The special element Besides ϑ we keep a second distinguished element

$$0 \ := \ 0 \neq \vartheta,$$

while all other products and sums that involve 0 are evaluated by the usual component rules $(u, v) \cdot (u', v') = (uu',\, uv' + u'v + vv')$.

Uniqueness of the additive zero. Since $\vartheta + x = x$ for every $x \in \mathbb{NL}$ and 0 fails to satisfy this identity ($0 + x \neq x$ whenever $x \neq \vartheta$), the field axiom "there is exactly one additive neutral" is preserved. Hence

$$\text{the only additive zero is } \vartheta, \qquad \text{while 0 is a regular (non-neutral) element.}$$

Remark. The conventional real numbers embed via $\iota :\ t \mapsto (t, 0)$. Under this embedding the two distinguished elements become $\iota(0) = \vartheta$ and an 0, so they are indistinguishable inside $\mathbb{R}$. Only in the extended algebra $\mathbb{NL}$ the two roles separate.

2.1.7 Distributivity.

For all $a, b, c \in \mathbb{NL}$ one has $(a+b)\,c = a\,c + b\,c$.
Proof:
$a = (u, v)$, $b = (u', v')$, $c = (u'', v'')$. Using the product rule we expand:

$$(a+b)c = (u+u',\, v+v')\cdot(u'', v'') = \big((u+u')u'',\, (u+u')v''+u''(v+v')+(v+v')v''\big).$$

Grouping terms gives:
$(uu'',\, uv'' + u''v + vv'')\ +\ (u'u'',\, u'v'' + u''v' + v'v'') = a\,c + b\,c.$

To summarize

The set of nonliquent numbers constitutes a field as it satisfies all nine axioms of a field. Specifically:

- It is commutative under both addition and multiplication.
- It is associative under both addition and multiplication.
- It contains an additive identity and additive inverses for all elements.
- It contains a multiplicative identity and multiplicative inverses for all non-zero elements, except ϑ and $1 - \text{ï}/\text{ï} - 1$.
- It satisfies distributivity of multiplication over addition.

The pseudo-field has the following special elements:

- **Multiplicative identity (one):** $1 = 2 - 0\text{ï}$
- **Additive identity (zero):** $\vartheta = \frac{1}{\text{ï}} - \frac{0}{\text{ï}} \cdot \text{ï}$

By definition, this set satisfies the properties of a pseudo-field construct, including the presence of a zero and a one.

2.2 Theories of Functions of Nonliquent Variables

2.2.1 Before Analysis

Before proceeding to the analysis, we establish the following rules:

$$\int \vartheta = \vartheta, \quad a - a = \vartheta, \quad a' = \vartheta, \quad \text{ï} - \text{ï} = \vartheta, \quad f(x) - f(x) = \vartheta, \quad \sin(\vartheta) = \vartheta, \quad \cos(\vartheta) = 1$$

These conclusions are derived from practical calculations. Adherence to these rules is necessary to ensure correct results.

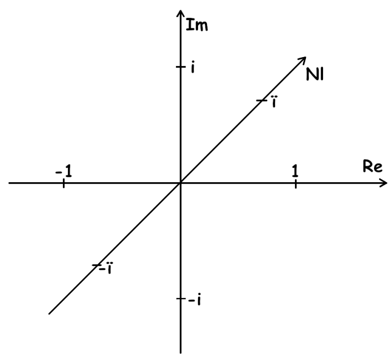


Figure 1: Nonliquent space

2.2.2 Limits and Differentiability

Limits: Let $f(z)$ be a function defined in a neighborhood of a point a. We say that $f(z)$ has a limit A as $z \to a$, and write:

$$\lim_{z \to a} f(z) = A,$$

if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that:

$$|f(z) - A| < \varepsilon \quad \text{for all } z \text{ satisfying } 0 < |z - a| < \delta$$

Continuity: A function $f(z)$ is continuous at a point a if:

$$\lim_{z \to a} f(z) = f(a)$$

A function is called continuous if it is continuous at every point in its domain. From the properties of limits: - The sum, product, and composition of continuous functions are continuous. - The quotient $\frac{f(z)}{g(z)}$ is continuous in a neighborhood of a if $g(z) \neq 0$ in that neighborhood.

