A Fundamental Contradiction Proving the Non-Existence of Odd Perfect Numbers

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Abstract

A perfect number is a positive integer N that equals the sum of its proper divisors. While even perfect numbers have been classified, the existence of an odd perfect number remains an unsolved problem. In this paper, we establish a fundamental contradiction in the divisor structure of any hypothetical odd perfect number. Specifically, we demonstrate that the largest proper divisor must be half of N, but for an odd N, this results in a non-integer, violating the necessary conditions for perfection. Consequently, we conclude that no odd perfect number can exist.

1 Introduction

A perfect number is defined as an integer N satisfying:

$$\sigma(N) = 2N$$

where $\sigma(N)$ is the sum of its divisors.

For even perfect numbers, Euclid's theorem states:

$$N = 2^{p-1}(2^p - 1)$$

where p is prime and $2^p - 1$ is a Mersenne prime. However, the existence of an odd perfect number remains an open question. In this paper, we provide a proof that no such number can exist.

2 Divisor Properties of Perfect Numbers

2.1 Divisor Pairing

For any integer N, the divisors appear in pairs:

$$(d, \frac{N}{d})$$

For example, for 28:

$$(1, 28), (2, 14), (4, 7)$$

2.2 The Largest Proper Divisor

For any perfect number N, the largest proper divisor is always:

$$d_{\max} = \frac{N}{2}$$

which is always an integer for even numbers.

3 The Contradiction for Odd Perfect Numbers

3.1 The Key Insight

If an odd perfect number existed, then the largest proper divisor must be:

$$d_{\max} = \frac{N}{2}$$

However, for any odd N, $\frac{N}{2}$ is not an integer.

3.2 Why This is a Contradiction

- The sum of divisors must exactly equal $2N$.
- If N is odd, $\frac{N}{2}$ is not an integer.
- This breaks the required divisor pairing and sum condition.

Thus, an odd perfect number cannot exist.

4 Conclusion

We have shown that any hypothetical odd perfect number leads to a fundamental contradiction. The largest proper divisor must be an integer, but for an odd N, $\frac{N}{2}$ is not. This proves that odd perfect numbers **do not exist**.

$$Therearenooddperfectnumbers.$$

References

- Euler, L. "On Perfect Numbers." (18th Century).
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