Exact Relativistic Corrections to the Quantization of a Classical Spinning Particle with Constant Electric and Magnetic fields

John French

email: johnf10035@yahoo.com

Abstract

This paper is an extension of a paper by the author where a quantization of classical spinning particle equations is carried out using the Euler angles of the particle. Relativistic corrections are found and compared to the Foldy-Wouthuysen transformation of the Dirac equation. In the first section we only consider constant weak electric and magnetic fields, and find agreement to all orders of $1/c^2$. We then look at a more general situation and compare our quantization method to that of Chiou and Chen again for constant weak fields. In the last section we consider a general constant magnetic field with no electric field and look at the difference between this result and the Foldy-Wouthuysen transformation when strong magnetic field terms are included.

Keywords Classical spin equation, Canonical quantization

.

I. Introduction

This paper is an extension of a previous paper by the author [1], referred to as the initial paper in this article. The initial paper is based on the equations of motion for a relativistic spinning charge in an external electric and magnetic field. These equations can be found in Jackson [2]. Only weak constant external fields are considered when an electric field is included, that is we are only keeping linear field terms. Using the Euler angles and particle position as degrees of freedom, a Lagrangian and Hamiltonian are found for these equations and then the system is quantized using the method of Bopp and Haag [3].

In the initial paper we found that we could quantize the system up to third order in $1/c^2$, and found agreement with the Foldy-Wouthuysen transformation [4] of the Dirac equation. More on the Dirac equation can be found in Sakurai [5]. In this extension of the initial paper we find a solution to all orders of $1/c^2$ which agrees with the Foldy-Wouthuysen transformation as found in Chiou and Chen [6].

We then look at our quantization method in a more general situation and find an expression of the quantization method that agrees with that of Chiou and Chen [7]. In the last section we consider a situation where there is no electric field, but a large constant magnetic field is considered. In this case we obtain a quantum equation which is different from the Foldy-Wouthuysen transformation if large magnetic fields are allowed.

II. Review of initial paper

From Jackson [2] we have the equations of motion for a spinning charge in a general inertial frame

$$m\frac{d\boldsymbol{\beta}}{dt} = \frac{q}{\gamma c}(\mathbf{E} + \boldsymbol{\beta} \times \mathbf{B} - \boldsymbol{\beta}(\boldsymbol{\beta} \cdot \mathbf{E})) \tag{1}$$

$$\frac{d\mathbf{s}}{dt} = \frac{q}{mc}\mathbf{s} \times \{\left(\frac{g}{2} - 1 + \frac{1}{\gamma}\right)\mathbf{B} - (\frac{g}{2} - 1)\frac{\gamma}{\gamma+1}(\boldsymbol{\beta}\cdot\mathbf{B})\boldsymbol{\beta} - (\frac{g}{2} - \frac{\gamma}{\gamma+1})\boldsymbol{\beta}\times\mathbf{E}\} \qquad (2)$$

where **s** is the spin angular momentum in the particle's rest frame and **E** and **B** are the electric and magnetic fields in a general frame. In these expressions $\boldsymbol{\beta} = \mathbf{v}/c$ and $\gamma = (1 - \beta^2)^{-1/2}$ where **v** is the velocity of the particle in the general frame and c is the speed of light. q is the charge of the particle, m is its rest mass, and t represents the time in a general inertial frame. g is the gyromagnetic factor which based on the Dirac equation is taken to be 2. We use a bold symbol to indicate a vector.

Now set $\mathbf{s} = I\gamma\boldsymbol{\omega}$ where I is the moment of inertia for the particle and $\boldsymbol{\omega}$ is its angular velocity. A factor of γ is included because in this paper $\boldsymbol{\omega}$ represents the derivatives of the Euler angles with respect to the time t in a general frame, not the time in the rest frame of the particle.

