A story of a function

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Abstract

We dedicate this text to a study of a function with interesting properties: besides being, by construction, well suited for a certain type of approximations, the function happens to be lacunary, i.e. without analytic continuation outside the complex unit disk. It however satisfies a functional equation which can be (with some restriction) solved everywhere except the origin. Unfortunately, this solution cannot be understood as its natural continuation beyond the disk.

MSC: 30B10, 30B40

We dedicate this text to a study of a function with interesting properties. It was "discovered" in a specific approximation approach when asking for a simple coefficient formula. Briefly: in the derivative-matching expansions of the form [1]

$$f(z) \approx f(0) + \sum_{n=1}^{\infty} a_n g(z^n), \quad g(0) = 0, \quad g'(0) \neq 0, \tag{1}$$

the coefficients a_n are determined as $a_n = \sum_{d|n} f_d h_{\frac{n}{d}}$, where $d|n$ means "d divides n". Numbers $\{f_d\}$ represent the power-expansion coefficients of f and the sequence $\{h_j\}$ is the inverse of $\{g_i\}$ with respect to the Dirichlet convolution, $\{g_i\}$ being the power expansion coefficients of g. From various possibilities for g one may search for a one giving a simple expression for a_n. Avoiding the trivial choice $h = \mathbf{1} = \{\delta_{1,i}\}_i$, $g(z) = z$ (i.e. the Taylor series), one can consider $h_n|_{n=1,2,3,\ldots} = 1, 1, 0, 0, 0, 0, \ldots$ leading to $(k, n \in \mathbb{N}_0)$

$$g_n = \begin{cases} (-1)^k & \text{for } n = 2^k \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad a_n = \begin{cases} f_n & \text{if } n \text{ is odd} \\ f_n + f_{\frac{n}{2}} & \text{if } n \text{ is even} \end{cases}. \tag{2}$$

One then has

$$\text{yo}\,(z) \equiv g\,(z) = \sum_{n=0}^{\infty} (-1)^n z^{2^n}. \tag{3}$$

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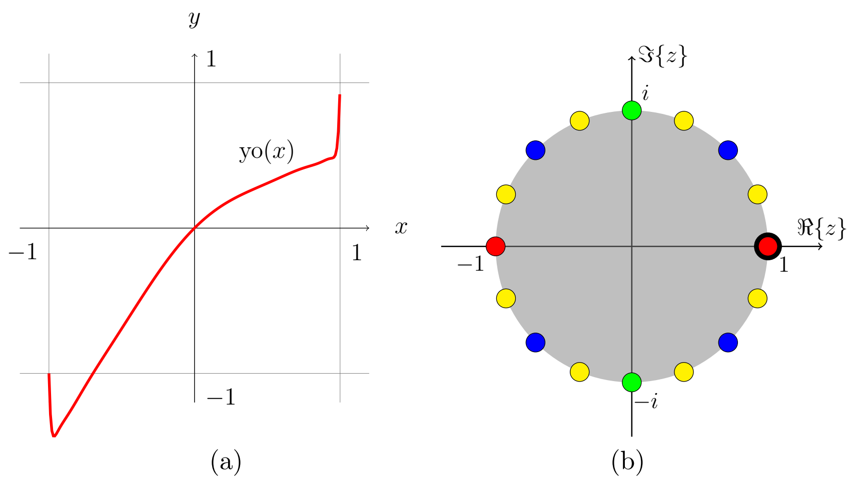


Figure 1:
a) Graph of $\sum_{n=0}^{6}(-1)^n x^{2^n} \approx \mathrm{yo}(x)$.
b) Repeated square roots of the unity (black annulus): the first generation is represented by the red circles, then second, third and fourth generations by green, blue and yellow circles respectively. The set of all consecutive square roots of unity is dense on a unit circle.

1 Unit circle

The definition (3) converges for all complex $|z| < 1$. On the real segment $\langle 0, 1)$ it is a series with alternating signs whose terms decrease in norm and tend to zero. With the domain of convergence being always a disk around the expansion point, (3) necessarily converges for all $|z| < 1$. For $z \geq 1$ the series diverge because the individuals summands do not tend to zero. And yet, the behavior at the unit circle is interesting: The graph of the function on the real axis (Figure 1-a) seems to tend to a finite value when approaching one.

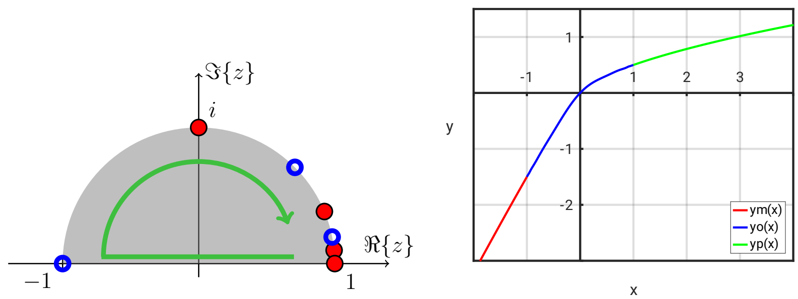
Before a more detailed investigation of the behavior on the unit circle let us state one observation: yo is a lacunary function [2] with the unit circle being its natural boundary. Indeed, the spacing between the powers of z with nonzero coefficients grows rapidly enough for the function to fulfill the conditions of the *Ostrowski–Hadamard gap theorem* (OHGP) implying that the function cannot be analytically continued to and beyond the unit circle.

To investigate the behavior on the unit circle one can notice that the definition (3) implies the validity of the functional equation

$$\mathrm{yo}(z) = z - \mathrm{yo}(z^2), \tag{4}$$

with two self-consistency points $z = 0$ and $z = 1$. The first gives $\mathrm{yo}(0) = 0$ (in accordance with (3)). The second point leads to

$$\mathrm{yo}(1) = 1 - \mathrm{yo}(1) \Rightarrow \mathrm{yo}(1) = 1/2,$$



(a) (b)

Figure 2: a) We study the continuity of yo at the unit circle by successively taking the square root starting at $z = 1$. Choosing always the root in the upper-half of the complex plane (all circles), we evaluate yo at every second point (full circles). Doing a round trip, the points come back and approach unity.
b) Functions ym(red), yo (blue) and yp (green).

which extends the definition of the function. Next one proceeds recursively

$$\text{yo}(-1) = -1 - \text{yo}(1) = -3/2, \qquad \text{yo}(i) = i - \text{yo}(-1) = 3/2 + i,$$
$$\text{yo}(-i) = -i - \text{yo}(-1) = 3/2 - i, \qquad \text{yo}(e^{i\pi/4}) = e^{i\pi/4} - \text{yo}(i) = \dots$$

In this way one computes function values at arbitrary square roots of the unity $z = e^{im\pi/2^n}$, $m, n \in N_0$, which represent a dense subset of the unit circle ("square-root points" or SR points), see Figure 1-b. This extension is of course only formal and non-analytic.

