New proof of dark numbers by means of the thinned out harmonic series

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Abstract: It is shown by the intersection of the complements of all Kempner series belonging to definable natural numbers that not all natural numbers can be defined. We call them dark.

1. Dark Numbers

Not all numbers can be chosen, expressed, and communicated as individuals such that the receiver knows what the sender has meant. We call those numbers dark numbers. Much evidence concerning dark numbers has been collected and discussed [1]. But in the following we will present the shortest proof of their existence. Of course the facilities to express numbers depend on the environment and the power of the applied system. But this proof shows that, independent of the system, infinitely many natural numbers will remain dark forever.

A simple example is provided already by the denominators of the harmonic series $(1/n)_{n\in\mathbb{N}}$. Whatever attempts are made to express denominators m as large as possible, the sum from 1/1 to $1/m$ is finite while the remaining part of the series diverges.

2. Kempner Series

The harmonic series diverges. But as Kempner [2] has shown in 1914, deleting all terms containing the digit 9 turns it into a converging series, the Kempner series, here abbreviated as $K(9)$. That means that the complement $C(9)$ of removed terms

$$C(9) = \frac{1}{9} + \frac{1}{19} + \frac{1}{29} + ... + \frac{1}{89} + \frac{1}{90} + \frac{1}{91} + \frac{1}{92} + ... + \frac{1}{98} + \frac{1}{99} + \frac{1}{109} + ... \ ,$$

all containing the digit 9, carries the divergence alone. *All* other terms can be removed. Same is true when all terms containing the digit 8 are removed. That means that the complement $C(8)$

$$C(8) = \frac{1}{8} + \frac{1}{18} + \frac{1}{28} + ... + \frac{1}{78} + \frac{1}{80} + \frac{1}{81} + \frac{1}{82} + ... + \frac{1}{88} + \frac{1}{89} + \frac{1}{98} + \frac{1}{108} + ...$$

of the Kempner series $K(8)$ carries the divergence alone. Since here those terms containing the digits 9 without digits 8 belong to the converging series $K(8)$ we can conclude that the divergence is caused by the intersection only, i.e., by all terms containing the digits 8 and 9 *simultaneously*:

$$C(8) \cap C(9) = \frac{1}{89} + \frac{1}{98} + \frac{1}{189} + \frac{1}{198} + ... \ .$$

3. Proof

But not all terms containing 8 and 9 are needed. We can continue and remove all terms containing 1, 2, 3, 4, 5, 6, 7, or 0 in the denominator without changing this result because the ten corresponding Kempner series $K(0)$, $K(1)$, ..., $K(9)$ are converging and their complements $C(0)$, $C(1)$, ..., $C(9)$ are diverging. But only the intersection of all complements carries the divergence. That means that only the terms containing *all* the digits 0 to 9 simultaneously constitute the diverging series.

But that is not the end! We can remove any natural number k, like 2025, and the remaining Kempner series will converge. For proof use base 2026 where 2025 is a digit. This extends to every defined number, i.e., every number k that can be defined, chosen, and communicated such that the receiver knows what the sender has meant. When the terms containing k are deleted, then the remaining series converges.

4. Result

The diverging part of the harmonic series is constituted only by the intersection of all complements $C(k)$ of Kempner series $K(k)$ of defined natural numbers k, i.e., by all the terms containing the digit sequences of *all* defined natural numbers. No defined natural number exists which must be left out. Terms which, although being larger than every defined number, do not contain all defined digit sequences, for instance not Ramsey's number, belong to converging Kempner series and not to the diverging series of the intersection of all complements. All infinitely many terms containing not the digit 1 or not the digit sequence 2025 or not the digit sequence of Ramsey's number can be deleted without violating the divergence.

All Kempner series $K(k)$ of defined natural numbers k split off in this way are converging and therefore the sum of their always finite sums is finite too, although it can be very large [3]. The divergence however remains. It is carried only by terms which are dark and greater than all digit sequences of all defined numbers – we can even say greater than all digit sequences of all defin*able* numbers because, when larger numbers will be defined in future, they will behave in the same way. It is impossible to choose a natural number such that the intersection of the complements of all Kempner series of larger numbers is finite.

This is a proof of the huge set of undefinable or dark numbers.

Literature

[1] W. Mückenheim: "Evidence for Dark Numbers", Eliva Press, Chisinau, 2024, pp. 1-36.
[2] A. J. Kempner: "A Curious Convergent Series", American Mathematical Monthly 21 (2), 1914, pp. 48–50.
[3] T. Schmelzer, R. Baillie: "Summing a Curious, Slowly Convergent Series", American Mathematical Monthly 115 (6), 2008, pp. 525–540.