Using the Euler angles and particle position as the degrees of freedom, we have from the initial paper the Hamiltonian H and conjugate momentum $\mathbf{p_v}$ and $\mathbf{p_\omega}$

$$H = \left(m_0c^2 + \frac{1}{2}I\gamma^2\omega^2\right)\gamma + \frac{Iq}{mc}\frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}\cdot(\boldsymbol{\omega}\times\mathbf{E}) + q\Phi \qquad (3)$$

$$\mathbf{p_v} = \left(m_0c^2 + \frac{1}{2}I\gamma^2\omega^2\right)\frac{1}{c}\gamma\boldsymbol{\beta} + \frac{q}{c}\mathbf{A} + \frac{Iq}{mc^2}(\frac{\gamma}{\gamma+1}\boldsymbol{\omega}\times\mathbf{E} + \frac{\gamma^3}{(\gamma+1)^2}(\boldsymbol{\beta}\cdot(\boldsymbol{\omega}\times\mathbf{E}))\boldsymbol{\beta}) \qquad (4)$$

$$\mathbf{p_\omega} = I\gamma\boldsymbol{\omega} + \frac{Iq}{mc}(\mathbf{B} - \frac{\gamma}{\gamma+1}\boldsymbol{\beta}\times\mathbf{E}) \qquad (5)$$

where $m_0 = m - \frac{1}{2c^2}I\gamma^2\omega^2$ and can be considered the non-rotating rest mass. **A** is the vector potential and Φ is the scalar potential.

To quantize the system we want to express the Hamiltonian H in terms of $\mathbf{p_v}$ and $\mathbf{p_\omega}$. From the initial paper we were able to do this exactly for only the case of a non-zero **B** field where we obtained

$$H = \{m'^2c^4 + c^2\pi^2 - \frac{2q}{m}m'c\mathbf{p}_{\boldsymbol{\omega}} \cdot \mathbf{B}\}^{\frac{1}{2}}$$

$$= m'c^2(1 + \frac{\pi^2}{m'^2c^2})^{\frac{1}{2}} - \frac{q}{mc}(1 + \frac{\pi^2}{m'^2c^2})^{-\frac{1}{2}}\mathbf{p}_{\boldsymbol{\omega}} \cdot \mathbf{B} \qquad (6)$$

where $\boldsymbol{\pi} = \mathbf{p}_{\mathbf{v}} - \frac{q}{c}\mathbf{A}$ and $m' = m_0 + \frac{1}{2Ic^2}p_{\boldsymbol{\omega}}{}^2$. We have only kept linear fields.

When we include a non-zero **E** field we were able to find a solution up to third order in $1/c^2$ which is given by

$$H = m'c^2\left(1 + \frac{1}{2}\frac{\pi^2}{m'^2c^2} - \frac{1}{8}\frac{\pi^4}{m'^4c^4} + \frac{1}{16}\frac{\pi^6}{m'^6c^6}\right) + \frac{q}{mc}\left(-1 + \frac{1}{2}\frac{\pi^2}{m'^2c^2} - \frac{3}{8}\frac{\pi^4}{m'^4c^4} + \frac{5}{16}\frac{\pi^6}{m'^6c^6}\right)\mathbf{p}_{\boldsymbol{\omega}} \cdot \mathbf{B}$$

$$- \frac{1}{2}\frac{q}{mm'c^2}\left(1 - \frac{3}{4}\frac{\pi^2}{m'^2c^2} + \frac{5}{8}\frac{\pi^4}{m'^4c^4}\right)\boldsymbol{\pi} \cdot (\mathbf{p}_{\boldsymbol{\omega}} \times \mathbf{E}) + q\Phi$$

$$= m'c^2\left(1 + \frac{1}{2}\frac{\pi^2}{m'^2c^2} - \frac{1}{8}\frac{\pi^4}{m'^4c^4} + \frac{1}{16}\frac{\pi^6}{m'^6c^6}\right) - \frac{q}{mc}\left(1 + \frac{\pi^2}{m'^2c^2}f(\frac{\pi^2}{m'^2c^2})\right)\mathbf{p}_{\boldsymbol{\omega}} \cdot \mathbf{B}$$

$$+ \frac{q}{mm'c^2}f(\frac{\pi^2}{m'^2c^2})\boldsymbol{\pi} \cdot (\mathbf{p}_{\boldsymbol{\omega}} \times \mathbf{E}) + q\Phi \qquad (7)$$

where

$$f\left(\frac{\pi^2}{m'^2c^2}\right) = -\frac{1}{2} + \frac{3}{8}\frac{\pi^2}{m'^2c^2} - \frac{5}{16}\frac{\pi^4}{m'^4c^4} \qquad (8)$$

By comparing equation (6) to equation (7) we find

$$f\left(\frac{\pi^2}{m'^2c^2}\right) = \left(\frac{\pi^2}{m'^2c^2}\right)^{-1}\left(\left(1+\frac{\pi^2}{m'^2c^2}\right)^{-\frac{1}{2}} - 1\right) \tag{9}$$

to second order in $1/c^2$.