The OHGP tells us that in no point of the unit circle yo is analytic. We can prove this independently thanks to the functional equation (4) and get some more insight. For this purpose it is convenient to apply the functional equation twice so as to get a value difference in two points $\text{yo}(z^4) = z^2 - z + \text{yo}(z)$. We choose a specific path when taking square roots, starting from 1 we go to -1 and then we remain on the upper complex semi-plane, always taking as the square root the point with half-argument, see Figure 2-a. We have after N steps

$$\text{yo}\,(1) = \left[e^{i\pi} - e^{i\pi/2}\right] + \text{yo}(e^{i\pi/2}) = \left[e^{i\pi} - e^{i\pi/2}\right] + \left[e^{i\pi/4} - e^{i\pi/8}\right] + \text{yo}(e^{i\pi/8})$$
$$= \dots = \left\{\sum_{n=0}^{N}\left[e^{i\pi/2^{2n}} - e^{i\pi/2^{2n+1}}\right]\right\} + \text{yo}(e^{i\pi/2^{2N+1}}).$$

So

$$\text{yo}\,(1) - \text{yo}(e^{i\pi/2^{2N+1}}) = \sum_{n=0}^{N}\left[e^{i\pi/2^{2n}} - e^{i\pi/2^{2n+1}}\right] \equiv \sum_{n=0}^{N} \Delta_n \equiv D_N. \qquad (5)$$

The function arguments on the left-hand side of (5) become arbitrary close when N rises, $\lim_{N\to\infty} e^{i\pi/2^{2N+1}} = 1$. But what about the function values? Let us compute the real part of Δ_n

$$\Re(\Delta_n) = \Re\left(e^{i\pi/2^{2n}}\right) - \Re\left(e^{i\pi/2^{2n+1}}\right) = \cos\frac{\pi}{2^{2n}} - \cos\frac{\pi}{2^{2n+1}}.$$

For all $n \geq 0$ we have $0 < \frac{\pi}{2^{2n}} \leq \pi$, which is an interval where the cosine is monotonic and decreasing. Therefore

$$\cos\frac{\pi}{2^{2n+1}} > \cos\frac{\pi}{2^{2n}} \Rightarrow \Re(\Delta_n) < 0.$$

If each term in (5) has a strictly negative real part then their sum is strictly negative too, implying that $D \equiv D_\infty$ is nonzero. Using the root test we can in addition prove that $\sum_{n=0}^{N} \Delta_n$ converges (absolutely)

$$\begin{aligned}\limsup_{n\to\infty} \sqrt[n]{|\Delta_n|} &= \limsup_{n\to\infty} \sqrt[n]{\left|e^{i\pi/2^{2n+1}}\left(e^{i\pi/2^{2n+1}} - 1\right)\right|} \\ &= \lim_{n\to\infty} \sqrt[n]{\left|e^{i\pi/2^{2n+1}}\right|}\sqrt[n]{\left|\left(e^{i\pi/2^{2n+1}} - 1\right)\right|} = 1 \times \frac{1}{4} < 1,\end{aligned}$$

where the second root tends to $1/4$ since for small positive angles φ one has $\left|e^{i\varphi} - 1\right| \sim \varphi$. The approximate value is $D \approx -(1.2321 + 0.5458i)$. We can conclude that yo is not continuous at one because in its arbitrary small neighborhood the value of yo is significantly different $\text{yo}(1) = \text{yo}\left(e^{i\varepsilon_1}\right) + D$, where $\varepsilon_1 > 0$ is infinitesimal but D is not. This discontinuity is then propagated to all SR points. Indeed, let $e^{i(\pi+\varepsilon_2)}$ be the square root of $e^{i\varepsilon_1}$ which lies in the proximity of -1 in the lower complex half-plane, $\varepsilon_2 = \varepsilon_1/2$. We have

$$\begin{aligned}\text{yo}\left[e^{i(\pi+\varepsilon_2)}\right] &= e^{i(\pi+\varepsilon_2)} - \text{yo}\left(e^{i\varepsilon_1}\right) = -1 - \text{yo}(1) + D \\ &= -1 - [1 - \text{yo}(-1)] + D = -2 + \text{yo}(-1) + D,\end{aligned}$$

where we were manipulating terms intuitively by setting $e^{i(\pi+\varepsilon_2)} = -1$. The result implies $\text{yo}(-1) - \text{yo}\left[e^{i(\pi+\varepsilon_2)}\right] = 2 - D$, i.e. the discontinuity is also situated at $z = -1$. The procedure can be repeated recursively from $z = -1$ to all daughter SR points, meaning that the function is discontinuous on a dense subset. Strictly speaking, for this to be rigorously proven, one should show that the appearing numerical terms (such as $2 - D$) do not exactly cancel in some points. Nevertheless one can do two (and more) loops in the upper half plane: Stopping in an infinitesimal neighborhood of $z = 1$ after the first round trip, i.e. at $z = e^{i\varepsilon_1}$, we start the second round trip by taking this time the square root in the proximity of -1, i.e. $z = e^{i(\pi+\varepsilon_2)} = \sqrt{e^{i\varepsilon_1}}$. Then the second round trip continues in the upper half-plane, as show previously in Figure 2-a. It is evident that the result after approaching $z = 1$ for the second time will be just getting one additional D, $\text{yo}(1) = \text{yo}\left(e^{i\varepsilon_3}\right) + 2D$ for some appropriate and infinitesimal ε_3. This discontinuity will also propagate to all remaining SR points, producing an additional difference of D. If, in the previous case, the numerical factors somehow cancel at some points, then they do not cancel this time, i.e. the function is on the unit circle indeed discontinuous everywhere. Moreover, by adding more round trips one adds more Ds, meaning that the norm of the function takes arbitrarily large values in any nonzero neighborhood of each point of the unit circle.

Another interesting topic is the analysis of the derivatives. It suffices to differentiate (4):

$$\mathrm{yo}'(z) = 1 - 2z\mathrm{yo}'(z^2), \quad \mathrm{yo}''(z) = -2\mathrm{yo}'(z^2) - 4z^2\mathrm{yo}''(z^2), \quad \mathrm{yo}'''(z) = \ldots . \tag{6}$$

Again, there are two self-consistency points $z = 0, 1$. Although seemingly trivial, the case $z = 0$ allows us to determine derivatives at zero meaning we can reconstruct the power series (3), i.e. it tells us that the functional equation (4) is an equivalent way of defining yo inside the circle. At $z = 1$ we solve (6) and get

$$\mathrm{yo}'(1) + 2\,\mathrm{yo}'(1) = 1 \Rightarrow \mathrm{yo}'(1) = \frac{1}{3}, \quad \mathrm{yo}''(1) = -\frac{2}{15}, \quad \mathrm{yo}'''(1) = \frac{8}{45}, \ \ldots \tag{7}$$

Arbitrary high derivatives can be determined in this way. Thanks to (6) the derivatives can be propagated to an arbitrary SR point. Thus, for example,

$$\mathrm{yo}'(-1) = 1 + 2\,\mathrm{yo}'(1) = 1 + 2 \times (1/3) = 5/3. \tag{8}$$

One should be aware that these derivatives are purely formal, they do not exist in the sense in which the derivative is defined.