Differentiability: The derivative of a function $f(z)$ at a point a is defined as:

$$f'(a) = \lim_{z \to a} \frac{f(z) - f(a)}{z - a},$$

provided the limit exists. This definition is analogous to that of the derivative in real analysis but extends to nonliquent variables.

For a function $f(z) = u(x, y) + \ddot{\imath}v(x, y)$, where $z = x + \ddot{\imath}y$, we compute partial derivatives to establish differentiability:

$$f'(z) = \lim_{\Delta x \to \vartheta, \Delta y = \vartheta} \frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0) + \ddot{\imath}\big(v(x_0 + \Delta x, y_0) - v(x_0, y_0)\big)}{\Delta x}$$

Alternatively, by varying y:

$$f'(z) = \lim_{\Delta y \to \vartheta, \Delta x = \vartheta} \frac{u(x_0, y_0 + \Delta y) - u(x_0, y_0) + \ddot{\imath}\big(v(x_0, y_0 + \Delta y) - v(x_0, y_0)\big)}{\ddot{\imath}\Delta y}$$

The derivative becomes:

$$f'(z) = \frac{\partial u}{\partial x} + \ddot{\imath}\frac{\partial v}{\partial x},$$

or equivalently:

$$f'(z) = \frac{1}{\ddot{\imath}} \cdot \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

Differentiability of Complex-Nonliquent Functions: It is also possible to derive the conditions for differentiating a complex-nonliquent function. Let $f(z)$ be represented as:

$$f(z) = iu(x, y) + \ddot{\imath}v(x, y),$$

where $z = ix + \ddot{\imath}y$, and $u(x, y)$ and $v(x, y)$ are real-valued functions.

The derivative of $f(z)$ can be computed by considering limits as follows:

$$f'(z) = \lim_{\Delta x \to \vartheta, \Delta y = \vartheta} \frac{i\big(u(x_0 + \Delta x, y_0) - u(x_0, y_0)\big) + \ddot{\imath}\big(v(x_0 + \Delta x, y_0) - v(x_0, y_0)\big)}{i\Delta x}$$

Similarly, when varying y, the derivative is:

$$f'(z) = \lim_{\Delta y \to \vartheta, \Delta x = \vartheta} \frac{i\big(u(x_0, y_0 + \Delta y) - u(x_0, y_0)\big) + \ddot{\imath}\big(v(x_0, y_0 + \Delta y) - v(x_0, y_0)\big)}{\ddot{\imath}\Delta y}$$

In both cases, we observe that the conditions for differentiability are preserved. This ensures consistency in the treatment of complex-nonliquent functions, confirming that the derivative is well-defined under the established rules.

Conditions for Differentiability: For $f(z) = u(x, y) + \ddot{\imath}v(x, y)/iu(x, y) + \ddot{\imath}v(x, y)$ to be differentiable at $z = x + \ddot{\imath}y/ix + \ddot{\imath}y$, it must satisfy the following:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y},$$

and:

$$v(x, y) = v(y)$$

2.2.3 Taylor Series and Expansion of Basic Functions

In nonliquent analysis, Taylor series expansions differ from complex analysis due to the properties of ï. Expansions are performed around $(0 - 0)$.

Exponential Function: For $e^{\text{ï}x}$, we begin with the standard Taylor series for e^x:

$$e^x = \sum_{n=\vartheta}^{\infty} \frac{x^n}{n!}$$

Substituting ïx:

$$e^{\text{ï}x} = \sum_{n=\vartheta}^{\infty} \frac{(\text{ï}x)^n}{n!} = 1 + \text{ï}x + \frac{(\text{ï}x)^2}{2!} + \dots$$

Simplifying:

$$e^{\text{ï}x} = 1 + \text{ï}(e^x - 1)$$

Sine and Cosine Functions: For $\sin(\text{ï}x)$:

$$\sin(x) = \sum_{n=\vartheta}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!},$$

$$\sin(\text{ï}x) = \text{ï}x - \text{ï}\frac{x^3}{3!} + \text{ï}\frac{x^5}{5!} = \text{ï}\sin(x)$$

For $\cos(\text{ï}x)$:

$$\cos(x) = \sum_{n=\vartheta}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!},$$

$$\cos(\text{ï}x) = 1 + \text{ï}(\cos(x) - 1)$$

Logarithmic Function: For $\ln(1 + x\text{ï})$:

$$\ln(1 + x\text{ï}) = x\text{ï} - \frac{x^2\text{ï}}{2} + \frac{x^3\text{ï}}{3} - \dots = \text{ï}\ln(1 + x)$$

