THE LIGHT INTERACTION WITH LIGHT

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March 23, 2025

Abstract

The scattering of light by light is considered in case where the internal particles of this process have spin 0. The first calculation of this process was performed by Karplus et al. (1950). The pedagogical explanation of this proces was realized for instance by Akhiezer et al. (1965), or, by Beresteteskii et al. (1982). We use here the model with the Green function for the spin 0 particles. The article is written with the mathematical simplicity and the Schwinger pedagogical clarity.

1 Introduction

The scattering of light by light which is considered here is in no case interference of light, or, quantum interfrence because interference is a phenomenon in which two waves superimpose to form a resultant wave of greater or lesser amplitude. Interference usually refers to the interaction of waves that are correlated (coherent) with each other because they originate from the same source, or they have the same or nearly the same frequency. When two or more waves are incident on the same point, the total displacement at that point is equal to the vector sum of the displacements of the individual waves. If a crest of one wave meets a crest of another wave of the same frequency at the same point, then the magnitude of the displacement is the sum of the individual magnitudes. This is constructive interference and occurs when the phase difference between the waves is a multiple of 2π. Destructive interference occurs when the crest of one wave meets a trough of another wave. In this case, the magnitude of the displacements is equal to the difference in the individual magnitudes, and occurs when this difference is an odd multiple of π.

We here do not consider the light interference but interaction of light by light. The first calculation of this process was performed by Karplus et al. (1950). The pedagogical explanation of this proces was realized, for instance, by Akhiezer et al. (1965), or, by Beresteteskii et al. (1982). We use here the model with the spin 0. The phenomenon scattering of light by light is problem of quantum electrodynamics and we use here the

Schwinger quantum field theory methods (Schwinger, 1969; 1970) at the calculation of this physical process.

2 Scattering of light by light

The scattering of light by light is considered where the internal particles of this process have spin 0. The only difference with the situation with spin 1/2 particles is in using the Green function for the spin 0 particles. Instead of the Green function G_+^A for spin 1/2 particles we work with the function Δ_+^A. We follow the Schwinger monograph (Schwinger, 1973).

The proceeds involving the two spin 0 particle exchange and various numbers of photons is contained in the coupling term

$$W_{2..} = \frac{1}{2}\int (dx)(dx')K(x)\Delta_+^A(x,x')K(x'). \tag{1}$$

This formula generates the photon sources in terms of an effective two-particle field in the form

$$i\varphi(x)\varphi(x')|_{eff} = \Delta_+^A(x,x'). \tag{2}$$

The effective two-particle source follows from comparison

$$\langle 0_+|0_-\rangle = i\int (dx)\varphi(x)eq\frac{1}{i}\partial^\mu\varphi(x)\delta A_\mu(x) \tag{3}$$

with

$$\langle 0_+|0_-\rangle = \frac{1}{2}\left[i\int (dx)K(x)\varphi(x)\right]^2 =$$

$$-\frac{1}{2}\int (dx)(dx')\varphi(x)K(x)K(x')\varphi(x'), \tag{4}$$

which gives

$$iK(x)K(x')|_{eff} = eq\left(\delta A^\mu(x) + \delta A^\mu(x')\right)\frac{1}{i}\partial_\mu\delta(x-x'), \tag{5}$$

or,

$$iK(x)K(x')|_{eff} = eq\left(p\delta A + \delta Ap\right)(x)\delta(x-x'). \tag{6}$$

The vacuum amplitude of causal coupling between two photon sources symbolized by δA and A is analogical to the spin 1/2 situation, or,

$$\langle 0_+|0_-\rangle = i\delta W(A) = \frac{1}{2}\int (dx)(dx')\mathrm{tr}\left[iK(x)K(x')|_{eff}\ i\varphi(x)\varphi(x')|_{eff}\right] =$$

$$\frac{1}{2}\mathrm{Tr}\left[eq(p\delta A + \delta Ap)\Delta_+^A\right], \tag{7}$$

where Tr is the more compact notation in which the space-time coordinates join spin and charge indices as matrix labels.

2

For Δ_+^A we can use the well known integral equation with the formal solution (Schwinger, 1970)

$$\Delta_+^A = \left[1 - \Delta_+\left(eq(pA + Ap) - e^2A^2\right)\right]^{-1}\Delta_+ \quad =$$

$$\Delta_+\left[1 - \left(eq(pA + Ap) - e^2A^2\right)\Delta_+\right]^{-1}. \tag{8}$$

Now, instead of eq. (7) we write

$$i\delta W(A) = \frac{1}{2}\mathrm{Tr}\left[\left(eq(p\delta A + \delta Ap) - 2e^2\delta AA\right)\Delta_+^A\right]. \tag{9}$$

because δA and A are disjoint and their product vanishes in the causal arrangement for which eq. (7) is derived.