2 Beyond the unit circle

Quite some body of literature focuses on the question of non-analytic continuation of functions beyond their natural boundary, see e.g. [3]. Looking at the latter (remark 6.9.12) and on other sources, it seems that the lacunary functions are among the most resistant in this regard, no "natural" way of extending them seems to be commonly accepted. In what follows we make an attempt to extend yo using (4), yet we basically confirm the previous statement.

Anticipating our results, we are going to use for the attempted extension a different function label, namely $\mathrm{yp}\,(z)$. We start by considering the real axis outside the unit circle (using x instead z). For $\mathrm{yp}\,(x)$ we assume only that it satisfies the functional equation (4) and the value at one: $\mathrm{yp}\,(1) = 1/2$. Again, we apply the functional equation (4) twice so as to make appear the difference of yp values at two points $\mathrm{yp}\,(z) = z - z^2 + \mathrm{yp}\left(z^4\right)$. We start in the proximity of one, $x = 1 + h$, and we will be interested in the limit $h \to 0^+$. We have

$$\begin{aligned}\mathrm{yp}(1+h) &= \left[(1+h) - (1+h)^2\right] + \mathrm{yp}\left[(1+h)^4\right] \\ &= \left[(1+h) - (1+h)^2\right] + \left[(1+h)^4 - (1+h)^8\right] + \mathrm{yp}\left[(1+h)^{16}\right] \\ &= \ldots = \left\{\sum_{n=0}^{N}\left[(1+h)^{2^{2n}} - (1+h)^{2^{2n+1}}\right]\right\} + \mathrm{yp}\left[(1+h)^{2^{2N+2}}\right].\end{aligned} \tag{9}$$

Here two limits appear, $h \to 0^+$ and $N \to \infty$. To proceed in a consistent way we relate them choosing $h = \alpha/2^{2N+2}$ where $\alpha > 0$ is a real parameter. Applying the limit we have

$$\mathrm{yp}(1) = \lim_{N\to\infty} \left\{ \sum_{n=0}^{N} \left[(1 + \frac{\alpha}{2^{2N+2}})^{2^{2n}} - (1 + \frac{\alpha}{2^{2N+2}})^{2^{2n+1}} \right] \right\} + \mathrm{yp}\left(e^{\alpha}\right),$$

or, equivalently

$$\begin{aligned} \mathrm{yp}\left(e^{\alpha}\right) &= \frac{1}{2} + \lim_{N\to\infty} \sum_{n=0}^{N} \left[(1 + \frac{\alpha}{2^{2N+2}})^{2^{2n+1}} - (1 + \frac{\alpha}{2^{2N+2}})^{2^{2n}} \right] \\ &\equiv \frac{1}{2} + \lim_{N\to\infty} \sum_{n=0}^{N} \omega_{N,n} = \frac{1}{2} + \lim_{N\to\infty} \Theta_N\left(\alpha\right). \end{aligned} \tag{10}$$

Our main concern now is to show that the sum converges to a finite value. The proof of the latter is important to our text but technical, and thus we present it in Appendix. There we also derive the boundaries ($x \geq 1$)

$$\mathcal{B}\left[\ln\left(x\right)/2\right] \leq \mathrm{yp}\left(x\right) \leq \mathcal{B}\left[2\ln\left(x\right)\right], \quad \mathcal{B}\left(q\right) = \frac{1}{\ln\left(4\right)} \left[\mathrm{Ei}\left(q\right) - \mathrm{Ei}\left(\frac{q}{2}\right) \right], \tag{11}$$

where Ei is the exponential integral. The limiting functions are shown in Figure 3-a. Furthermore, the Appendix contains the derivation of an elegant alternative expression [4] for an efficient computation of yp

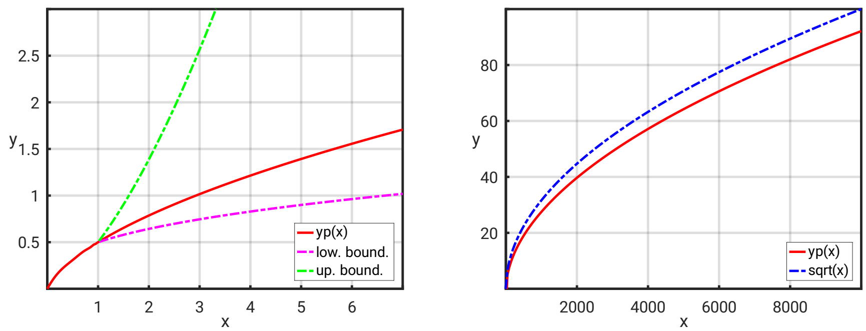
$$\mathrm{yp}\left(e^{\alpha}\right) = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{\alpha^k}{k!\left(2^k+1\right)} = \frac{1}{2} + \Theta\left(\alpha\right), \ \alpha \in \mathbb{R}. \tag{12}$$

There are interesting remarks to make:

- Function $\mathrm{yp}\left(e^{\alpha}\right) = 1/2 + \Theta\left(\alpha\right)$ is, as function of α, an entire function. Indeed, for $\alpha \geq 0$ all terms in the sum (12) are positive, thus partial sums rise. Yet, they are bounded by the exponential $\Theta\left(\alpha\right) \leq \sum_{k=0}^{\infty} \alpha^k/k! = \exp\left(\alpha\right)$. This implies convergence for all $\alpha \geq 0$. Because the convergence domain is a disk centered at the expansion point, the yp series converges for all $\alpha \in \mathbb{C}$.

- We used the functional equation (4) only to determine the value at $x = 1$, we did not use the derivatives (7). Yet, because we derived the expression for yp respecting (4), yp reproduces these values and they are not formal (as for yo), but true derivatives of $\mathrm{yp}(x)$ at $x = 1$.

- Writing $\mathrm{yp}\left(z\right) = 1/2 + \Theta\left(\ln z\right)$ one sees that analytic properties of $\mathrm{yp}\left(z\right)$ are driven by the logarithm in the argument. It implies yp is not defined at $z = 0$ and that it has a cut starting at zero with infinitely many Riemann sheets.



(a) (b)

Figure 3: The function $\mathrm{yp}(x)$ with its bounds (a) given by (11) for $x > 1$ and its behavior compared to $\sqrt{x}$ on a large interval (b).