Other Functions: Additional expansions:

$$\sinh(\text{ï}x) = \text{ï}\sinh(x), \quad \cosh(\text{ï}x) = 1 + \text{ï}(\cosh(x) - 1),$$

$$\tan(\text{ï}x) = \text{ï}\tan(x), \quad \cot(\text{ï}x) = \frac{1}{\text{ï}x} + \text{ï}(\cot(x) - \frac{1}{x})$$

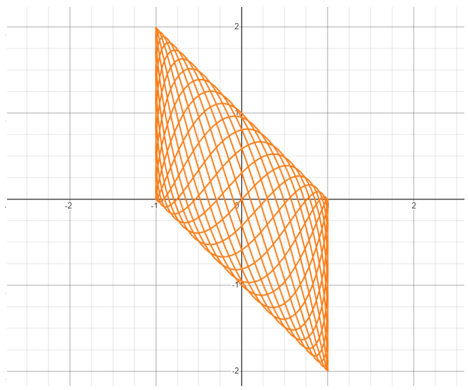


Figure 2: sin (f(z)) The visualization was made using the free website Desmos

Parametrization Definition

Let γ be a smooth curve in the nonliquent plane, defined parametrically as:

$$z(t) = x(t) + \ddot{\imath}y(t), \quad t \in [a, b],$$

where $x(t)$ and $y(t)$ are smooth real-valued functions of t. The curve γ represents the trajectory of a point in the nonliquent plane, with $x(t)$ as the real part and $y(t)$ as the nonliquent part.

For clarity, consider the parameterization of a specific function, such as $\sin(f(z))$. A visual representation of this function's graph is shown below:

Derivation of the Parameterization We start with the definition:

$$\sin(f(z)) = \sin(x + \ddot{\imath}y),$$

and apply the formula for the sine of a sum:

$$\sin(x + \ddot{\imath}y) = \sin x \cos(\ddot{\imath}y) + \cos x \sin(\ddot{\imath}y).$$

Using the previously derived series expansions for $\cos(\ddot{\imath}y)$ and $\sin(\ddot{\imath}y)$:

$$\cos(\ddot{\imath}y) = 1 + \ddot{\imath}(\cos y - 1), \quad \sin(\ddot{\imath}y) = \ddot{\imath}\sin y,$$

we substitute these into the equation:

$$\sin(x + \ddot{\imath}y) = \sin x\big(1 + \ddot{\imath}(\cos y - 1)\big) + \cos x(\ddot{\imath}\sin y).$$

Simplify the expression:

$$\sin(x + \ddot{\imath}y) = \sin x + \ddot{\imath}\sin x \cos y - \ddot{\imath}\sin x + \ddot{\imath}\cos x \sin y.$$

Group terms involving $\ddot{\imath}$:

$$\sin(x + \ddot{\imath}y) = \sin x + \ddot{\imath}(\sin x \cos y - \sin x + \cos x \sin y).$$

Recognizing a trigonometric relationship, rewrite:

$$\sin(x + \ddot{\imath}y) = \sin x + \ddot{\imath}(\sin x + \sin(x + y)).$$

Parametrization Setup Let t represent the parameter. The real and nonliquent components of the function are parameterized as:

$$u(t) = \sin t,$$
$$v(t) = \sin t + \sin(t + f(t)),$$

where $f(t)$ is a linear or other user-defined function.

By setting $f(t)$ to be a family of linear functions, the resulting graph corresponds to the visualization in Figure 2. This approach demonstrates how parameterization simplifies the representation and analysis of nonliquent functions in practical applications.