We then write eq. (9) as

$$i\delta W(A) \quad =$$

$$\frac{1}{2}\mathrm{Tr}\left[\left(eq(p\delta A + \delta Ap) - 2e^2\delta AA\right)\left(1 - \Delta_+(eq(pA + Ap) - e^2A^2)\right)^{-1}\Delta_+\right] \quad =$$

$$-\frac{1}{2}\delta\mathrm{Tr}\ln\left[1 - \left(eq(pA + Ap) - 2e^2A^2\right)\Delta_+\right]. \tag{10}$$

There are other representation of $W(A)$ which are useful in special situations. For instance, we get the specific form of $W(A)$ if we use the proper-time representation of Δ_+^A (Schwinger, 1973):

$$\Delta_+^A = \frac{1}{\Pi^2 + m^2 - i\varepsilon} = i\int_0^\infty ds e^{-is(\Pi^2 + m^2)}, \tag{11}$$

where $\varepsilon \to 0_+$ is implicit in the integral as a convergence factor $\exp(-\varepsilon s)$. After insertion of eq. (11) into eq. (9) we get:

$$\delta W(A) = -\frac{1}{2}i\int_0^\infty ds\mathrm{Tr}\left[\delta(\Pi)^2 e^{-is(\Pi^2 + m^2)}\right], \tag{12}$$

or,

$$W(A) = -\frac{i}{2}\int\frac{ds}{s}\mathrm{Tr}e^{-is(\Pi^2 + m^2)}. \tag{13}$$

Now, the goal is to evaluate the trace of of the corresponding term in (13) and to express $W(A)$ in terms of the effective Lagrange function, or in other words, to express it in the form

$$W(A) = \int(dx)\mathcal{L}(F), \tag{14}$$

where $\mathcal{L}(F)$ is the effective Lagrange function. First, let us try to find Tr of $\exp(-is(\Pi^2 + m^2))$.

Using the commutator

we get

$$[\Pi_\mu, \Pi_\nu] = ieqF_{\mu\nu}, \tag{15}$$

we get

$$\left[\Pi_\mu, \Pi^2\right] = 2ieqF_{\mu\nu}\Pi^\nu \tag{16}$$

and therefore

$$\Pi_\mu(s) = e^{is\Pi^2}\Pi_\mu e^{-is\Pi^2}, \tag{17}$$

which can be transcribed in the equivalent form

$$\frac{d\Pi_\mu(s)}{ds} = 2eqF_{\mu\nu}\Pi^\nu(s) \tag{18}$$

with the matrix solution

$$\Pi(s) = e^{2eqFs}\Pi = \Pi e^{-2eqFs}, \tag{19}$$

because of the antisymmetry of $F_{\mu\nu}$.

Now, let us introduce the following tensor

$$T_{\mu\nu} = \mathrm{Tr}'\left[\Pi_\mu\Pi_\nu e^{-is\Pi^2}\right] = \mathrm{Tr}'\left[\Pi_\mu e^{-is\Pi^2}\Pi_\nu(s)\right] =$$

$$\mathrm{Tr}'\left[\Pi_\nu(s)\Pi_\mu e^{-is\Pi^2}\right], \tag{20}$$

where Tr' does not refer to charge space.

The equivalent form of $T_{\mu\nu}$ is as follows:

$$T_{\mu\nu} = \mathrm{Tr}'\left[\Pi_\mu\Pi_\nu(s)e^{-is\Pi^2}\right] - \mathrm{Tr}'\left[[\Pi_\mu, \Pi_\nu(s)]\,e^{-is\Pi^2}\right], \tag{21}$$

where for the commutator that appeared it is

$$[\Pi_\mu, \Pi_\nu(s)] = \left[\Pi_\mu, \Pi^\lambda(s)\left(e^{-2eqFs}\right)_{\lambda\nu}\right] = ieq\left\{Fe^{-2eqFs}\right\}_{\mu\nu}. \tag{22}$$

Now, using eqs. (19) and (22) we can express eq. (21) in the matrix form as follows:

$$T = Te^{-2eqFs} - ieqFe^{-2eqFs}\mathrm{Tr}'e^{-is\Pi^2}, \tag{23}$$

or,

$$T\left(1 - e^{-2eqFs}\right) = -ieqFe^{-2eqFs}\mathrm{Tr}'e^{-is\Pi^2}, \tag{24}$$

or,

$$\mathrm{Tr}'\Pi\Pi e^{-is\Pi^2} = -ieqF\frac{F}{e^{2eqFs} - 1}\mathrm{Tr}'e^{-i\Pi^2}. \tag{25}$$

We use this result to get

$$i\frac{d}{ds}\mathrm{Tr}'e^{-is\Pi^2} = \mathrm{Tr}'\Pi^2 e^{-is\Pi^2} = -ieq\mathrm{Tr}'\frac{F}{e^{2eqFs} - 1}\mathrm{Tr}'e^{-is\Pi^2}, \tag{26}$$

4

which is the differential equation for $\text{Tr}' \exp(-is\Pi^2)$.

