# SURFACE AREA OF THE MÖBIUS STRIP

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ABSTRACT. The (half) area of the surface of the Möbius strip is the expected product of the length of the circular spine times the width of the sweep line times a positive correction factor. The manuscript writes down this correction factor as a Taylor series of the ratio of width over circle radius; the factor approaches one if that ratio approaches zero. [vixra:2503.0103]

## 1. Incentive

The first Guldin rule (Pappus' theorem) provides a formula for the surface generated by revolving a planar curve with known center of mass around a circle [1, (8.72)]. The naïve expectation is that the Möbius strip has an area equal to the product of length of a circular center line by the width which originates from the idea to take a rectangular piece of paper and to attach its short sides after a twist. This manuscript corrects this hypothesis and evaluates a correction factor for this product of width and length.

## 2. The Straight Screw

The simplest toy model of a twisted surface is screw with a straight long axis along the horizontal $+y$ coordinate plus a straight sweep line of length $w$ attached with its center to the $y$-axis. The sweep line is rotated at an angle $\theta$ around the $y$-axis, $\theta$, measured in radians, increasing linearly with $y$:

$$\theta(y) = \frac{2\pi y k}{D}. \tag{1}$$

The screw is essentially constructed as if a ship with a 2-blade propeller (mathematically just a line of length $w$—with apologies to real engineers) would travel along the $y$-axis from 0 to $D$ and all points of the stick were recorded.

The parameter $k$ represents how often the propeller revolves while the ship moves forward by $D$ units. Figure 1 is an example for $k = 3$, $D = 5$, $w = 0.6$.

A point on the twisted ribbon of the screw has Cartesian coordinates

$$\vec{r} = \begin{pmatrix} t\cos\frac{2\pi y k}{D} \\ y \\ t\sin\frac{2\pi y k}{D} \end{pmatrix} \tag{2}$$

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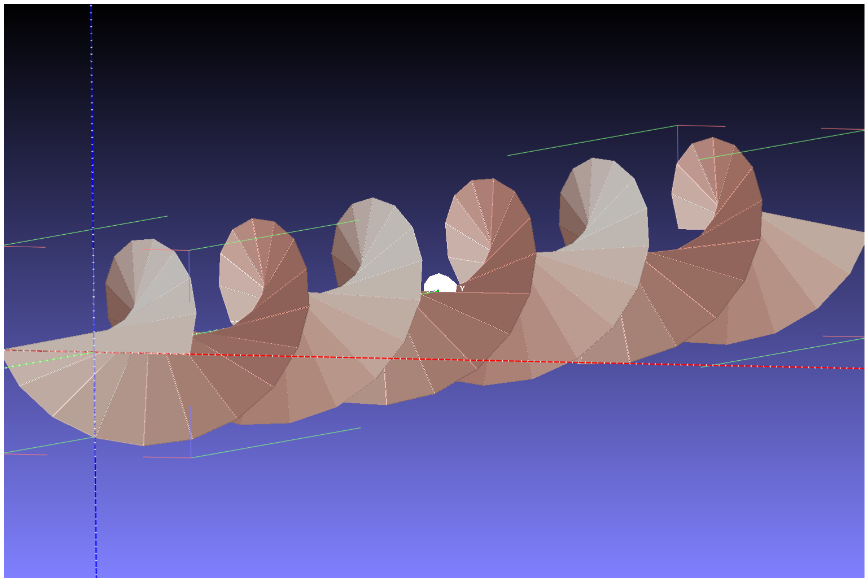

FIGURE 1. A straight screw with direction line along $+y$. $x$-axis along red, $y$-axis along green, $z$-axis along blue line.

with two parameters $-w/2 \le t \le w/2$, $y \ge 0$. The sine of $\theta$ multiplied by the parameter $t$ shows how much a point of the propeller is above the horizontal $x-y$-plane. Tangential directions are defined by the partial derivatives,

$$\frac{\partial \vec{r}}{\partial y} \equiv \vec{r}_y = \begin{pmatrix} -\frac{2\pi k}{D} t \sin \frac{2\pi y k}{D} \\ 1 \\ \frac{2\pi k}{D} t \cos \frac{2\pi y k}{D} \end{pmatrix}; \tag{3}$$

$$\frac{\partial \vec{r}}{\partial t} \equiv \vec{r}_t = \begin{pmatrix} \cos \frac{2\pi y k}{D} \\ 0 \\ \sin \frac{2\pi y k}{D} \end{pmatrix}. \tag{4}$$

These are orthogonal with Gaussian parameter $F$:

$$F = \vec{r}_y \cdot \vec{r}_t = 0. \tag{5}$$

The cross product defines a direction of the surface normal (with a basically arbitrary choice of the sign):

$$\vec{r}_t \times \vec{r}_y = \begin{pmatrix} -\sin \frac{2\pi k y}{D} \\ -\frac{2\pi k}{D} t \\ \cos \frac{2\pi k y}{D} \end{pmatrix}. \tag{6}$$