2.2.4 Integral Over a Closed Contour

Let us evaluate the integral of a smooth function over a closed contour:

Let f be a continuous function in the domain D. Then the following statements hold:

- If $f(z)\,dz$ is an exact differential in the domain D, then for any closed piecewise-smooth curve $\gamma \subset D$, the following equality is satisfied:

$$\oint_\gamma f(z)\,dz = \oint_\gamma u\,dx$$

Proof: Let us expand and group the components of $f(z)$ to utilize Green's theorem:

$$\oint_\gamma f(z)\,dz = \oint_\gamma (u + \ddot{\imath}v)\,d(x + \ddot{\imath}y) = \oint_\gamma u\,dx + \ddot{\imath}\oint_\gamma v\,dx + \ddot{\imath}\oint_\gamma u\,dy + \ddot{\imath}\oint_\gamma v\,dy$$

Applying Green's theorem to the second and third terms, we obtain:

$$\oint_\gamma u\,dx + \ddot{\imath}\iint_\Omega \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dxdy + \ddot{\imath}\iint_\Omega \frac{\partial v}{\partial x}\,dxdy$$

Now, recalling the conditions for differentiability:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad v(x,y) = v(y),$$

we see that both the second and third terms vanish because:

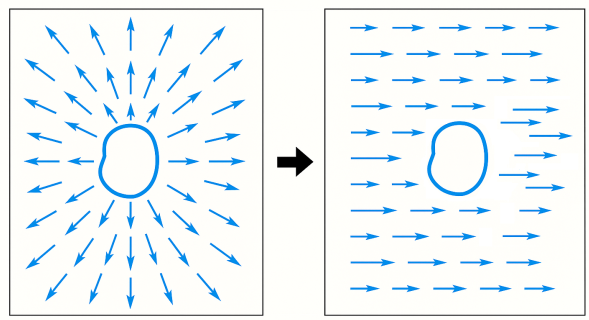
$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = \vartheta, \quad \frac{\partial v}{\partial x} = 0 \quad (\text{as } v \text{ depends only on } y)$$

Thus, the expression simplifies to:

$$\oint_\gamma u\,dx + \ddot{\imath}\iint_\Omega \vartheta\,dxdy + \ddot{\imath}\iint_\Omega \vartheta\,dxdy$$

Since the additional terms are zero, we are left with:

$$\oint_\gamma f(z)\,dz = \oint_\gamma u\,dx$$



Contour integration of nonliquent function
transforming a vector field

Extension: Now consider a function of the form $f(z) = iu + \ddot{\imath}v$ with $dz = d(ix + \ddot{\imath}y)$. The integral over the closed contour becomes:

$$\oint_\gamma f(z)\,dz = \oint_\gamma (iu + \ddot{\imath}v)\,d(ix + \ddot{\imath}y) = -\oint_\gamma u\,dx + i\ddot{\imath}\oint_\gamma v\,dx + i\ddot{\imath}\oint_\gamma u\,dy + i\ddot{\imath}\oint_\gamma v\,dy$$

Grouping the terms and applying Green's theorem, we obtain:

$$-\oint_\gamma u\,dx + i\ddot{\imath}\iint_\Omega \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dxdy + i\ddot{\imath}\iint_\Omega \frac{\partial v}{\partial x}\,dxdy$$

Using the same conditions for differentiability:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial v}{\partial x} = 0,$$

we find that the additional terms vanish, leaving:

$$\oint_\gamma f(z)\,dz = -\oint_\gamma u\,dx$$

This result demonstrates that for a complex-nonliquent function, the integral over a closed contour differs only by a sign in this case.

2.3 Orthonormal Systems in the NL–Hilbert Space

2.3.1 Inner Product and Nonliquent Norm

Let

$$f(x) = u_f(x) + \ddot{\imath}\,v_f(x), \quad g(x) = u_g(x) + \ddot{\imath}\,v_g(x),$$

where $u_f, u_g, v_f, v_g \colon [-\pi, \pi] \to \mathbb{R}$ are real–valued and square–integrable. Define

$$\langle f, g \rangle_{\mathrm{NL}} = \frac{1}{\pi}\int_{-\pi}^{\pi} u_f(x)\,u_g(x)\,\mathrm{d}x.$$

The nonliquent norm of f is

$$\|f\|_{\mathrm{NL}} = \sqrt{\langle f, f\rangle_{\mathrm{NL}}}\,.$$

The mapping $\|\cdot\|_{\mathrm{NL}}$ satisfies:

1. $\|f\|_{\mathrm{NL}} \geq 0$, and $\|f\|_{\mathrm{NL}} = 0$ if and only if $f \equiv 0$.
2. Homogeneity: for any real scalar α, $\|\alpha\, f\|_{\mathrm{NL}} = |\alpha|\, \|f\|_{\mathrm{NL}}$.
3. Triangle inequality: $\|f + g\|_{\mathrm{NL}} \leq \|f\|_{\mathrm{NL}} + \|g\|_{\mathrm{NL}}$.