The relationship yo $\leftrightarrow$ yp deserves an additional investigation, but let us before study the asymptotic behavior of yp on the real axis using only rough (non-rigorous) arguments. Let us assume that on the positive real axis the behavior of its derivative can be approximated by an exponent $\mathrm{yp}'(x)|_{x\to+\infty} \sim ax^b$, $a, b \in \mathbb{R}$. If we plug this into the functional equation for the first derivative (6) we get for large x

$$\mathrm{yp}'(x) = 1 - 2\,x\,\mathrm{yp}'(x^2) \quad \overset{x\to+\infty}{\Longrightarrow} \quad ax^b = 1 - 2ax^{2b+1}.$$

The latter is in the limit $x \to +\infty$ exactly satisfied only for $a = 1/2$ and $b = -1/2$. Thus

$$\mathrm{yp}'(x)|_{x\to+\infty} \sim \frac{1}{2\sqrt{x}},$$

meaning that for a large positive x one has[1] $\mathrm{yp}(x)|_{x\to+\infty} \sim \sqrt{x}$ (Figure 3-b). As we will argue later, it is non trivial to apply (6) to the last expression in order to get the behavior of the function at large negative x. One can do so, but by precaution we change the function name again yp $\to$ ym

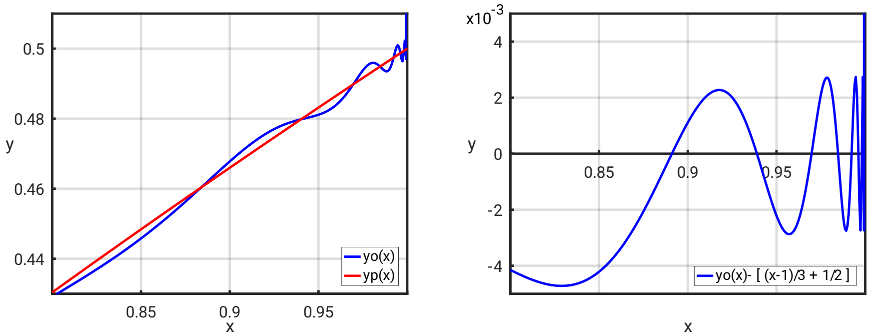
$$\mathrm{ym}'(x)|_{x\to-\infty} = 1 - 2x\frac{1}{2\sqrt{x^2}} = 2,$$

i.e. ym approaches a straight line. Here ym is defined as $\mathrm{ym}(x) = x - \mathrm{yp}(x^2)$, $x \leq -1$. A common picture of ym, yo and yp and is shown in Figure 2-b.

3 Yo and yp

The situation as we have it now can be summarized:

[1]Using $\mathrm{yp}(x) < \sqrt{x} \Rightarrow \mathrm{yp}(x^2) < x = \sqrt{x^2}$, it is not difficult to show that $\sqrt{x} > \mathrm{yp}(x)$ for $x \geq 1$. It suffices to recognize this property as true in a small right neighborhood of one and then recursively transport it to an arbitrary large x. The square root is a significantly better upper boundary than $\mathcal{B}[2\ln(x)]$.



(a) (b)

Figure 4: The left neighborhood of one: yo and yp (a) and the difference of yo from the tangent (at $x = 1$) to yp (b).

- The yo function is analytic inside the unit circle and can be expanded in one of the two self-consistency points of (4), namely at $x = 0$. There the derivatives are in agreement with (4). The derivatives one obtains from (4) at $x = 1$ define a different function (yp) which is analytic there and is not the analytic continuation of yo.
- The function yp is analytic in the whole complex plane besides the cut on the negative real axis. Particularly, it is analytic at $x = 1$, implying it is different from yo inside the unit circle (cut excepted). By consequence, the functional equation (4) has on this domain at least two different solutions. The function yp cannot be defined at $x = 0$ since the set of derivatives given by (4) at $x = 0$ is unique. Having these derivatives there means being yo, which yp is not. Yp can be interpreted as an analytic function on a Riemann surface with an infinite number of Riemann sheets generated by the branch point situated at zero.

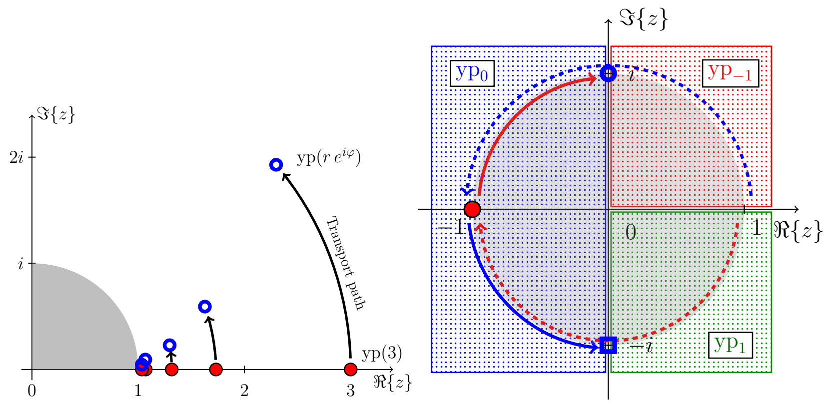
On $x \in (0, 1)$ yo and yp behave similarly, in Figure 2-b they cannot be distinguished and both can be represented by the right half of the blue segment. However, a more precise numerical inspection shows non-vanishing differences, see Figure 4. The graph suggests that yo has no limit at one, it shows an oscillatory behavior with limited but non-vanishing amplitude.

There is another important difference between yo and yp: Inside the unit circle the function yo fulfills the functional equation (4) without exception, for any z_1 and z_2 such that $z_2 = z_1^2$ one has

$$\mathrm{yo}(z_1) = z_1 - \mathrm{yo}(z_2).$$

This is no longer true for yp, where z_2 cannot be an arbitrary square root, but the root with half-argument:

$$\mathrm{yp}(re^{i\varphi}) = re^{i\varphi} - \mathrm{yp}(r^2 e^{i2\varphi}), \quad r \in \mathbb{R}_{0+}. \tag{13}$$



(a) (b)

Figure 5: a) The argument of yp is moved away from the real axis, together with all its daughter square roots, along a continuous path to a complex point $re^{i\varphi}$. We require the function value to be transported also in a continuous manner which determines our choice of square roots we use when computing $\mathrm{yp}(re^{i\varphi})$. b) When computing $\mathrm{ym}(z_0)$ we ask for a smooth transition of the value from $z = -1$ along a continuous line. The value of z^2 in the argument of yp (see (14)) may leave the physical sheet for z_0 far from -1. The blue area represents arguments of ym for which the argument of yp is on the physical sheet. If we evaluate ym for some z in the red area then arguments of yp are from the sheet -1, in the green area from the sheet $+1$. Computing e.g. $\mathrm{ym}(i)$ we follow the red line coming to the border of the blue and the red domain. For z transported in this way z^2 follows the dashed red path approaching $-1 = e^{i(-\pi)}$, i.e. $\alpha = -i\pi$ in (12). The evaluation of $\mathrm{ym}(-i)$ corresponds to the blue line which gives dashed blue line for the transport of the yp value, approaching the branch cut from above, $-1 = e^{i(+\pi)}$, i.e. $\alpha = i\pi$ in (12).