Its length and Gaussian parameter $G$ are

$$|\vec{r}_t \times \vec{r}_y| = \sqrt{1 + (2t\pi k/D)^2} = \sqrt{G}. \tag{7}$$

For a line on the surface that keeps a constant distance $t$ to the $y$-axis, the line element $ds$ is the length of (3), which happens to be the same as $\sqrt{G}$ for this geometry. The length of such a line is

$$S_k(t) = \int_0^D ds = \int_0^D dy \sqrt{1+(2\pi t k/D)^2} = \sqrt{D^2+(2\pi t k)^2}. \tag{8}$$

So a point on the propeller at distance $t \neq 0$ from the axis moves by a distance $S_k(t) > D$ while the ship moves by the distance $D$. The (one-sided) surface area of the screw is [6, 2.262.1]

$$A_k = \int_{-w/2}^{w/2} dt S_k(t) = \frac{w}{2}\sqrt{D^2+(\pi k w)^2} + \frac{D^2}{2\pi k} \operatorname{arsinh} \frac{w\pi k}{D} \tag{9}$$
$$\approx Dw \left[1 + \frac{\pi^2 k^2}{6}(w/D)^2 - \frac{\pi^4 k^4}{40}(w/D)^4 + \frac{\pi^6 k^6}{112}(w/D)^6 - \frac{5\pi^8 k^8}{1152}(w/D)^8 + \cdots\right].$$

The limit where the propeller does not rotate: $A_0 = wD$.

## 3. Möbius Strips

### 3.1. **Mathematical Model, Coordinates.**

We look at a Möbius strip of directing circle radius $R$ located in the $x-y-$plane with a propeller of width $w$ staying with its middle at the directing/guide curve. In the picture of the ship of Section 2, its steering wheel is now fixed at an angle that lets the ship run in a circle of radius $R$. A point on the directing curve has the Cartesian coordinates

$$\begin{pmatrix} R\cos\lambda \\ R\sin\lambda \\ 0 \end{pmatrix} \tag{10}$$

parameterized by an azimuthal angle $0 \le \lambda \le 2\pi$. The tangent line to the circle points into the orthogonal direction

$$\begin{pmatrix} -\sin\lambda \\ \cos\lambda \\ 0 \end{pmatrix}. \tag{11}$$

A point on the strip at a distance $t$ to the directing curve has a torsion angle $\theta$ relative to the $x-y$-plane, such that its $z$-coordinate is $t\sin\theta$ in the range $-w/2 \le t \le w/2$. This leaves the factor $t\cos\theta$ for the $x$ and $y$ coordinates. Since the propeller is obtained by rotation around the tangent (11), its direction must be orthogonal to that, so dispersion of the $t\cos\theta$ factor gives a propeller vector of

$$\begin{pmatrix} t\cos\theta\cos\lambda \\ t\cos\theta\sin\lambda \\ t\sin\theta \end{pmatrix}. \tag{12}$$

Attaching it to the circle (10) gives the Cartesian coordinates of a point on the strip parameterized by $\lambda$ and $t$:

$$\vec{r}(\lambda, t) = \begin{pmatrix} R\cos\lambda \\ R\sin\lambda \\ 0 \end{pmatrix} + \begin{pmatrix} t\cos\theta\cos\lambda \\ t\cos\theta\sin\lambda \\ t\sin\theta \end{pmatrix} = \begin{pmatrix} (R+t\cos\theta)\cos\lambda \\ (R+t\cos\theta)\sin\lambda \\ t\sin\theta \end{pmatrix}. \tag{13}$$

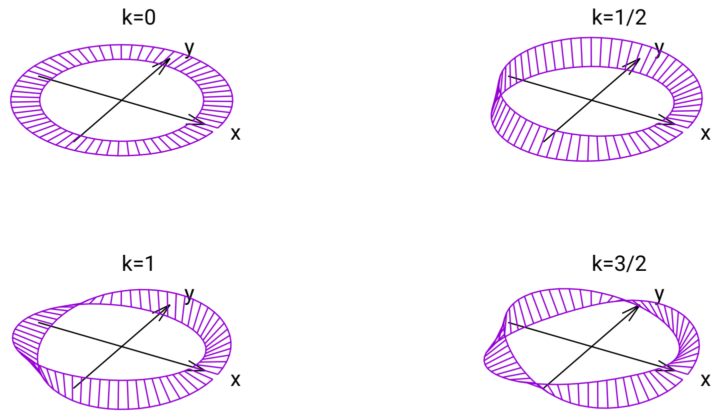


FIGURE 2. Möbius ribbons for twist numbers $k = 0 \ldots 3/2$.