2.3.2 The NL–Hilbert Space

The set

$$\mathcal{H}_{\mathrm{NL}} = \big\{\, f(x) = u(x) + \ddot{\imath}\, v(x) \;\big|\; \|f\|_{\mathrm{NL}} < \infty \big\}$$

with the operations $\langle\cdot,\cdot\rangle_{\mathrm{NL}}$ and $\|\cdot\|_{\mathrm{NL}}$ is called the NL–Hilbert space. It is complete with respect to the metric induced by $\|\cdot\|_{\mathrm{NL}}$.

2.3.3 Orthogonality and Orthonormality

Functions $f, g \in \mathcal{H}_{\mathrm{NL}}$ are orthogonal if

$$\langle f, g\rangle_{\mathrm{NL}} = 0.$$

A family $\{\varphi_n\}_{n=1}^{\infty} \subset \mathcal{H}_{\mathrm{NL}}$ is orthonormal if

$$\langle \varphi_m, \varphi_n\rangle_{\mathrm{NL}} = \delta_{mn}, \quad m, n \in \mathbb{N},$$

where δ_{mn} is the Kronecker delta.

2.4 Proof of Orthogonality for ψ_m and ψ_n

We employ the NL–inner product in its conjugate form:

$$\langle \psi_m, \psi_n\rangle_{\mathrm{NL}} = \frac{1}{\pi}\int_{-\pi}^{\pi} \psi_m(x)\,\overline{\psi_n(x)}\,\mathrm{d}x,$$

with the NL–conjugation rule

$$\overline{a + b\,\ddot{\imath}} = a + b - b\,\ddot{\imath}.$$

Hence

$$\overline{\psi_n(x)} = \sin(nx) + e^{-n|x|} - \ddot{\imath}\, e^{-n|x|}.$$

Multiply:

$$\psi_m(x)\,\overline{\psi_n(x)} = \big(\sin(mx) + \ddot{\imath}e^{-m|x|}\big)\big(\sin(nx) + e^{-n|x|} - \ddot{\imath}e^{-n|x|}\big).$$

Expanding yields

$$\sin(mx)\sin(nx) + \sin(mx)\,e^{-n|x|} - ï\,\sin(mx)\,e^{-n|x|}$$
$$+\,ï\,e^{-m|x|}\sin(nx) + (ï - ï^2)e^{-(m+n)|x|}.$$

Since $ï^2 = ï$, the last term vanishes exactly. Each mixed term of the form $e^{-k|x|}\sin(\ell x)$ is an odd function on $[-\pi, \pi]$ and integrates to zero. Thus only the product $\sin(mx)\sin(nx)$ remains, which for $m \neq n$ integrates to zero by standard sine orthogonality:

$$\int_{-\pi}^{\pi} \sin(mx)\sin(nx)\,\mathrm{d}x = 0.$$

Therefore

$$\langle \psi_m, \psi_n \rangle_{\mathrm{NL}} = 0 \quad (m \neq n).$$

2.5 Differential equation

2.5.1 Universal substitution for obtaining a nonliquent root

To obtain a nonliquent root, let us suppose that the function y depends on two functions $u(x)$ and $v(x)$:

$$y' + by = p(x)$$

$$y = u + vï$$

Then we get one function that is purely real, and the other purely nonliquent:

$$u' + v'ï + bu + bvï = p(x)$$

From this, we extract two equations using the method of undetermined coefficients:

$$\begin{cases} u' + bu = p(x) \\ v' + bv = \vartheta \end{cases}$$

From this, we obtain two equalities, the first one corresponds to the original equation. And the second guarantees that the nonliquent part will be zeroed.

2.6 Application to Fluid Dynamics and the Divergence Equation

Having developed the foundational algebraic structure for nonliquent numbers, the next objective was to explore a possible physical application. A natural candidate for such an application is the Navier–Stokes equation, which governs the motion of viscous fluids. If we can successfully reinterpret this equation within the framework of nonliquent differentiation, it would imply that nonliquent functions are capable of describing fluid flows.

To simplify the problem, we begin by considering a static flow field with no external forces. This assumption eliminates time dependence and body forces, thereby reducing the number of variables and focusing our attention on the structural properties of the flow.