III.Exact Solution and Quantization

To extend the solution given in equation (7), assume that equation (9) is valid to all orders in $1/c^2$ so that equation (7) becomes

$$H = m'c^2\left(1+\frac{\pi^2}{m'^2c^2}\right)^{\frac{1}{2}} - \frac{q}{mc}\left(1+\frac{\pi^2}{m'^2c^2}\right)^{-\frac{1}{2}}\mathbf{p}_{\boldsymbol{\omega}}\cdot\mathbf{B}$$

$$+\frac{q}{mm'c^2}\left(\frac{\pi^2}{m'^2c^2}\right)^{-1}\left(\left(1+\frac{\pi^2}{m'^2c^2}\right)^{-\frac{1}{2}} - 1\right)\boldsymbol{\pi}\cdot(\mathbf{p}_{\boldsymbol{\omega}}\times\mathbf{E}) + q\Phi \tag{10}$$

Then using the relation

$$x\left\{\frac{1}{1+(1+x)^{1/2}} - \frac{1}{(1+x)^{1/2}}\right\} = \frac{1}{(1+x)^{1/2}} - 1 \tag{11}$$

for some number x, set $x = \frac{\pi^2}{m'^2c^2}$ so that we can express equation (10) for H as

$$H = m'c^2\left(1+\frac{\pi^2}{m'^2c^2}\right)^{\frac{1}{2}} - \frac{q}{mc}\left(1+\frac{\pi^2}{m'^2c^2}\right)^{-\frac{1}{2}}\mathbf{p}_{\boldsymbol{\omega}}\cdot\mathbf{B}$$

$$+\ \frac{q}{mm'c^2}\{\frac{1}{1+(1+\frac{\pi^2}{m'^2c^2})^{1/2}}-\frac{1}{(1+\frac{\pi^2}{m'^2c^2})^{1/2}}\}\ \boldsymbol{\pi}\cdot(\mathbf{p}_{\boldsymbol{\omega}}\times\mathbf{E})+q\Phi \tag{12}$$

To test our assumption of the validity of equation (12) we can use equations (4) and (5) for $\mathbf{p}_{\mathbf{v}}$ and $\mathbf{p}_{\boldsymbol{\omega}}$ in equation (12) to see if we obtain equation (3) for H. Using equation (5) we can write

$$m_0c^2+\frac{1}{2}I\gamma^2\omega^2=m'c^2-\frac{Iq}{mc}\gamma\boldsymbol{\omega}\cdot(\mathbf{B}-\frac{\gamma}{\gamma+1}\boldsymbol{\beta}\times\mathbf{E}) \tag{13}$$

Then using equation (13) in equation (4) along with $\boldsymbol{\pi}=\mathbf{p}_{\mathbf{v}}-\frac{q}{c}\mathbf{A}$ we find

$$\boldsymbol{\pi}=m'c\gamma\boldsymbol{\beta}+\frac{Iq}{mc^2}\{-\gamma^2(\boldsymbol{\omega}\cdot\mathbf{B})\boldsymbol{\beta}+\frac{\gamma}{\gamma+1}\boldsymbol{\omega}\times\mathbf{E}-\frac{\gamma^4}{(\gamma+1)^2}\big(\boldsymbol{\beta}\cdot(\boldsymbol{\omega}\times\mathbf{E})\big)\boldsymbol{\beta}\} \tag{14}$$

and that

$$(1+\frac{\pi^2}{m'^2c^2})^{\frac{1}{2}}=\gamma+\frac{qI}{mm'c^3}\{-\gamma^2\beta^2\boldsymbol{\omega}\cdot\mathbf{B}+\frac{\gamma+\gamma^2-\gamma^3}{1+\gamma}\ \boldsymbol{\beta}\cdot(\boldsymbol{\omega}\times\mathbf{E})\} \tag{15}$$

