Another Approximation for Prime Counting Function

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Abstract

This paper presents a new approximation for the prime counting function, denoted $\pi(x)$, which counts the number of prime numbers less than x. While the classical approximation $P(x) \approx \frac{x}{\ln x}$ (Prime Number Theorem) is widely used, we propose a more accurate approximation for relatively small numbers: $p(x) \approx \frac{x}{\log(x^2)}$, where log is the base-10 logarithm. We explore the connection between this approximation and the Riemann zeta function $\zeta(s)$, demonstrating how the terms of the zeta function can be interpreted geometrically as measures of regular angles. Additionally, we derive several lemmas related to the sum of reciprocals of the counting function and provide a geometric interpretation of the results. This work contributes to the ongoing study of prime number distribution and its relationship with analytic number theory.

1 Approximation of the Prime Counting Function

The prime counting function, denoted $\pi(x)$, is defined as the number of prime numbers less than x.

A classical approximation is given by:

$$P(x) \approx \frac{x}{\ln x} \quad (PrimeNumberTheorem - PNT).$$

We propose a new approximation:

$$p(x) \approx \frac{x}{\log(x^2)},$$

where log denotes the base-10 logarithm. This approximation is more accurate than $\frac{x}{\ln x}$ for relatively small numbers (experimentally below 2500).

For the sequence $x_n = 10^n$, we have:

$$p(10^n) = \frac{10^n}{\log(10^{2n})} = \frac{10^n}{2n}.$$

Lemma 1 $p(10^n) = \frac{10^n}{2n}$, *for* $n \in N^*$.

2 Connection with the Zeta Function

Consider the sequence $x_n = 10^{n^s}$.

From Lemma 1:

$$p(10^{n^s}) = \frac{10^{n^s}}{2n^s}.$$

By dividing by 10^{n^s}:

$$\frac{p(10^{n^s})}{10^{n^s}} = \frac{1}{2n^s}.$$

Summing from $n = 1$ to infinity:

$$\sum_{n=1}^{+\infty} \frac{p(10^{n^s})}{10^{n^s}} = \sum_{n=1}^{+\infty} \frac{1}{2n^s} = \frac{1}{2}\zeta(s).$$

Lemma 2 $\sum_{n=1}^{+\infty} \frac{p(10^{n^s})}{10^{n^s}} = \frac{1}{2}\zeta(s)$ *for* $s > 1$.

3 Sum of Reciprocals of the Counting Function

From Lemma 1:

$$p(10^n) = \frac{10^n}{2n}.$$

Thus:

$$\frac{1}{p(10^n)} = \frac{2n}{10^n}.$$

Summing:

$$\sum_{n=1}^{+\infty} \frac{2n}{10^n} = 2 \cdot \frac{10^{-1}}{(1 - 10^{-1})^2} \approx 0.2469135802.$$

Lemma 3 $\sum_{n=1}^{+\infty} \frac{1}{p(10^n)} \approx 0.2469135802.$

4 Geometric Interpretation

In an orthogonal coordinate system, the expression $\frac{p(x)}{x}$ represents the tangent of the angle α formed by the curve $p(x)$ with the abscissa axis:

$$\tan(\alpha) = \frac{p(x)}{x}.$$

When α is small, we approximate:

$$\tan(\alpha) \approx \alpha.$$

Applying this property to the sequence $x_n = 10^{n^s}$:

$$\tan(x_n) = \frac{p(x_n)}{x_n} = \frac{1}{2n^s}.$$

When x_n is sufficiently small:

$$x_n \approx \frac{1}{2n^s}.$$

Summing:

$$\sum_{n=1}^{+\infty} x_n = \sum_{n=1}^{+\infty} \frac{1}{2n^s} = \frac{1}{2}\zeta(s).$$

Lemma 4 *For the sequence $x_n = 10^n$ and the approximate counting function $p(x) \approx \frac{x}{\log(x^2)}$, we have:*

$$\sum_{n=1}^{+\infty} x_n = \frac{1}{2}\zeta(s) \quad for s > 1.$$

Thus, we can interpret the terms of the zeta function as measures of regular angles.

References

[1] Davenport, H. (1980). *Multiplicative Number Theory* (3rd ed.). Springer-Verlag. Reference work on analytic number theory, including in-depth analyses of the functions $\zeta(s)$ and $\pi(x)$.

[2] Edwards, H. M. (1974). *Riemann's Zeta Function.* Academic Press. Detailed study of Riemann's Zeta Function, its history, and applications in number theory.