We begin with the continuity equation, which expresses the conservation of mass. We assume that the velocity field $\mathbf{V}$ can be decomposed into real and nonliquent components, and we impose the condition of incompressibility, which requires that the divergence of the velocity field be zero. As a starting point, we consider the continuity equation from fluid dynamics, which expresses the conservation of mass:

$$\nabla \cdot (\rho \mathbf{V}) = \vartheta.$$

We assume a two-dimensional velocity field $\mathbf{V} = (u, v)$, and that the density $\rho = \rho(y)$ is a function only of the vertical coordinate. This assumption is necessary, as more general forms do not yield a solvable structure under the constraints of nonliquent differentiation.

We now impose the nonliquent conditions of differentiability, namely: - If $f = u + \ddot{\imath} v$, then differentiability is defined under the conditions:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad v = v(y).$$

These conditions are necessary to ensure compatibility with the nonliquent derivative structure.

Applying the divergence operator in Cartesian coordinates, we expand the continuity equation as:

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = \vartheta.$$

Since $\rho = \rho(y)$, the partial derivative with respect to x vanishes:

$$\rho \frac{\partial u}{\partial x} + \frac{d\rho}{dy} \cdot v + \rho \frac{\partial v}{\partial y} = \vartheta.$$

Using the nonliquent conditions:

$$\frac{\partial u}{\partial x} = f'(y), \qquad \frac{\partial v}{\partial y} = f'(y), \qquad v = f(y),$$

we substitute into the equation:

$$\rho f'(y) + \frac{d\rho}{dy} \cdot f(y) + \rho f'(y) = \vartheta.$$

This simplifies to:

$$2\rho f'(y) + \frac{d\rho}{dy} \cdot f(y) = \vartheta.$$

Solving for $\frac{d\rho}{dy}$, we obtain:

$$\frac{d\rho}{dy} = -2\rho \frac{f'(y)}{f(y)},$$

which leads to the general solution:

$$\rho(y) = \frac{1}{f^2(y)} + C,$$

where $C \in \mathbb{R}$ is an integration constant.

This result shows how the fluid density $\rho(y)$ is directly determined by the structure of the nonliquent velocity component $f(y)$ under the imposed differential constraints.

2.6.1 Derivation from the Navier–Stokes Equation in Two Dimensions

We now consider the Navier–Stokes equations for incompressible, viscous, and stationary flow in two dimensions. The general form of the Navier–Stokes equations for incompressible, Newtonian fluid flow in Cartesian coordinates is given by:

$$\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla)\mathbf{V} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \mathbf{V},$$

where:

- $\mathbf{V} = (V_x, V_y)$ is the velocity field,
- p is the pressure,
- ρ is the fluid density,
- ν is the kinematic viscosity,
- $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$,
- $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the Laplacian.

Substituting the real and nonliquent V into the equation, we obtain the following system:

$$\begin{cases} \frac{\partial V_x}{\partial x} \cdot V_x + \frac{\partial V_x}{\partial y} \cdot V_y = \frac{\partial p}{\partial x}\left(-\frac{1}{\rho}\right) + \nu\left(\frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_x}{\partial y^2}\right), \\ \\ \frac{\partial V_y}{\partial x} \cdot V_x + \frac{\partial V_y}{\partial y} \cdot V_y = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 V_y}{\partial x^2} + \frac{\partial^2 V_y}{\partial y^2}\right). \end{cases}$$

We now introduce the nonliquent differentiability conditions, where the velocity field is expressed through a single scalar nonliquent function $f(y)$. We define:

$$V_x = x f'(y), \qquad V_y = f(y).$$

2.6.2 Combining the Two Equations under a Single Differential

Starting from the reduced two equations obtained after substituting $V_x = x\, f'(y)$, $V_y = f(y)$ into the stationary Navier–Stokes system, we have:

$$E_1(y): \quad (f'(y))^2 + f(y)\, f''(y) - \nu\, f^{(3)}(y) = -\frac{1}{\rho\, x}\frac{\partial p}{\partial x},$$

$$E_2(y): \quad f(y)\, f'(y) - \nu\, f''(y) = -\frac{1}{\rho}\frac{\partial p}{\partial y}.$$