Yet, yp fulfills this equation on a much larger domain. This restriction follows from how yp was constructed. Considering (9)–(10) one sees, that the functional equation was applied in steps, starting close to one. We followed the real axis and a pair of consecutive arguments always respected the restricted equation (13). We can of course apply the procedure also to an h with nonzero imaginary part, but we want the function yp to be continuous when going off the positive real axis. With a sequence of points $\{x_n\}_{n=0}^{\infty}$, $x_n \overset{n\to\infty}{\to} 1$ such that $x_{n+1} = \sqrt{x_n}$, where the usual square root is used, we associate a sequence of complex points $\{z_n\}_{n=0}^{\infty}$, $z_n \overset{n\to\infty}{\to} 1$ meant to compute the value of yp at some $z_0 \in \mathbb{C}$. For that we transport each x_n along a smooth path to become z_n. Along the path connecting x_0 and z_0 we transport also the value of yp, see Figure 5-a. For this transport to be smooth (yp is analytic), we need to keep our convention for taking the square root. This naturally explains the origin of the restriction, yp does not respect the equation (4) for an inappropriate root.

At last one can analyze the behavior of yp with respect to (13) on the whole Riemann surface. We will name the zeroth sheet as "physical" and put the index of the sheet in the subscript next to the function name. The index increases when going around zero in the counter-clock direction. For the physical sheet $\varphi = 0$ the equation (13) stands

$$\mathrm{yp}_0(1) = 1 - \mathrm{yp}_0(1) \quad \text{i.e.} \quad 1/2 = 1 - 1/2.$$

This changes for different sheets. With $\varphi = 2\pi$ one for example has (a numerical approximation is shown)

$$\mathrm{yp}_1(1) = 1 - \mathrm{yp}_2(1) \quad \text{i.e.} \quad (-0.7321 - 0.5458i) = 1 - (1.7321 + 0.5458i)\,.$$

The key point is of course $\mathrm{yp}_1(1) \neq \mathrm{yp}_2(1)$. The functional equation is thus genuinely satisfied only by yp on the physical sheet. Estimates for $\mathrm{yp}_{1,2}(1)$ come from the numerical evaluation of $\Theta\,(2\pi i)$ and $\Theta\,(4\pi i)$ using (12)[2].

4 Ym

We use the expression

$$\mathrm{ym}(z) = z - \mathrm{yp}(z^2) = 2z + \mathrm{yp}(-z), \quad z \approx -1 \tag{14}$$

to define ym in the neighborhood of $z = -1$. This definition is unambiguous because for any z its square z^2 is unique. The expansion point of ym is $x = -1 \in \mathbb{R}$, there its derivatives can be determined, see (8). Because we understand ym as defined by its expansion at $x = -1$ and because we require continuity, we compute the value of ym at some complex point z_0 by transporting its value from $x = -1$ to z_0 along some continuous path, see Figure 5-b. Any point at this path has a unique partner point z^2 which follows a different path starting at $x = 1$ and which is the argument of yp. By its relation to yp, one sees that ym is an analytic function on the whole infinitely-sheeted Riemann surface, the branch point situated at $x = 0$ excepted. One can constrain ym to a single sheet (complex plane) by introducing a cut, its most natural place may be the positive real axis, situated symmetrically to the expansion point. The function ym also inherits from yp its behavior with respect to the functional equation (4). Let φ be a small angle and $r \in \mathbb{R}_{0+}$. One has:

$$\mathrm{yp}(r^2 e^{i2\varphi}) = r^2 e^{i2\varphi} - \mathrm{yp}(r^4 e^{i4\varphi}).$$

Then (14) leads to

$$\mathrm{ym}(-z) = -z - 2z^2 - \mathrm{ym}(-z^2) \quad \text{or} \quad \mathrm{ym}(z) = z - 2z^2 - \mathrm{ym}(-z^2), \tag{15}$$

where z is the root of z^2 with the half-argument (left equation) or the one with the half-plus-π argument (right equation). An analogous approach to what is presented here can be used for any function defined in a way sketched in (6) at some SR point.

[2]Interestingly $\Re\,\mathrm{yp}_2(1) \approx \sqrt{3}$ and $\Im\,\mathrm{yp}_2(1) \approx \sin\left(\frac{1}{\sqrt{3}}\right)$.

5 Discussion

Considering z inside the unite circle, $z \neq 0$, one observes that

$$\mathrm{y}\lambda\left[z, \lambda\left(z\right)\right] \equiv \lambda\left(z\right)\mathrm{yo}(z) + \left[1 - \lambda\left(z\right)\right]\mathrm{yp}(z)$$

is (as a function of its first argument) also a solution to the restricted equation for an arbitrary function $\lambda(z)$. We do not know about other continuous solutions (analytic or not). If existing, they are not defined at zero and one. This follows from our solution-constructing procedure (9)–(10) which provides a unique result (without any specific assumptions, e.g. about analyticity). We presume an analogical construction exists at $z = 0$ giving yo as the unique solution.

For what concerns approximation properties, monomials are easy to express in terms of yo[3]:

$$\mathrm{yo}(x^m) = x^m - \mathrm{yo}(x^{2m}) \quad \Rightarrow \quad x^m = \mathrm{yo}(x^m) + \mathrm{yo}(x^{2m}), \tag{16}$$

which agrees with (2). The approximation (1) is based on matching derivatives at zero. Yet, at least polynomials can be constructed also using yp, $x^m = \mathrm{yp}(x^m) + \mathrm{yp}(x^{2m})$, despite yp not being at zero defined. Using yp, polynomials can be expressed[4] outside the unit circle too.

Also, one might ask whether the expansion (1) is equivalent to the Taylor series in the sense that (1) can be built by replacing each power x^m in the Taylor series by (16). The replacement itself is of course allowed, what is questionable is the re-arrangement of the resulting series by powers of x in the argument of yo. Since such a re-arrangement concerns an infinite number of terms one has to provide a rigorous justification for it. We consider this question as open in general, for specific situations the proof can be performed. Consider for example a function with positive expansion coefficients on the positive real axis (such as $\exp\left(x\right)$, Figure 6 left)

$$f\left(x\right) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \left[a_n \mathrm{yo}(x^n) + a_n \mathrm{yo}(x^{2n})\right], \quad a_n, x \in \mathbb{R}_{0+}.$$

Evaluated at some point $x_0 \in \mathbb{R}_{0+}$ strictly within the convergence radius, power series are known to be absolutely convergent. We also have $\mathrm{yo}(x_0^{(2)n}) > 0$, thus the positive number $a_n x_0^n \geq 0$ is a sum of two positive numbers $a_n \mathrm{yo}(x_0^n) \geq 0$ and $a_n \mathrm{yo}(x_0^{2n}) \geq 0$. Necessarily $0 \leq a_n \mathrm{yo}(x_0^{(2)n}) \leq a_n x_0^n$ meaning that the two sequences $\sum_{n=0}^{\infty} a_n \mathrm{yo}(x^n)$ and $\sum_{n=0}^{\infty} a_n \mathrm{yo}(x^{2n})$ are each absolutely convergent and so is their sum. We then re-arrange the sum, ordering it by powers appearing in the argument of yo

$$f(x) = \sum_{n=0}^{\infty} \left[\left(a_{2n} + a_n\right)\mathrm{yo}(x^{2n}) + a_{2n+1}\mathrm{yo}(x^{2n+1})\right].$$

[3]We use yo in examples but what is presented applies also to yp or yλ.