The principle of the definition now lets the torsion angle $\theta$ increase linearly with $\lambda$ such that a point of constant $t$ initially at

$$\vec{r}(0, w/2) = \begin{pmatrix} R + w/2 \\ 0 \\ 0 \end{pmatrix} \tag{14}$$

ends up at

$$\vec{r}(2\pi, w/2) = \begin{pmatrix} R - w/2 \\ 0 \\ 0 \end{pmatrix} \tag{15}$$

after one $\lambda$-rotation through the circle. This is achieved by setting

$$\theta = \lambda/2. \tag{16}$$

Continuous surfaces with larger numbers of twists as in Figure 2 can be constructed by selecting other positive half-integer $k = 0, 1/2, 1, 3/2, \ldots$:

$$\theta = k\lambda. \tag{17}$$

Insertion into (13) defines a family of Möbius strips [3, 8]:

$$\vec{r} = \begin{pmatrix} [R + t\cos(k\lambda)]\cos\lambda \\ [R + t\cos(k\lambda)]\sin\lambda \\ t\sin(k\lambda) \end{pmatrix}. \tag{18}$$

3.2. **Gaussian Parameters.** Two tangential directions on the surface are constructed as the partial derivatives:

$$\frac{\partial\vec{r}}{\partial t} \equiv \vec{r}_t = \begin{pmatrix} \cos(k\lambda)\cos\lambda \\ \cos(k\lambda)\sin\lambda \\ \sin(k\lambda) \end{pmatrix}; \quad E = |\vec{r}_t| = 1; \tag{19}$$

$$\frac{\partial \vec{r}}{\partial \lambda} \equiv \vec{r}_\lambda = \begin{pmatrix} -tk\sin(k\lambda)\cos\lambda - R\sin\lambda - t\sin\lambda\cos(k\lambda) \\ -tk\sin(k\lambda)\sin\lambda + R\cos\lambda + t\cos\lambda\cos(k\lambda) \\ tk\cos(k\lambda) \end{pmatrix}. \tag{20}$$

These are orthogonal:

$$F = \vec{r}_\lambda \cdot \vec{r}_t = 0. \tag{21}$$

The cross product (direction of the surface normal, not of unit length) is

$$\vec{r}_t \times \vec{r}_\lambda = \begin{pmatrix} tk\sin\lambda - R\sin(k\lambda)\cos\lambda - t\cos\lambda\sin(k\lambda)\cos(k\lambda) \\ -tk\cos\lambda - R\sin(k)\sin\lambda - t\sin\lambda\sin(k\lambda)\cos(k\lambda) \\ [R + t\cos(k\lambda)]\cos(k\lambda) \end{pmatrix}. \tag{22}$$

The length of the cross product is

$$|\vec{r}_t \times \vec{r}_\lambda| = |\vec{r}_\lambda| = \sqrt{G} = \sqrt{[R + t\cos(k\lambda)]^2 + (tk)^2}. \tag{23}$$

3.3. **Edge length.** The derivatives of the position as a function of the $\lambda$ parameter in (20) define the line segment

$$\begin{aligned} &\sqrt{(\partial r_x/d\lambda)^2 + (\partial r_y/d\lambda)^2 + (\partial r_z/d\lambda)^2} \\ &\qquad = \sqrt{[R + t\cos(k\lambda)]^2 + (tk)^2} = R\sqrt{[1 + \frac{t}{R}\cos(k\lambda)]^2 + (tk/R)^2} \end{aligned} \tag{24}$$

for curves that run at constant distance $t$ to the circular backbone.

The length $S_k$ of such a line along the ribbon (up to the usual debatable factor of 2 if $k$ is non-integer) is

$$S_k(t) = \int_{\lambda=0}^{2\pi} R\sqrt{[1 + \frac{t}{R}\cos(k\lambda)]^2 + (tk/R)^2}d\lambda. \tag{25}$$

The $\lambda$-integral leads to Elliptic integrals which we shall avoid here (App. C).

The Taylor expansion of the kernel in powers of small $t/R$ is

$$\begin{aligned} \sqrt{[1 + \frac{t}{R}\cos(k\lambda)]^2 + (\frac{tk}{R})^2} &= 1 + \cos(k\lambda)\frac{t}{R} + \frac{k^2}{2}(\frac{t}{R})^2 - \frac{k^2}{2}\cos(k\lambda)(\frac{t}{R})^3 \\ &+ \frac{k^2}{8}[2\cos(k\lambda) - k][2\cos(k\lambda) + k](\frac{t}{R})^4 - \frac{k^2}{8}\cos(k\lambda)[4\cos^2(k\lambda) - 3k^2](\frac{t}{R})^5 \\ &+ \frac{k^2}{16}[8\cos^4(k\lambda) - 12k^2\cos^2(k\lambda) + k^4](\frac{t}{R})^6 \\ &- \frac{k^2}{16}\cos(k\lambda)[8\cos^4(k\lambda) - 20k^2\cos^2(k\lambda) + 5k^4](\frac{t}{R})^7 + \cdots \end{aligned} \tag{26}$$

Term-by-term integration of the power series over $\int_0^{2\pi} d\lambda$ yields

$$S_0 = 2\pi R\big[1 + \frac{t}{R}\big]; \tag{27}$$

$$S_{1/2} = 2\pi R\big[1 + \frac{1}{8}(\frac{t}{R})^2 + \frac{7}{128}(\frac{t}{R})^4 + \frac{25}{1024}(\frac{t}{R})^6 + \frac{75}{32768}(\frac{t}{R})^8 - \frac{2793}{262144}(\frac{t}{R})^{10} + \cdots\big]; \tag{28}$$

$$S_1 = 2\pi R\big[1 + \frac{1}{2}(\frac{t}{R})^2 + \frac{1}{8}(\frac{t}{R})^4 - \frac{1}{8}(\frac{t}{R})^6 - \frac{15}{128}(\frac{t}{R})^8 + \frac{21}{128}(\frac{t}{R})^{10} + \cdots\big]; \tag{29}$$