The solution of the derived differential equation (26) is possible express as follows (Schwinger, 1973):

$$\text{Tr}'e^{-is\Pi^2} = C \exp\left\{-\frac{1}{2}\text{Tr}' \ln\left(\frac{\sinh eqFs}{eqF}\right)\right\} =$$

$$\frac{C}{s^2}\left[\det \frac{eFs}{\sinh eFs}\right]^{1/2}, \tag{27}$$

where we have used identity $\text{Tr} \ln = \ln \det$, the dimensionality of space-time in the latter form and the fact that the sign of q is immaterial.

The constant C can be determined from considering the small s limit. This situation with the small s is dominated by large Π values and the non-commutativity of different Π components cases to be significant. Using four-dimensional forms of conventional quantum relations, we get

$$s \to 0: \quad \text{Tr}'e^{-is\Pi^2} = \int (dx)\langle x|e^{-isp^2}|x\rangle =$$

$$\int \frac{(dp)}{(2\pi)^4}(dx)e^{-isp^2}, \tag{28}$$

where

$$\int \frac{(dp)}{(2\pi)^4}e^{-isp^2} = \left(\int_{-\infty}^{\infty} \frac{(dp_1)}{(2\pi)}e^{-isp_1^2}\right)^3 \int_{-\infty}^{\infty} \frac{(dp_0)}{(2\pi)}e^{-isp_0^2}. \tag{29}$$

With regard to Laplace relation

$$\int_{-\infty}^{\infty} dp_1 e^{-isp_1^2} = \left(\frac{\pi}{is}\right)^{1/2}; \quad s > 0 \tag{30}$$

we have

$$\int \frac{(dp)}{(2\pi)^4} = \frac{1}{4\pi^2}\frac{1}{is^2}. \tag{31}$$

After insertion of eq. (31) into eq. (28) and then into eq. (27), we get

$$C = -\frac{1}{(4\pi)^2}i \int (dx). \tag{32}$$

Now, we can write for $W(A)$:

$$W(A) = -\int (dx)\frac{1}{(4\pi)^2}\int_0^{\infty} \frac{ds}{s^3}e^{-ism^2}\left[\det \frac{eFs}{\sinh eFs}\right]^{1/2} = \int (dx)\mathcal{L}(F), \tag{33}$$

where

$$\mathcal{L}_{spin\, 0}(F) = -\frac{1}{(4\pi)^2}\int_0^{\infty} \frac{ds}{s^3}e^{-ism^2}\left[\det \frac{eFs}{\sinh eFs}\right]^{1/2}. \tag{34}$$

The reality of the Lagrange function is obvious after deforming the path integration according to transformation $s \to is$. In other words

$$\mathcal{L}_{spin\,0}(F) = \frac{1}{(4\pi)^2} \int_0^\infty \frac{ds}{s^3} e^{-sm^2} \left\{ \left[\det \frac{eFs}{\sinh eFs} \right]^{1/2} - 1 - \frac{1}{3}(es)^2 \tilde{F} \right\}, \tag{35}$$

where we have used notation

$$\tilde{F} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} = \frac{1}{2}(\mathbf{E}^2 - \mathbf{H}^2) \tag{36}$$

to which we add

$$\tilde{G} = -\frac{1}{4} \, {}^*F^{\mu\nu} F_{\mu\nu} = \mathbf{E} \cdot \mathbf{H} \tag{37}$$

and

$$\tilde{H}_\pm = 2(\tilde{F} \pm i\tilde{G}) = (\mathbf{E} \pm i\mathbf{H})^2. \tag{38}$$