Then using equations (14) and (15) in equation (12) we obtain

$$H=m'c^2\gamma+\ \frac{q}{mc}\{I\left\{-\gamma^2\beta^2\boldsymbol{\omega}\cdot\mathbf{B}+\frac{\gamma+\gamma^2-\gamma^3}{1+\gamma}\ \boldsymbol{\beta}\cdot(\boldsymbol{\omega}\times\mathbf{E})\right\}-\frac{1}{\gamma}\mathbf{p}_{\boldsymbol{\omega}}\cdot\mathbf{B}$$

$$-\ \frac{1}{1+\gamma}\boldsymbol{\beta}\cdot(\mathbf{p}_{\boldsymbol{\omega}}\times\mathbf{E})\}+q\Phi \tag{16}$$

Then using equations (13) and (5) in equation (16) we obtain equation (3). In these calculations we have used the relation $\gamma^2 - 1 = \gamma^2\beta^2$ and only kept linear field terms. Since we can obtain equation (3) in this way it shows that equation (12) is a valid solution.

Following the initial paper we can quantize the solution for H in equation (12) by replacing m' by $m = m_0 + \frac{3\hbar^2}{8Ic^2}$, $\boldsymbol{\pi}$ by $\widehat{\boldsymbol{\pi}} = -i\hbar\boldsymbol{\nabla} - \frac{q}{c}\mathbf{A}$, and $\mathbf{p}_{\boldsymbol{\omega}}$ by $\frac{1}{2}\hbar\boldsymbol{\sigma}$ where $\boldsymbol{\sigma}$ are the Pauli spin matrices in vector form and a hat represents an operator. Thus equation (12) becomes the corresponding quantum equation

$$i\hbar\frac{\partial\Psi}{\partial t} = [mc^2\hat{\gamma}_{\boldsymbol{\pi}} - \frac{q\hbar}{2mc}\frac{1}{\hat{\gamma}_{\boldsymbol{\pi}}}\boldsymbol{\sigma}\cdot\mathbf{B} + \frac{q\hbar}{2m^2c^2}\{\frac{1}{1+\hat{\gamma}_{\boldsymbol{\pi}}} - \frac{1}{\hat{\gamma}_{\boldsymbol{\pi}}}\}\,\widehat{\boldsymbol{\pi}}\cdot(\boldsymbol{\sigma}\times\mathbf{E}) + q\Phi]\Psi \tag{17}$$

where $\hat{\gamma}_{\boldsymbol{\pi}} = (1 + \frac{\widehat{\pi}^2}{m^2c^2})^{\frac{1}{2}}$. This agrees with the Foldy-Wouthuysen expansion of the Dirac equation given by Chiou and Chen [6].

IV.General method for linear fields

To extend this method to a general form, write eq. (2) in the form

$$\frac{d\mathbf{s}}{dt} = \frac{q}{mc\gamma}\mathbf{s}\times\mathbf{B}' \tag{18}$$

where $\mathbf{B}'$ is a general function linear in the $\mathbf{E}$ and $\mathbf{B}$ fields and depends upon γ and $\boldsymbol{\beta}$ so that $\mathbf{B}' = \mathbf{B}'(\gamma, \boldsymbol{\beta}, \mathbf{E}, \mathbf{B})$. Following the initial paper we can write a Lagrangian for this in the form

$$L = -m_0c^2\gamma^{-1} + \frac{1}{2}I\gamma\omega^2 + q(\frac{1}{c}\mathbf{v}\cdot\mathbf{A} - \Phi) + \frac{Iq}{mc}\boldsymbol{\omega}\cdot\mathbf{B}' \tag{19}$$

where we have only kept linear field terms and kept $\mathbf{E}$ and $\mathbf{B}$ constant. The conjugate momentum $\mathbf{p_v}$, $\mathbf{p_\omega}$ and Hamiltonian H take the form

$$\mathbf{p_v} = \boldsymbol{\nabla}_\mathbf{v}\mathrm{L} = \left(\mathrm{m_0c^2} + \frac{1}{2}\mathrm{I}\gamma^2\omega^2\right)\frac{1}{\mathrm{c}}\gamma\boldsymbol{\beta} + \frac{\mathrm{q}}{\mathrm{c}}\mathbf{A} + \frac{\mathrm{Iq}}{\mathrm{mc}}\boldsymbol{\nabla}_\mathbf{v}(\boldsymbol{\omega}\cdot\mathbf{B}') \tag{20}$$