(30)
$$S_{3/2}=2\pi R\big[1+\frac{9}{8}(\frac{t}{R})^2-\frac{9}{128}(\frac{t}{R})^4-\frac{783}{1024}(\frac{t}{R})^6+\frac{37035}{32768}(\frac{t}{R})^8+\frac{267183}{262144}(\frac{t}{R})^{10}+\cdots\big];$$

$$S_2=2\pi R\big[1+2(\frac{t}{R})^2-(\frac{t}{R})^4-\frac{5}{4}(\frac{t}{R})^6+\frac{75}{8}(\frac{t}{R})^8-\frac{1533}{64}(\frac{t}{R})^{10}+\cdots\big]; \tag{31}$$

(32)
$$S_{5/2}=2\pi R\big[1+\frac{25}{8}(\frac{t}{R})^2-\frac{425}{128}(\frac{t}{R})^4+\frac{1825}{1024}(\frac{t}{R})^6+\frac{928875}{32768}(\frac{t}{R})^8-\frac{56366625}{262144}(\frac{t}{R})^{10}+\cdots\big].$$

The fact that these $S_k(t)$ are larger than $2\pi R$ for $k>0$ is no surprise, because these are basically lengths measured along the cutting edges of screws for screws that do not have straight but circular axes of length $2\pi R$.

The length of the rim of the stripe is obtained by inserting $t=\pm w/2$. (The sign obviously matters only for the planar case $S_0$.)

3.4. **Area.** The area is [10, (8.19)][1, (3.498b)]

$$\begin{aligned} A_k &= \iint \sqrt{EG-F^2}d\lambda dt = \iint |\vec{r}_t\times\vec{r}_\lambda| d\lambda dt = \int_{-w/2}^{w/2} dt S_k(t) \\ &= \int_0^{2\pi} d\lambda \int_{-w/2}^{w/2} dt\sqrt{[R+t\cos(k\lambda)]^2+(tk)^2} \\ &= R\int_0^{2\pi} d\lambda \int_{-w/2}^{w/2} dt\sqrt{[1+\frac{t}{R}\cos(k\lambda)]^2+(\frac{tk}{R})^2} \\ &= \frac{wR}{2}\int_0^{2\pi} d\lambda\int_{-1}^{1} dx\sqrt{[1+\frac{xw}{2R}\cos(k\lambda)]^2+(\frac{xwk}{2R})^2}. \end{aligned} \tag{33}$$

**Remark 1.** *Optionally one could multiply this by 2 to cover the 'back-side' area, i.e., to sweep this in the range $0\le\lambda\le 4\pi$.*

**Remark 2.** *The $t$-integral may be executed* [6, 2.262.1,2.262.2]

$$\begin{aligned} &\int_{-w/2}^{w/2} dt\sqrt{R^2+2Rt\cos(k\lambda)+t^2\cos^2(k\lambda)+t^2k^2} \\ &= \frac{(\cos^2(k\lambda)+k^2)t+R\cos(k\lambda)}{2(\cos^2(k\lambda)+k^2)}\sqrt{[R+t\cos(k\lambda)]^2+t^2k^2} \\ &+ \frac{R^2k^2}{2(\cos^2(k\lambda)+8k^2)^{3/2}}\operatorname{arsinh}\frac{(\cos^2(k\lambda)+k^2)t+R\cos(k\lambda)}{kR}\Big|_{t=-w/2}^{w/2} \end{aligned} \tag{34}$$

*but since this still leaves a pending $\lambda$-integration, this analysis is not continued from there.*

The case $k=0$ is the trivial planar hollow circle, difference of areas of circles with radii $R\pm w/2$, with $A_0=\pi[(R+w/2)^2-(R-w/2)^2]=2\pi wR$.

The further strategy is to utilize the power series expansion of $S_k(t)$ assuming $w$ is small, where the integration over the powers of $t$ is elementary.

**Definition 1.**

$$\hat{w}=w/R \tag{35}$$

*is the unitless ratio of the strip width by the radius of the backbone circle.*

The terms with odd powers of $t$ disappear while integrating because the $t-$limits are symmetric. $A_k$ is $2\pi Rw$ multiplied by an even function of $\hat{w}$.

Insertion of the $S_k$-series into (33) and term-by-term integration of (28) over $-w/2 \le t \le w/w$ yields

$$A_{1/2} = 2\pi wR\big[1 + \frac{1}{96}\hat{w}^2 + \frac{7}{10240}\hat{w}^4 + \frac{25}{458752}\hat{w}^6 + \frac{25}{25165824}\hat{w}^8 - \frac{2793}{2952790016}\hat{w}^{10} - \frac{53277}{223338299392}\hat{w}^{12} + \cdots\big] \tag{36}$$

There is an apparent discrepancy between this formula and the usual manual construction of a Möbius model which attaches two ends of a rectangular stripe of dimension $2\pi R \times w$ after bending/twisting. In fact the paper model does not keep the center line of the rectangular stripe on a planar circle; its 2-dimensional surface is even more complex than the mathematical model (13) [11, 7, 12].