Multiply by rho and substitute:

$$E_1(y): \quad x\frac{(f'(y))^2 + f(y)\, f''(y) - \nu\, f^{(3)}(y)}{f^2} = -\frac{\partial p}{\partial x},$$

$$E_2(y): \quad \frac{(f(y)\, f'(y) - \nu\, f''(y))}{f^2} = -\frac{\partial p}{\partial y}.$$

Since $\partial_x p$ depends only on x, multiplying by x and differentiating with respect to y must give zero:

$$E_2(y): \frac{\partial}{\partial x}\left[\frac{f\, f' - \nu\, f''}{f^2}\right] = -\frac{\partial p}{\partial y \partial x}. = \vartheta$$

$$x\frac{d}{dy}\left[-\frac{\left((f')^2 + f\, f'' - \nu\, f^{(3)}\right)}{f^2}\right] = \vartheta.$$

Hence, the unified differential equation is:

$$f'(ff'' - (f')^2 + 2\nu f''') + f(ff''' - \nu f'''') = \vartheta.$$

2.7 Analytical Intractability and Particular Solutions

The nonlinear differential equation

$$f'(ff'' - (f')^2 + 2\nu f''') + f(ff''' - \nu f'''') = \vartheta$$

arises as a one-dimensional reduction of the two-dimensional Navier–Stokes system under nonliquent differentiability conditions. Due to its nonlinearity and the presence of higher-order derivatives, this equation does not admit a general closed-form solution using classical analytical techniques.

However, by employing symbolic computation and systematic trial methods, two particular solutions were identified:

Solution 1: Inviscid case ($\nu = \vartheta$) When the viscosity parameter is set to zero, the equation simplifies to:

$$f'(ff'' - (f')^2) + fff''' = \vartheta.$$

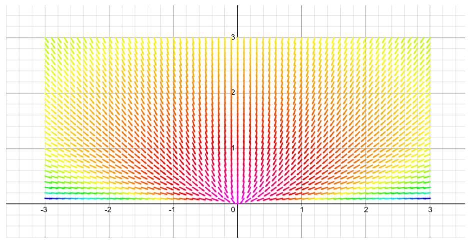


Figure 3: Field that follows the rules of derivative The visualization was made using the free website Desmos

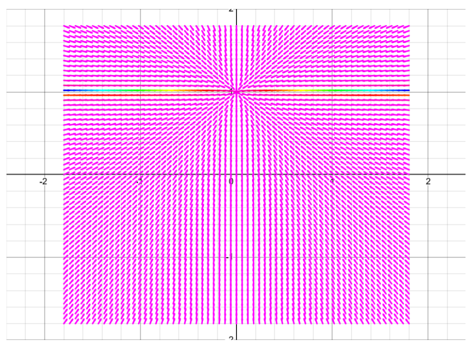


Figure 4: Field that follows the rules of derivative The visualization was made using the free website Desmos

In this case, the following function satisfies the equation:

$$f(x) = Cx^{2\pm\sqrt{2}},$$

where $C \in \mathbb{R}$ is an arbitrary constant. This solution corresponds to a quadratic flow profile with a constant shift, representing a simple parabolic structure.

Solution 2: Viscous case with $\nu \neq 0$

For a particular nonzero value of viscosity, the equation admits another exact solution of rational form:

$$f(x) = -\frac{12\nu}{7(x-C)},$$

where $C \in \mathbb{R}$ is an integration constant. This function features a singularity at $x = 0$, indicating a concentrated flow behavior akin to a point source or sink.

These solutions demonstrate that, while the general solution remains inaccessible analytically, specific structured profiles consistent with physical intuition can be extracted from the equation under simplifying assumptions. These results also validate the internal consistency of the nonliquent framework when applied to classical models of fluid dynamics.

Conclusions and Final Remarks

Key Achievements:

1. **Introduction of Nonliquent Numbers**: - A new number, ï, is defined as $ï = \frac{1}{0}$, with associated properties such as $ï^n = ï, \forall n \in \mathbb{N}$. - The construct $0 \cdot ï = 1$ was introduced axiomatically, resolving contradictions in the arithmetic involving ï.

2. **Field Properties**: - The set of nonliquent numbers forms a field satisfying all nine axioms, including commutativity, associativity, distributivity, and the existence of additive and multiplicative inverses. - Special elements like the multiplicative identity $1 = 2 - 0 \cdot ï$ and additive identity $\vartheta = 0 - 0$ were defined.