$$\mathbf{p_\omega} = \boldsymbol{\nabla}_{\boldsymbol{\omega}}\mathrm{L} = \mathrm{I}\gamma\boldsymbol{\omega} + \frac{\mathrm{Iq}}{\mathrm{mc}}\mathbf{B}' \tag{21}$$

$$\mathrm{H} = \mathbf{p_v}\cdot\mathbf{v} + \mathbf{p_\omega}\cdot\boldsymbol{\omega} - \mathrm{L} = \left(\mathrm{m_0c^2} + \frac{1}{2}\mathrm{I}\gamma^2\omega^2\right)\gamma + \frac{\mathrm{Iq}}{\mathrm{m}}\boldsymbol{\beta}\cdot\boldsymbol{\nabla}_\mathbf{v}(\boldsymbol{\omega}\cdot\mathbf{B}') + \mathrm{q}\Phi \tag{22}$$

where $\boldsymbol{\nabla}_\mathbf{v}$ represents the gradiatat with respect to $\mathbf{v}$ and $\boldsymbol{\nabla}_{\boldsymbol{\omega}}$ represents the gradiant with respect to $\boldsymbol{\omega}$. Then using eq. (21) we can write

$$\mathrm{m_0c^2} + \frac{1}{2}\mathrm{I}\gamma^2\omega^2 = \mathrm{m'c^2} - \frac{\mathrm{Iq}}{\mathrm{mc}}\gamma\boldsymbol{\omega}\cdot\mathbf{B}' \tag{23}$$

and using eq. (23) in eq. (20) we find

$$\boldsymbol{\pi} = \mathrm{m'c}\gamma\boldsymbol{\beta} + \frac{\mathrm{Iq}}{\mathrm{mc}}\{\boldsymbol{\nabla}_\mathbf{v}(\boldsymbol{\omega}\cdot\mathbf{B}') - \frac{1}{\mathrm{c}}\gamma^2(\boldsymbol{\omega}\cdot\mathbf{B}')\boldsymbol{\beta}\} \tag{24}$$

and

$$\gamma_{\boldsymbol{\pi}} = (1 + \frac{\pi^2}{\mathrm{m'^2c^2}})^{\frac{1}{2}} = \gamma + \frac{\mathrm{qI}}{\mathrm{mm'c^2}}\{\boldsymbol{\beta}\cdot\boldsymbol{\nabla}_\mathbf{v}(\boldsymbol{\omega}\cdot\mathbf{B}') - \frac{1}{\mathrm{c}}\gamma^2\beta^2\boldsymbol{\omega}\cdot\mathbf{B}'\} \tag{25}$$

where we have defined $\gamma_{\boldsymbol{\pi}}$ by eq. (25) and ignored non-linear field terms. Using eq. (23) in eq. (22) the Hamiltonian can be written as

$$H = m'c^2\gamma + \frac{Iq}{mc}\{\boldsymbol{\beta}\cdot\boldsymbol{\nabla}_{\mathbf{v}}(\boldsymbol{\omega}\cdot\mathbf{B}') - \frac{1}{c}\gamma^2\boldsymbol{\omega}\cdot\mathbf{B}'\} + q\Phi \tag{26}$$

Then using eq. (25) for γ, eq. (26) becomes

$$H = m'c^2\gamma_{\boldsymbol{\pi}} - \frac{Iq}{mc}\boldsymbol{\omega}\cdot\mathbf{B}' + q\Phi \tag{27}$$

where we have used the relation $\gamma^2 - 1 = \gamma^2\beta^2$. Then using eq. (21) for $\boldsymbol{\omega}$ and ignoring second order field terms eq. (27) becomes

$$H = m'c^2\gamma_{\boldsymbol{\pi}} - \frac{q}{mc\gamma}\,\mathbf{p}_{\boldsymbol{\omega}}\cdot\mathbf{B}' + q\Phi \tag{28}$$