No new aspect arises in the analysis if twist numbers $k \ge 1$ are computed—besides the fact that for integer $k$ the computed area is indeed the area of only one of two sides.

$$A_1 = 2\pi wR\left[1 + \frac{1}{24}\hat{w}^2 + \frac{1}{640}\hat{w}^4 - \frac{1}{3584}\hat{w}^6 - \frac{5}{98304}\hat{w}^8 + \frac{21}{1441792}\hat{w}^{10} + \frac{105}{27262976}\hat{w}^{12} + \cdots\right]; \tag{37}$$

$$A_{3/2} = 2\pi wR\big[1 + \frac{3}{32}\hat{w}^2 - \frac{9}{10240}\hat{w}^4 - \frac{783}{458752}\hat{w}^6 + \frac{4115}{8388608}\hat{w}^8 + \frac{267183}{2952790016}\hat{w}^{10} - \frac{28573965}{223338299392}\hat{w}^{12} + \cdots\big]; \tag{38}$$

$$A_2 = 2\pi wR\left[1 + \frac{1}{6}\hat{w}^2 - \frac{1}{80}\hat{w}^4 - \frac{5}{1792}\hat{w}^6 + \frac{25}{6144}\hat{w}^8 - \frac{1533}{720896}\hat{w}^{10} - \frac{399}{6815744}\hat{w}^{12} + \cdots\right]; \tag{39}$$

$$A_{5/2} = 2\pi wR\big[1 + \frac{25}{96}\hat{w}^2 - \frac{85}{2048}\hat{w}^4 + \frac{1825}{458752}\hat{w}^6 + \frac{309625}{25165824}\hat{w}^8 - \frac{56366625}{2952790016}\hat{w}^{10} + \frac{3746147475}{223338299392}\hat{w}^{12} + \cdots\big]. \tag{40}$$

## 4. Summary

The (quasi one-sided) surface area of the Möbius strip of width $w$ swept along a planar directing circle of radius $R$ is given by (36), where (35) denotes the unitless ratio of the two main parameters. The main message is the same as for the surface of a straight screw: the area is *not* the product of the length of the sweep line by the length of the directing curve if the area is not flat.

## Appendix A. Acknowlegements

Figures 1 and 3 are `meshlab` renderings of STL files.

## Appendix B. Embedding of Möbius strips

The parameters of the second quadratic fundamental normal form are listed here [1, (3.503c)][10, (8.26)]. The normal vector of the plane is via (22)

$$\vec{n} = \frac{1}{\sqrt{G}}\vec{r}_t \times \vec{r}_\lambda. \tag{41}$$

The products of partial derivatives are

$$L = -\vec{n}_\lambda \cdot \vec{r}_\lambda = \frac{1}{\sqrt{G}}\sin(k\lambda)\left[[R+t\cos(k\lambda)]^2 + 2t^2k^2\right]; \tag{42}$$

(the factor in the square brackets is *not* the same as the discriminant of the root in (23); it is not $G$.)

$$N = -\vec{n}_t \cdot \vec{r}_t = 0; \tag{43}$$

$$M = -(\vec{n}_\lambda \cdot \vec{r}_t + \vec{n}_t \cdot \vec{r}_\lambda)/2 = \frac{kR}{\sqrt{G}}. \tag{44}$$

## Appendix C. Elliptic Integrals

The integrals (33) along the spine of the strip have the shape

$$\int_0^{2\pi} d\lambda \sqrt{[1+\frac{xw}{2R}\cos(k\lambda)]^2 + (\frac{xwk}{2R})^2} \tag{45}$$
$$= 2\int_0^{\pi} d\phi \sqrt{(1+\frac{xw}{2R}\cos\phi)^2 + (\frac{xwk}{2R})^2}.$$

The substitution $\cos\phi = \xi$ rephrases this as an elliptic integral

$$\ldots = 2\int_{-1}^{1} d\xi \sqrt{(1+\frac{xw}{2R}\xi)^2 + (\frac{xwk}{2R})^2}\frac{1}{\sqrt{1-\xi^2}} \tag{46}$$
$$= \frac{2xw}{2R}\int_{-1}^{1} d\xi \sqrt{(\frac{2R}{xw}+\xi)^2 + k^2}\frac{1}{\sqrt{1-\xi^2}}$$
$$= \frac{xw}{R}\int_{-1}^{1} d\xi \sqrt{(ik+\frac{2R}{xw}+\xi)(-ik+\frac{2R}{xw}+\xi)}\frac{1}{\sqrt{(1-\xi)(1+\xi)}}$$
$$= x\hat{w}\sum_{m=0}^{2}\beta_m \int_{-1}^{1} d\xi \frac{\xi^m}{\sqrt{(ik+\frac{2R}{xw}+\xi)(-ik+\frac{2R}{xw}+\xi)(1-\xi)(\xi-(-1))}}$$

with imaginary unit $i$. According to the Byrd-Friedmann formula [2, 259.00] the case $m = 0$ is an elliptic integral of the first kind; The other two cases $m = 1, 2$ are linear combinations of elliptic integrals of the first, second and third kind [2, 259.03].