The general evaluation of the determinant can be realized by means of the eigenvalues of tensor F. It is convenient to introduce the selfdual tensors as follows:

$$F_\pm = F \pm i \, {}^*F, \quad {}^*F_\pm = \mp i F_\pm. \tag{39}$$

Considered as matrices, the two tensors commute, and the square of each is multiple of the unit matrix. It is possible to checked it by explicit use of the small number of independent components. The squares are

$$(F_\pm^2)_{\mu\nu} = g_{\mu\nu} \tilde{H}_\pm \tag{40}$$

where the coefficients $\tilde{H}_\pm$ are found by forming the trace. Using the equivalent relations

$$\frac{1}{2}(F^2 - {}^*F^2) = g_{\mu\nu} \tilde{F}, \quad ({}^*FF)_{\mu\nu} = g_{\mu\nu} \tilde{G}, \tag{41}$$

the eigenvalues appear in oppositely signet pairs, $\pm F', \pm F''$, where

$$F', F'' = \frac{1}{2} \left[\tilde{H}_+^{1/2} \pm \tilde{H}_-^{1/2} \right]. \tag{42}$$

Accordingly

$$\left[\det \frac{eFs}{\sin eFs} \right]^{1/2} = \frac{eF's}{\sin eF's} \frac{eF''s}{\sin eF''s} =$$

$$\frac{2(es)^2 i\tilde{G}}{\cos(es\tilde{H}_-^{1/2}) - \cos(es\tilde{H}_+^{1/2})} = \frac{(es)^2 \tilde{G}}{\text{Im} \cos(es\tilde{H}^{1/2})}, \tag{43}$$

where we have finally written just $\tilde{H}$ in place of $\tilde{H}_-$.

In such a way the spin 0 result Lagrangian is

$$\mathcal{L}_{spin\,0}(F) = \frac{1}{(4\pi)^2} \int_0^\infty \frac{ds}{s^3} e^{-sm^2} \left[\frac{(es)^2 \tilde{G}}{\text{Im} \cos(es\tilde{H}^{1/2})} - 1 - \frac{1}{3}(es)^2 \tilde{F} \right]. \tag{44}$$

The Lagrange function (35) can be expressed approximately in terms quartic in the fields. To obtain it we use the determinant expansion according to the algorithm:

$$\det(1 + A) = 1 + \operatorname{tr}A + \frac{1}{2}\left((\operatorname{tr}A)^2 - \operatorname{tr}A^2\right) + \ldots \tag{45}$$

This gives

$$\left[\det\frac{eFs}{\sin eFs}\right]^{1/2} = 1 + \frac{1}{3}(es)^2\tilde{F} + \frac{1}{90}(es)^4(7\tilde{F}^2 + \tilde{G}^2) + \ldots \tag{46}$$

and

$$\mathcal{L}_{04;spin\,0} = \frac{\alpha^2}{90}\frac{1}{m^4}(7\tilde{F}^2 + \tilde{G}^2) = \frac{\alpha^2}{90}\frac{1}{m^4}\left[\frac{7}{4}(\mathbf{E}^2 - \mathbf{H}^2)^2 + (\mathbf{E}\cdot\mathbf{H})^2\right]. \tag{47}$$

3 Discussion

In the preceding text we exhibited the space-time form of couplings that involve only the electromagnetic field, and we also used these forms directly for calculations, in the special circumstance of slowly varying fields. With more general situations, however, it is usually preferable to consider an appropriate causal arrangement and then perform the space-time extrapolation. We are recognizing now that source theory is flexible; it is not committed to any special calculational method and is free to choose the most convenient one. Indeed, it is the interplay and synthesis of various calculational devices, each adapted to specific circumstances, that constitutes the general source theory computational method

The arrangement is, two photons collide to create a charged particle pair, and then the two photons emitted in the subsequent annihilation of the particles are detected. For spin 0 particles, we can use the analogy with the preceding methods with some extensions (Schwinger, 1973) and considering the strong magnetic field situation (Schwinger, 1989).

References

Akhiezer, A. I. and Berestetskii, V. B., *Quantum Electrodynamics* (Wiley, New York, 1965)

Berestetskii, V. B., Lifschitz, E. M., and Pitaevskii, L. P. *Quantum Electrodynamics*,(2nd ed. Oxford, England: Pergamon Press, p. 596, 1982).

Karplus, R. and Neuman, M. (1950). Non-Linear Interactions between Electromagnetic Fields, Phys. Rev. **80**, No. 3, pp. 380-385.

Schwinger, J. *Particles and Sources*,
(Gordon and Breach, Science Publishers, New york, London, Paris, 1969).

Schwinger, J. *it Particles, Sources and Fields I.*,
(Addison-Wesley Publishing Company, Reading, Mass. 1970).

Schwinger, J. *Particles, Sources and Fields II.*, (Addison-Wesley Publishing Company, Reading, Mass. 1973).

Schwinger, J. *Particles, Sources, and Fields, III.*, (AddisonWesley, Reading, Mass. 1989).