Then since $\mathbf{B}'$ is linear in the fields if we only keep linear fields we can replace γ by $\gamma_{\boldsymbol{\pi}}$ and $\boldsymbol{\beta}$ by $\frac{q}{m'c\gamma_{\boldsymbol{\pi}}}\boldsymbol{\pi}$ in $\mathbf{B}'$. Thus eq. (28) for the Hamiltonian becomes

$$H = m'c^2\gamma_{\boldsymbol{\pi}} - \frac{q}{mc\gamma_{\boldsymbol{\pi}}}\,\mathbf{p}_{\boldsymbol{\omega}}\cdot\mathbf{B}'(\gamma_{\boldsymbol{\pi}}, \frac{q}{m'c\gamma_{\boldsymbol{\pi}}}\boldsymbol{\pi}, \mathbf{E}, \mathbf{B}) + q\Phi \tag{29}$$

Following the initial paper we can quantize the solution for H in eq. (29) by replacing m' by $m = m_0 + \frac{3\hbar^2}{8Ic^2}$, $\boldsymbol{\pi}$ by $\hat{\boldsymbol{\pi}}$ and $\mathbf{p}_{\boldsymbol{\omega}}$ by $\frac{1}{2}\hbar\boldsymbol{\sigma}$. Thus eq. (29) becomes the corresponding quantum equation

$$i\hbar\frac{\partial\Psi}{\partial t} = [mc^2\hat{\gamma}_{\boldsymbol{\pi}} - \frac{q}{2mc}\frac{1}{\hat{\gamma}_{\boldsymbol{\pi}}}\hbar\boldsymbol{\sigma}\cdot\mathbf{B}'(\hat{\gamma}_{\boldsymbol{\pi}}, \frac{1}{mc\hat{\gamma}_{\boldsymbol{\pi}}}\hat{\boldsymbol{\pi}}, \mathbf{E}, \mathbf{B}) + q\Phi]\Psi \tag{30}$$

This leads to the method used by Chiou and Chen [7] where they use a conjugate momentum based only on the translational equation and for the Hamiltonian combine the translational and spin terms leading to an expression

$$H = mc^2\gamma - \frac{q}{mc}\mathbf{s} \cdot (\frac{1}{\gamma}\mathbf{B}') + q\Phi \tag{31}$$

They then replace γ by $\gamma_{\boldsymbol{\pi}}$ and $\boldsymbol{\beta}$ by $\frac{1}{mc\gamma_{\boldsymbol{\pi}}}\boldsymbol{\pi}$ based on their definition of conjugate momentum and quantize the system by replacing $\boldsymbol{\pi}$ by $\hat{\boldsymbol{\pi}}$, $\gamma_{\boldsymbol{\pi}}$ by $\hat{\gamma}_{\boldsymbol{\pi}}$ and $\mathbf{s}$ by $\frac{1}{2}\hbar\boldsymbol{\sigma}$ leading to eq. (30). They also look at the case where g is not 2 and obtain the F-W transformation of the Dirac-Pauli equation [8]. It is interesting that the two different methods lead to the same results.

V.Quantization for large constant magnetic fields

It is interesting to note that eqs. (3-5) are valid for any constant magnetic field if the electric field is zero. That is we do not need to restrict the equations to only a weak **B** field where only linear terms are kept. In that case following the initial paper and using eq. (4) we can write

$$\gamma = \{1 + c^2\pi^2\{m_0c^2 + \frac{1}{2}I\gamma^2\omega^2\}^{-2}\}^{1/2} \tag{32}$$

and then from eq. (5) we have

$$\gamma\omega = \frac{1}{I}\,\mathbf{p}_{\boldsymbol{\omega}} + \frac{q}{mc}\mathbf{B} \tag{33}$$

Using these relations in eq. (3) for H we have

$$H = ((m_0c^2 + \frac{1}{2I}\,p_{\boldsymbol{\omega}}{}^2 - \frac{q}{mc}\,\mathbf{p}_{\boldsymbol{\omega}} \cdot \mathbf{B} + \frac{1}{2}I\frac{q^2}{m^2c^2}B^2)^2 + c^2\pi^2)^{1/2} + q\Phi \tag{34}$$

Upon quantization eq. (34) becomes

$$i\hbar\frac{\partial\Psi}{\partial t} = [((m_0c^2 + \frac{3}{8I}\hbar^2 - \frac{q}{2mc}\hbar\boldsymbol{\sigma} \cdot \mathbf{B} + \frac{1}{2}I\frac{q^2}{m^2c^2}B^2)^2 + c^2\,\hat{\boldsymbol{\pi}}^2)^{1/2} + q\Phi]\Psi \tag{35}$$