## Appendix D. Outlook: Möbius Bodies

To enter a discussion of Möbius solids with finite volume, one may define sweep curves that are closed (so they have a definitive area) and move these along a directing circle.

The simplest example is a sweep curve that is a circle that stays with its center on the rim of the directing circle to create the torus with a well-known volume [1,

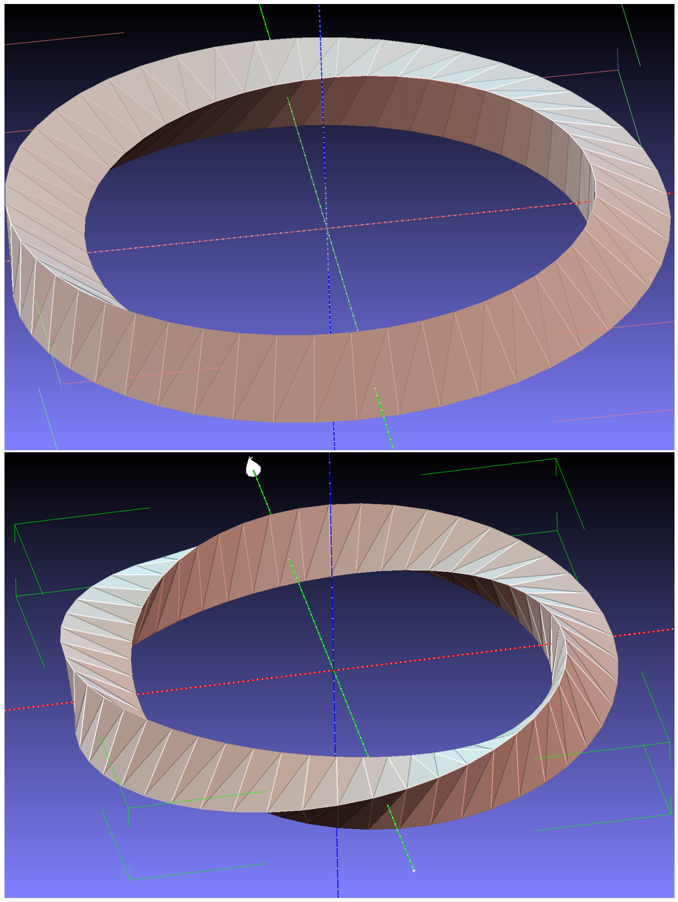

FIGURE 3. Möbius bodies defined by sweeping a regular triangle of edge length 0.3 along a circle of radius 1. $x$-axis in red, $y$-axis in green, $z$-axis in blue.

3.3.4]. Another category is a curve defined by a regular polygon of $n$ sides which is twisted by $\theta$-angles which are multiples of $2\pi/n$ while its center of mass stays on the directing circle through one rotation $\lambda = 0 \ldots 2\pi$. For a triangular cross-section (created by warming up and vandalizing one of the famous swiss chocolate bars), one may for example twist the triangle by 120° in one revolution or 360° in one revolution as illustrated in the upper and lower Figure 3.

(i) In the 120° or 240° twisted case, the body has only one surface: after traveling once or twice around the $z$-axis and staying on the smooth surface without passing over an edge, one ends up at different places on the surface (the analog of the Möbius strip with half-integer $k$-numbers); to return to the same place one must travel by multiples of 3 around the $z$-axis. (ii) In the 360°-twisted case, traveling once around the $z$-axis returns to the same place and the surface contains 3 individual areas separated by the 3 ridges.

The computation of the volume of the twisted triangle may be founded on the Gauss integral theorem that computes the triple integral of the divergences of a

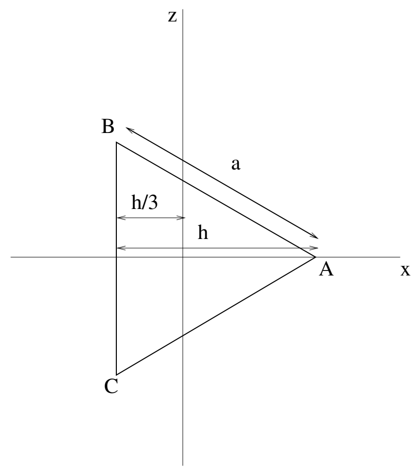


FIGURE 4. A regular triangle with side length $a$ with center at the origin as in (47).

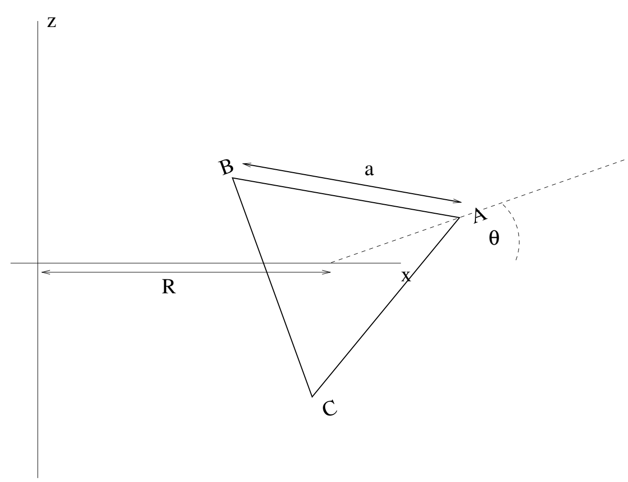


FIGURE 5. A regular triangle with side length $a$ in the rot-shifted version of (52).

vector field (which is $\vec{r}$) by the surface integral of the dot product of the vector field with the surface normal [1, 13.118b][6, 10.711]. The rest of the section is an explicit execution of this algebra.