[4]One notices, that for polynomials we have an exact finite expression, not an approximation.

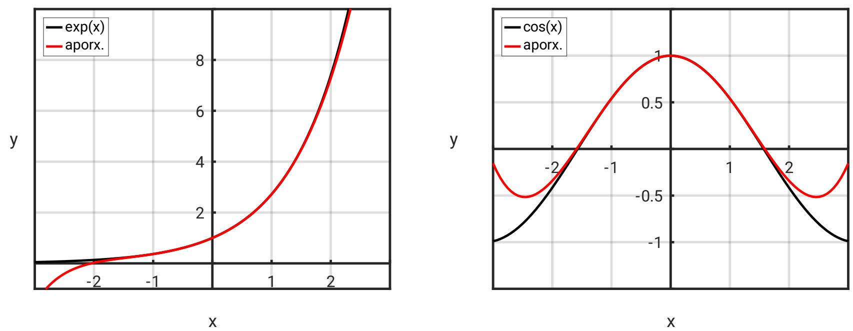


Figure 6: Approximation of $\exp(x)$ and $\cos(x)$ by $f(0) + \sum_{n=1}^{10} a_n \text{ya}(x^n)$.

If true for both, yo and yp, such an equivalence can play a role of the middlemen in the replacement yo $\rightarrow$ yp: $\text{yo}(x^m) + \text{yo}(x^{2m}) = x^m = \text{yp}(x^m) + \text{yp}(x^{2m})$ so that one gets an yp-based expansion that goes beyond the unit circle.

A cosmetic defect of using yp appears on the negative real axis where its values (from the upper or the lower edge of the cut) are complex. We therefore propose to build approximations using ya:

$$\text{ya}(x)\,|_{x\leq -1} = \text{ym}(x), \quad \text{ya}(x)\,|_{|x|<1} = \text{yo}(x), \quad \text{ya}(x)\,|_{x\geq 1} = \text{yp}(x),$$

$\text{ya}(x)$ already shown in Figure 2-b. Approximations of two example functions using $\text{ya}(x)$ are shown in Figure 6.

6 Summary, outlook

In this text we have introduced a new function yo and analyzed it. We have encountered several interesting properties: Yo is well suited for approximations of a new type (1) whose construction is, rather unexpectedly, related to the Dirichlet convolution. Yo turns out to be a member of a specific function class, namely it is lacunary and no natural way of extending it beyond the unit circle is known. Yet, realizing that yo obeys a functional equation which one is able to solve also outside the unit disc, one may wonder whether a natural continuation is possible. Unfortunately it is not: the solution should be interpreted as an independent function with the related functional equation restricted. Nevertheless one may say that the "events" took an interesting turn.

We believe that the set of functions with these interesting properties may be enlarged by studying other simple approximation forms. For example choosing $h = 1, 0, 1, 0, 0, 0, \ldots$ leads to $g(z) = \sum_{n=0}^{\infty} (-1)^n z^{3^n}$, which is another lacunary function with presumably similar properties to yo. This opens a large space for further investigations.

Appendix: Proof of convergence

We aim to prove the convergence of Θ_N when $N \to \infty$, where

$$\Theta_N(\alpha) = \sum_{n=0}^{N} \left[(1 + \frac{\alpha}{2^{2N+2}})^{2^{2n+1}} - (1 + \frac{\alpha}{2^{2N+2}})^{2^{2n}} \right] \equiv \sum_{n=0}^{N} \omega_{N,n}. \tag{17}$$

Upper boundary (UB)

First, we note that if n is interpreted as a smooth parameter and N is fixed then $\omega_{N,n}$ increases with increasing n. In this case one can set an upper boundary by replacing the sum with an integral where the upper integration limit is by one higher than the upper limit of the sum, see Figure 7-a. We have ($n \to x$)

$$\begin{aligned}\Theta_N(\alpha) \leq& \int_0^{N+1} \left[(1 + \frac{\alpha}{2^{2N+2}})^{2^{2x+1}} - (1 + \frac{\alpha}{2^{2N+2}})^{2^{2x}} \right] dx \\ =& \frac{1}{\ln(4)} \left\{ \mathrm{Ei}\left[2^{2x+1} \ln\left(1 + \frac{\alpha}{2^{2N+2}}\right)\right] - \mathrm{Ei}\left[2^{2x} \ln\left(1 + \frac{\alpha}{2^{2N+2}}\right)\right]\right\}_{x=0}^{x=N+1},\end{aligned}$$

here Ei is the exponential integral. On the upper and lower integration limit we get

$$\begin{aligned} U_N^{\mathrm{UB}}(\alpha) &= \frac{1}{\ln(4)} \left\{ \mathrm{Ei}\left[2^{2N+3} \ln\left(1 + \frac{\alpha}{2^{2N+2}}\right)\right] - \mathrm{Ei}\left[2^{2N+2} \ln\left(1 + \frac{\alpha}{2^{2N+2}}\right)\right]\right\}, \\ L_N^{\mathrm{UB}}(\alpha) &= \frac{1}{\ln(4)} \left\{ \mathrm{Ei}\left[2 \ln\left(1 + \frac{\alpha}{2^{2N+2}}\right)\right] - \mathrm{Ei}\left[\ln\left(1 + \frac{\alpha}{2^{2N+2}}\right)\right]\right\} \\ &= \frac{1}{\ln(4)} \left\{ L_N^{\mathrm{UB},1}(\alpha) - L_N^{\mathrm{UB},2}(\alpha) \right\}. \end{aligned}$$

For $U_N^{\mathrm{UB}}(\alpha)$ the limit $N \to \infty$ is easy to determine

$$2^{2N+3} \ln\left(1 + \frac{\alpha}{2^{2N+2}}\right) = \ln\left[\left(1 + \frac{\alpha}{2^{2N+2}}\right)^{2^{2N+3}} \right] = \ln\left[\left\{ \left(1 + \beta^{-1}\right)^{\beta} \right\}^{2\alpha} \right],$$

where $\beta = 2^{2N+2}/\alpha$. Similarly we have for the second term of $U_N^{\mathrm{UB}}(\alpha)$

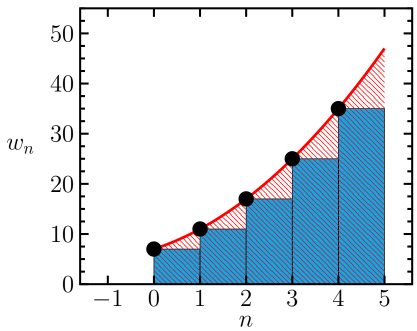
$$2^{2N+2} \ln\left(1 + \frac{\alpha}{2^{2N+2}}\right) = \ln\left[\left\{ \left(1 + \beta^{-1}\right)^{\beta} \right\}^{\alpha} \right].$$