Now we have a term proportional to I, and if we set $m = m_0 + \frac{3\hbar^2}{8Ic^2}$ then since $\frac{3\hbar^2}{8c^2}$ is finite if we look at the limit of I going to zero for a point like particle then m_0 would need to take on a negative value to account for finite m value. If we take I to be small enough so that we can ignore the $\frac{1}{2}I\frac{q^2}{m^2c^2}B^2$ term then setting $m = m_0 + \frac{3\hbar^2}{8Ic^2}$, eq. (35) takes the form

$$i\hbar\frac{\partial\Psi}{\partial t} = [(m^2c^4 - q\hbar c\boldsymbol{\sigma} \cdot \mathbf{B} + \frac{1}{4}\frac{q^2\hbar^2}{m^2c^2}(\boldsymbol{\sigma} \cdot \mathbf{B})^2 + c^2\,\hat{\boldsymbol{\pi}}^2)^{1/2} + q\Phi]\Psi \tag{36}$$

This agrees with the Foldy-Wouthuysen transformation with a constant magnetic field if we ignore the $\frac{1}{4}\frac{q^2\hbar^2}{m^2c^2}(\boldsymbol{\sigma} \cdot \mathbf{B})^2$ term, see Case [9]. Thus eq. (36) gives a different equation for a constant large magnetic field then the Foldy-Wouthuysen transformation of the Dirac equation. It would be interesting if an experiment of the motion of the electron in a strong magnetic field was carried out to see whether the eq. (36), or the Foldy-Wouthuysen transformation of the Dirac equation gave better agreement with the experimental results. It could also be interesting if we allow I to be non-zero then it might be possible by experiment to put a limit on its value. If the experiments agreed with the Foldy-Wouthuysen equations then it would show that the quantization method used here is only valid for low intensity fields.

Conclusion

As in the initial paper we have only considered the spin 1/2 case and compared our results to the Foldy-Wouthuysen transformation of the Dirac equation. In principle higher order spins could be considered and compared to non-relativistic expansions of higher order relativistic spin equations. It would be interesting if this method could also be extended to non-constant fields.

The fact that we get agreement to all orders of the Foldy-Wouthuysen transformation as given by Chiou and Chen makes it more probable that the electron can be considered as actually spinning, or at least that its spin is as real as its position. From this calculation it would appear that the Dirac equation is equivalent to the canonical quantization of a classical spinning charge, at least for constant fields of low intensity.

For large magnetic fields that fact that we obtain an equation different from the Foldy-Wouthuysen transformation indicates that either this approach is only valid for low intensity fields, or that the Foldy-Wouthuysen transformation is only valid for low intensity fields. The only way to tell which is correct would be to conduct an experiment of the electron in a high intensity magnetic field.

References

1. French, J.: Relativistic Corrections to the Quantization of a Classical Spinning Particle with Constant Electric and Magnetic fields. Found. of Phy., 55:21 (2025)
2. Jackson, J. D.: Classical Electrodynamics, 2ed ed., Wiley, New York (1975)
3. Bopp, F., Haag, R.: Uber die Moglichkeit von Spinmodellen. Z. Naturforschg., 5a, 644-653 (1950)
4. Foldy, L. L., Wouthuysen, S. A.: On the Dirac Theory of Spin 1/2 Particles and Its Non-Relativistic Limit. Phy. Rev. 78, 29-36 (1950)
5. Sakurai, J. J.: Advanced Quantum Mechanics. Addison-Wesley, Reading, Mass. (1967)

6. Chen, T. W., Chiou, D. W.: Exact Foldy-Wouthuysen transformation of the Dirac-Pauli Hamiltonian in the weak-field limit by the method of direct perturbation theory. Phys. Rev. A, 94, 052116 (2016)
7. Chen, T. W., Chiou, D. W.: Correspondence between classical and Dirac-Pauli spinors in view of the Foldy-Wouthuysen transformation. Phys. Rev. A 89, 032111 (2014)
8. Pauli, W.: Relativistic Field Theories of Elementary Particles. Rev. of Modern Phys. 13, 203-232 (1941)
9. Case, K. M.: Some Generalizations of the Foldy-Wouthuysen Transformation. Phy. Rev. 95, 1323-1328 (1954).