A regular triangle of side length $a$ has altitude $h = \sqrt{3}a/2$, area $A_{\triangleright} = \sqrt{3}a^2/4$, a reference position with center at the origin of coordinates, and Cartesian coordinates of 3 corners $\vec{A}$, $\vec{B}$ and $\vec{C}$

$$\begin{pmatrix} a/\sqrt{3} \\ 0 \\ 0 \end{pmatrix}; \quad \begin{pmatrix} -a/(2\sqrt{3}) \\ 0 \\ a/2 \end{pmatrix}; \quad \begin{pmatrix} -a/(2\sqrt{3}) \\ 0 \\ -a/2 \end{pmatrix}; \tag{47}$$

aligned as in Fig. 4. The triangle is rotated by the angle $\theta$ proportional to the azimuth $\lambda$ such that the surface is continuous after one revolution $\lambda = 0 \cdots 2\pi$,

$$\theta = k\lambda, \tag{48}$$

where the parameter $k = 0,\ 1/3,\ 2/3,\ 3/3,\ 4/3 \ldots$ is a multiple of $1/3$.

The triangle rotated counter-clock-wise by $\theta$ around the $y$-axis pushes the three $\vec{A}$, $\vec{B}$, $\vec{C}$ vertices to coordinates

(49)
$$\begin{pmatrix} a/\sqrt{3}\cos\theta \\ 0 \\ a/\sqrt{3}\sin\theta \end{pmatrix}; \quad \begin{pmatrix} -a/(2\sqrt{3})\cos\theta - a/2\sin\theta \\ 0 \\ -a/(2\sqrt{3})\sin\theta + a/2\cos\theta \end{pmatrix}; \quad \begin{pmatrix} -a/(2\sqrt{3})\cos\theta + a/2\sin\theta \\ 0 \\ -a/(2\sqrt{3})\sin\theta - a/2\cos\theta \end{pmatrix}$$

as in Fig. 5. These triangles are shifted to place their centers on the circle by adding $R$ to the $x$-coordinates, then rotated by $\lambda$ around the $z$-axis, so the edges of the Möbius body are at

$$\vec{A} = \begin{pmatrix} (R + a\sqrt{3}\cos\theta)\cos\lambda \\ (R + a/\sqrt{3}\cos\theta)\sin\lambda \\ a/\sqrt{3}\sin\theta \end{pmatrix}; \tag{50}$$

$$\vec{B} = \begin{pmatrix} (R - \sqrt{3}a/6\cos\theta - a/2\sin\theta)\cos\lambda \\ (R - \sqrt{3}a/6\cos\theta - a/2\sin\theta)\sin\lambda \\ -\sqrt{3}a/6\sin\theta + a/2\cos\theta \end{pmatrix}; \tag{51}$$

$$\vec{C} = \begin{pmatrix} (R - \sqrt{3}a/6\cos\theta + a/2\sin\theta)\cos\lambda \\ (R - \sqrt{3}a/6\cos\theta + a/2\sin\theta)\sin\lambda \\ -a/(2\sqrt{3})\sin\theta - a/2\cos\theta \end{pmatrix}. \tag{52}$$

Points on the Möbius surface have coordinates

$$\vec{r}_{AB} = \vec{A} + \zeta(\vec{B} - \vec{A}); \quad \vec{r}_{BC} = \vec{B} + \zeta(\vec{C} - \vec{B}); \quad \vec{r}_{CA} = \vec{C} + \zeta(\vec{A} - \vec{C}); \tag{53}$$

by linear interpolation with $0 \le \zeta \le 1$. By computing the partial derivatives $\partial\vec{r}_{AB}/\partial\lambda$, $\partial\vec{r}_{AB}/\partial\zeta$, their cross product

$$\vec{n}_{AB} \equiv \partial\vec{r}_{AB}/\partial\lambda \times \partial\vec{r}_{AB}/\partial\zeta, \tag{54}$$

and the same for the edges $BC$ and $CA$, we obtain the surface normals $\vec{n}_{AB}$, $\vec{n}_{BC}$, $\vec{n}_{CA}$ as functions of $\lambda$ and $\zeta$. The dot products are

$$\begin{aligned} \vec{r}_{AB}\cdot\vec{n}_{AB} = -\frac{1}{12}a\big[&-6R^2\cos(k\lambda) - 2a^2\cos(k\lambda) + 3aR\sin(2k\lambda) \\ &+ \sqrt{3}a^2\zeta\sin(k\lambda) - 3aR\zeta\sin(2k\lambda) + 3^{3/2}aR\zeta\cos(2k\lambda) + 3a^2\zeta\cos(k\lambda) \\ &- 3^{3/2}aR - \sqrt{3}aR\cos(2k\lambda) + 6\sqrt{3}R^2\sin(2k\lambda)\big]; \end{aligned} \tag{55}$$