So

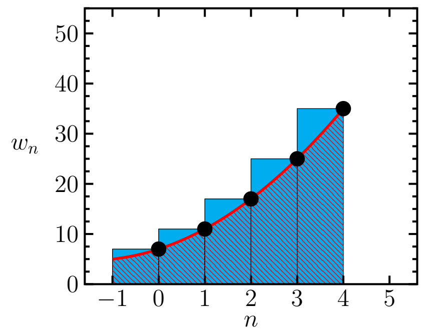
$$U^{\mathrm{UB}}(\alpha) \equiv \lim_{N \to \infty} U_N^{\mathrm{UB}}(\alpha) = \frac{\mathrm{Ei}\left[\ln\left(e^{2\alpha}\right)\right] - \mathrm{Ei}\left[\ln\left(e^{\alpha}\right)\right]}{\ln(4)} = \frac{\mathrm{Ei}(2\alpha) - \mathrm{Ei}(\alpha)}{\ln(4)}.$$

For the lower limit one needs to use small argument expansions

$$\mathrm{Ei}(h) = \gamma + \ln(h) + \mathcal{O}(h) \quad \text{and} \quad \ln(1+h) = h + \mathcal{O}\left(h^2\right)$$



(a)



(b)

Figure 7: The sum $\sum_{n=0}^{4} w_n$ is in both figures represented by black circles and is exactly given by the area of blue rectangles, constructed either on the right side (a) or on the left side (b) of the points, the histograms are just shifted. If n can be interpreted as a smooth index and w_n rises as function of n (red curve) then an upper boundary is given by the integral (red hatched area) that goes by one further than is the upper summation limit (a) and a lower boundary is given by an integral that starts at the number by one smaller than is the lower summation limit.

(γ is the Euler–Mascheroni constant). One has

$$\begin{aligned} L_N^{\mathrm{UB},1}(\alpha) &= \gamma + \ln\left[2\ln\left(1+\frac{\alpha}{2^{2N+2}}\right)\right] + \mathcal{O}\left[2\ln\left(1+\frac{\alpha}{2^{2N+2}}\right)\right] \\ &= \gamma + \ln(2\alpha) - \ln\left(2^{2N+2}\right) + \mathcal{O}\left(\frac{\alpha}{2^{2N+2}}\right). \end{aligned}$$

Similarly

$$L_N^{\mathrm{UB},2}(\alpha) = \gamma + \ln(\alpha) - \ln\left(2^{2N+2}\right) + \mathcal{O}\left(\frac{\alpha}{2^{2N+2}}\right).$$

Consequently

$$\lim_{N\to\infty} L_N^{\mathrm{UB}}(\alpha) = \lim_{N\to\infty} \frac{L_N^{\mathrm{UB},1}(\alpha) - L_N^{\mathrm{UB},2}(\alpha)}{\ln(4)} = \frac{\ln(2\alpha)-\ln(\alpha)}{\ln(4)} = \frac{1}{2}.$$

To summarize, we have an upper boundary

$$\Theta(\alpha) \equiv \Theta_\infty(\alpha) \leq \frac{\mathrm{Ei}(2\alpha) - \mathrm{Ei}(\alpha)}{\ln(4)} - \frac{1}{2},$$

which increases with increasing α and approaches zero for $\alpha \to 0^+$.

Lower boundary (LB)

Similarly, for a sum which rises with the summation index, one can construct a lower boundary by computing an integral, this time the lower integration limit being by one smaller than is the starting value of the summation index, see Figure 7-b

$$
\begin{aligned}
\Theta_N(\alpha) \geq & \int_{-1}^{N} \left[(1+\frac{\alpha}{2^{2N+2}})^{2^{2x+1}} - (1+\frac{\alpha}{2^{2N+2}})^{2^{2x}} \right] dx \\
= & \frac{1}{\ln(4)} \left\{ \mathrm{Ei}\left[2^{2x+1} \ln\left(1+\frac{\alpha}{2^{2N+2}}\right)\right] - \mathrm{Ei}\left[2^{2x} \ln\left(1+\frac{\alpha}{2^{2N+2}}\right)\right] \right\}_{x=-1}^{x=N}.
\end{aligned}
$$

Briefly, we have

$$
\begin{aligned}
U_N^{\mathrm{LB}}(\alpha) = & \frac{1}{\ln(4)} \left\{ \mathrm{Ei}\left[2^{2N+1} \ln\left(1+\frac{\alpha}{2^{2N+2}}\right)\right] - \mathrm{Ei}\left[2^{2N} \ln\left(1+\frac{\alpha}{2^{2N+2}}\right)\right] \right\}, \\
L_N^{\mathrm{LB}}(\alpha) = & \frac{1}{\ln(4)} \left\{ \mathrm{Ei}\left[\frac{1}{2} \ln\left(1+\frac{\alpha}{2^{2N+2}}\right)\right] - \mathrm{Ei}\left[\frac{1}{4} \ln\left(1+\frac{\alpha}{2^{2N+2}}\right)\right] \right\},
\end{aligned}
$$

from which follows

$$
U^{\mathrm{LB}}(\alpha) = \left[\mathrm{Ei}\left(\frac{\alpha}{2}\right) - \mathrm{Ei}\left(\frac{\alpha}{4}\right) \right] / \ln(4), \quad L^{\mathrm{LB}}(\alpha) = 1/2,
$$

where calculations are analogous to those presented for the upper boundary. Thus we conclude

$$
\frac{\mathrm{Ei}\left(\frac{\alpha}{2}\right) - \mathrm{Ei}\left(\frac{\alpha}{4}\right)}{\ln(4)} - \frac{1}{2} \leq \Theta(\alpha) \leq \frac{\mathrm{Ei}(2\alpha) - \mathrm{Ei}(\alpha)}{\ln(4)} - \frac{1}{2}.
$$

Convergence

Finite upper and lower boundaries do not imply the convergence, one needs to exclude oscillations between them for $N \to \infty$. To show that Θ_N increases with increasing N we will focus on the individual terms $\omega_{N,n}$ in (17) and compare $\omega_{N,n}$ with $\omega_{N+1,n+1}$. The sum Θ_{N+1} has one term more than Θ_N, we compare them as represented here

$$
\begin{array}{cccccccc}
\Theta_{N+1}: & \omega_{N+1,0} & \omega_{N+1,1} & \omega_{N+1,2} & \dots & \omega_{N+1,N} & \omega_{N+1,N+1} \\
 & & \updownarrow & \updownarrow & & \updownarrow & \updownarrow \\
\Theta_N: & & \omega_{N,0} & \omega_{N,1} & \dots & \omega_{N,N-1} & \omega_{N,N}
\end{array} \tag{18}
$$