3. **Extension of Calculus**: - Theories of limits, continuity, and differentiability were extended to nonliquent variables, preserving core mathematical principles. - Differentiability conditions, such as $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $v(x, y) = v(y)$, were established for functions $f(z) = u(x, y) + ïv(x, y)$.

4. **Taylor Series Expansions**: - Taylor series expansions for functions such as $e^{ïx}$, $\sin(ïx)$, and $\ln(1 + xï)$ demonstrated consistency with the new system, highlighting the interplay between real and nonliquent parts.

5. **Integration in Nonliquent Analysis**: - Integration over closed contours was analyzed, showing that $\oint_\gamma f(z)\, dz = \oint_\gamma u\, dx$ for exact differentials. - For complex-nonliquent functions, the integral differs by a sign: $\oint_\gamma f(z)\, dz = -\oint_\gamma u\, dx$.

6. **Zero Field**: - A unique "zero field" was identified, arising from reinterpretations of zero as an ordinary number with additive and multiplicative inverses. - This field has unique properties, such as its existence at an infinite distance from 1 and its closure to interactions involving zero or ï.

7. **Solutions to Differential Equations in the Nonliquent Framework**: - Nonliquent differentiability was successfully embedded into the continuity equation. - A modified form of the Navier–Stokes equation was derived:

$$f'(ff'' - (f')^2 + 2\nu f''') + f(ff''' - \nu f'''') = 0,$$

representing a two-dimensional fluid flow under nonliquent differentiation. - Two exact solutions were discovered:

- For $\nu = 0$: $f(x) = Cx^{2\pm\sqrt{2}}$,
- For $\nu \neq 0$: $f(x) = -\frac{12\nu}{7x+C}$,

demonstrating the solvability of certain nonlinear fluid models in the nonliquent domain.

Implications and Future Work:

1. **Mathematical Implications**: - The framework offers a novel perspective on division by zero, potentially resolving paradoxes in specific contexts. - It

introduces a rigorous extension of calculus and algebra to include nonliquent variables.

2. **Applications in Physics and Engineering**: - The field of nonliquent numbers may have applications in areas requiring new models for singularities, undefined operations, or extended numerical systems.

3. **Future Research Directions**: - Investigating applications of nonliquent numbers in differential equations, dynamical systems, and quantum mechanics. - Exploring connections between nonliquent numbers and complex or hypercomplex systems. - Developing computational methods and visualization tools for nonliquent variables.

Final Remarks:

The study lays a foundational framework for the theory of nonliquent numbers, demonstrating its consistency and mathematical elegance. While primarily theoretical, the potential applications in advanced mathematical and physical models merit further exploration. The introduction of ï represents a bold step in extending our understanding of numerical systems and the nature of mathematical abstraction.

Currently, work is underway to achieve full synchronization between the complex plane, the nonliquent plane, and the real plane. This effort aims to unify these mathematical constructs into a cohesive framework, allowing for seamless transitions and interactions between them. As a result, the full potential of three-dimensional parameterization and three-dimensional integration remains an area of active exploration. These advancements promise to extend the applicability of the theoretical framework, opening new pathways for analysis and computation in multidimensional spaces.

The nonliquent system offers direct advantages over the wheel system because, in it, nullity and infinity function more as states rather than as numbers. They are neither countable nor operable, and thus are not subject to any mathematical operations. Conversely, the wheel system destroys the axiomatic foundations of a field, unlike the nonliquent system. At the same time, the nonliquent system retains a fundamental drawback: division by 0–0 is still not feasible. I am actively working on resolving this issue.

References

[1] M. Bauer, P. Gianni, V. Shanmugam, "Wheels: A Generalization of Division by Zero," Journal of Algebra, vol. 307, pp. 120–151, 2007.

[2] J. Barwise, J. C. M. Moore, "Transreal Numbers," Notre Dame Journal of Formal Logic, vol. 37, no. 2, pp. 205–221, 1996.

[3] J. H. Conway, Functions of One Complex Variable, Graduate Texts in Mathematics, vol. 11, Springer, 1978.

[4] G. L. Litvinov and V. P. Maslov (eds.), Idempotent Mathematics and Mathematical Physics, Contemporary Mathematics, vol. 377, American Mathematical Society, 2005.