$$\begin{aligned} \vec{r}_{BC}\cdot\vec{n}_{BC} = \frac{1}{12}a\big[&2\sqrt{3}a^2\zeta\sin(k\lambda) - \sqrt{3}a^2\sin(k\lambda) + \sqrt{3}aR\cos(2k\lambda) \\ &+ 3^{3/2}aR - 12R^2\cos(k\lambda) - a^2\cos(k\lambda) + 3aR\sin(2k\lambda) \\ &- 6aR\zeta\sin(2k\lambda)\big]; \end{aligned} \tag{56}$$

$$\vec{r}_{CA}\cdot\vec{n}_{CA} = \frac{1}{12}a\big[-\sqrt{3}a^2\zeta\sin(k\lambda)+\sqrt{3}a^2\sin(k\lambda) - 2\sqrt{3}aR\cos(2k\lambda)+3^{3/2}aR+3a^2\zeta\cos(k\lambda)+6R^2\cos(k\lambda) - a^2\cos(k\lambda)+3^{3/2}aR\zeta\cos(2k\lambda)+6\sqrt{3}R^2\sin(k\lambda)+3aR\zeta\sin(2k\lambda)\big]. \tag{57}$$

The intermediate sum of all three sub-surfaces is divided by 3 (because $\nabla\cdot\vec{r}=3$ within the Gauss theorem)

$$\frac{1}{3}(\vec{r}_{AB}\cdot\vec{n}_{AB}+\vec{r}_{BC}\cdot\vec{n}_{BC}+\vec{r}_{CA}\cdot\vec{n}_{CA}) = \frac{\sqrt{3}}{4}a^2R \tag{58}$$

and does not depend on $k$, $\zeta$ or $\lambda$. The volume is

$$V_k = \int_0^1 d\zeta\int_0^{2\pi} d\lambda \frac{\sqrt{3}}{4}a^2R = \frac{\sqrt{3}\pi}{2}a^2R, \tag{59}$$

and this is exactly the value $2\pi RA_\triangleright$ that is also derived by the second Guldin rule [1, (8.75)].

**Remark 3.** *This is a feature of our construction of keeping the center of the twisted sweep curve on the directing circle [5, 9, 4]: The volume integral could be evaluated as* $\int_0^{2\pi}d\lambda\int_0^\infty xdx\int dz\Theta(x,z,\lambda)$ *in Cylinder Coordinates where* $\Theta$ *is a function which is one inside the solid, zero outside and* $x$ *is the associated Jacobian. In the inner double integral* $\iint xdxdz$ *the substitution* $x = x'+R$ *moves the cross section such that its center of mass is at* $x'=0$*, and splits into two integrals,* $\int_0^\infty x'dx'\int dz\Theta(x',z,\lambda)+R\int_0^\infty dx'\int dz\Theta(x',z,\lambda)$*. The first of these with kernel* $x\Theta(x',z,\lambda)$ *is the first moment in the horizontal direction, which is the* $x'$*-coordinate of the center of mass, which is zero because we defined the body by keeping the center at a distance* $R$*. The second of these is* $R$ *multiplied by the ordinary integral which is just the area. So the inner two integrals evaluate to* $RA_\triangleright$ *independent of* $\theta$*; the outer integral eventually contributes a factor* $2\pi$*. In particular, regularity of the twisted sweep curve is not required to keep the second Guldin rule for the volume intact.*

The surface of these twisted triangles is

$$S_k = \int_0^1 d\zeta\int_0^{2\pi} d\lambda[|\vec{n}_{AB}|+|\vec{n}_{BC}|+|\vec{n}_{CA}|] \tag{60}$$

for which we only give the results of leading terms of the power series expansions of small $a/R$:

$$S_0 = 6\pi aR; \tag{61}$$

$$S_{1/3} = 6\pi aR+\frac{\pi}{36R}a^3+\frac{\pi}{320R^3}a^5+\frac{2273\pi}{4354560R^5}a^7+\cdots; \tag{62}$$

$$S_{2/3} = 6\pi aR+\frac{\pi}{9R}a^3+\frac{\pi}{90R^3}a^5+\frac{29\pi}{27216R^5}a^7+\cdots; \tag{63}$$

$$S_1 = 6\pi aR+\frac{\pi}{4R}a^3+\frac{19\pi}{960R^3}a^5-\frac{11\pi}{10752R^5}a^7+\cdots. \tag{64}$$

The non-twisted case $S_0 = 6\pi aR$ is the first Guldin's rule applied to a regular triangle with edge length $a$ and angle $\theta = 0$ in Fig. 5, where the centers of the sides $AB$ and $CA$ have distances $R+h/6$ to the $z$-axis and the side $BA$ has distance $R-h/3$ to the $z$-axis, in total $2(R+h/6)+R-h/3 = 3R$.

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