We have

$$
\begin{aligned}
\omega_{N+1,n+1} - \omega_{N,n} = & \left[(1+\frac{q}{4})^{8t} - (1+\frac{q}{4})^{4t} \right] - \left[(1+q)^{2t} - (1+q)^{t} \right] \\
= & (1+\frac{q}{4})^{4t} \left[(1+\frac{q}{4})^{4t} - 1 \right] - (1+q)^{t} \left[(1+q)^{t} - 1 \right],
\end{aligned}
$$

where $q = \alpha/2^{2N+2} \geq 0$ and $t = 2^{2n} \geq 1$. To prove that the expression is positive we need to show that $(1+\frac{q}{4})^{4t} \geq (1+q)^t$. Considering q and t as fixed, we introduce the parametric function $\chi_{q,t}(\alpha) = (1+q/\alpha)^{\alpha t}$ with $\alpha > 1$. The above inequality is then equivalent to $[\chi_{q,t}(\alpha)]_{\alpha=1}^{\alpha=4} \geq 0$. It is now sufficient to demonstrate that $\chi_{q,t}(\alpha)$ rises for $\alpha \geq 1$ for arbitrary $q \geq 0$ and $t \geq 1$. We compute the derivative

$$\frac{d}{d\alpha}\chi_{q,t}(\alpha) = t\left(1+\frac{q}{\alpha}\right)^{\alpha t}\left[\ln\left(1+\frac{q}{\alpha}\right) - \frac{\frac{q}{\alpha}}{1+\frac{q}{\alpha}}\right] = t\left(1+\frac{q}{\alpha}\right)^{\alpha t} R.$$

The only suspicious term is the one in the square brackets. Using the substitution $\beta = 1+\frac{q}{\alpha}$ we get

$$R = \ln(\beta) - \frac{\beta-1}{\beta} = \ln(\beta) - \left(1-\frac{1}{\beta}\right).$$

The function $1-1/x$ is a known lower boundary of the logarithm function for $x>0$, thus R is positive. Moreover, the sum Θ_{N+1} has one additional term, namely $\omega_{N+1,0}$, see (18). All this implies that Θ_N strictly grows with N. A growing and bounded sequence is necessarily convergent.

Alternative proof

At last we want to mention an alternative proof of the convergence presented to us on the online forum [4] which is, indeed, short and elegant. One first re-arranges the sequence in the inverse order and gets ($H_{n<0}=0$, $H_{n\geq 0}=1$)

$$\Theta(\alpha) = \lim_{N\to\infty}\sum_{n=0}^{N}\omega_{N,n} = \lim_{N\to\infty}\sum_{m=0}^{N}\omega_{N,N-m} = \lim_{N\to\infty}\sum_{m=0}^{\infty}H_{N-m}\,\omega_{N,N-m}$$
$$= \lim_{N\to\infty}\sum_{m=0}^{\infty}H_{N-m}\left[(1+\frac{\alpha}{2^{2N+2}})^{2^{(2N+2)-(2m+1)}} - (1+\frac{\alpha}{2^{2N+2}})^{2^{(2N+2)-(2m+2)}}\right].$$

Next, after verifying the necessary conditions are met, one applies the Tannery's theorem and interchanges the summation and the limit. This gives

$$\Theta(\alpha) = \sum_{m=0}^{\infty}\left[e^{\frac{\alpha}{2^{2m+1}}} - e^{\frac{\alpha}{2^{2m+2}}}\right] \overset{n=m+1}{=} \sum_{n=1}^{\infty}\left[e^{\frac{\alpha}{2^{2n-1}}} - e^{\frac{\alpha}{2^{2n}}}\right]$$
$$= \sum_{n=1}^{\infty}\left[(+1)\left(e^{\frac{\alpha}{2^{2n-1}}}-1\right) + (-1)\left(e^{\frac{\alpha}{2^{2n}}}-1\right)\right] = \sum_{n=1}^{\infty}(-1)^{n+1}\left(e^{\frac{\alpha}{2^{n}}}-1\right).$$

Now, expanding the exponential, one checks the required assumptions are obeyed to change the order of the summations

$$\Theta(\alpha) = \sum_{n=1}^{\infty}(-1)^{n+1}\left(\sum_{k=1}^{\infty}\frac{\alpha^k}{k!2^{nk}}\right) = -\sum_{k=1}^{\infty}\frac{\alpha^k}{k!}\sum_{n=1}^{\infty}(-1)^n\frac{1}{2^{nk}},$$

where $\sum_{n=1}^{\infty}\left[-2^{-k}\right]^n$ is a geometric series. This gives an efficient way for computing $\Theta(\alpha)$ and $\mathrm{yp}(x=e^{\alpha})$

$$\Theta(\alpha) = \sum_{k=1}^{\infty}\frac{\alpha^k}{k!\,(2^k+1)}, \quad \mathrm{yp}(e^{\alpha}) = \frac{1}{2} + \sum_{k=1}^{\infty}\frac{\alpha^k}{k!\,(2^k+1)}. \tag{19}$$

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[4] metamorphy (https://math.stackexchange.com/users/543769/metamorphy). Convergence of a specific expression. Mathematics Stack Exchange. URL:https://math.stackexchange.com/q/4880653 (version: 2024-03-14).

Addendum: More solutions to $f(z) = z - f(z^2)$

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December 12, 2025

1 More solutions[1]

Let f_1 and f_2 be two solutions of the same differential equation

$$f(z) = z - f(z^2). \tag{1}$$

We define

$$h \coloneqq f_1 - f_2$$

and have

$$h(z) = \left[z - f_1(z^2)\right] - \left[z - f_2(z^2)\right] = -h(z^2).$$

This implies that any function h obeying the equation

$$h(z) = -h(z^2) \tag{2}$$

can be added to an existing solution of (1) to get a new solution g of (1)

$$g = f + h. \tag{3}$$

This helps us to understand the oscillating feature appearing on the interval $(0, 1)$ for the difference of functions

$$\mathrm{yd}(z) \coloneqq \mathrm{yo}(z) - \mathrm{yp}(z),$$

see the figure 1. The function $\mathrm{yd}(z)$ respects (2), and so, if we divide the interval $(0, 1)$ to sub-intervals with edge points $\{d_i\}$ such that $d_i = d_{i+1}^2$, the function $\mathrm{yd}(z)$ is quasi-periodic, it repeats itself on each interval (d_i, d_{i+2}), being stretched or squeezed. The amplitude of oscillations does not change, meaning that $\mathrm{yo}(z)$ has not a limit at $z = 1$ and $\mathrm{yp}(z)$ has not limit at $z = 0$. Indeed $\mathrm{yo}(z)$ is smooth and well-defined at zero. Thus, the intervals (d_i, d_{i+1}) being very short approaching zero, $\mathrm{yo}(z)$ is almost a constant on them. This implies $\mathrm{yp}(z)$ widely oscillates and has no limit. The vice versa argument is used for oscillations of $\mathrm{yo}(z)$ when approaching one from the left. This represents a proof that allows us to conclude the question of limits at $z = 0$ and $z = 1$.

Another benefit we get from previous ideas is that by solving (2) we get new solutions to (1). But what are the solutions of (2)?

[1] The Addendum was initiated by the discussion found on the online forum [1].

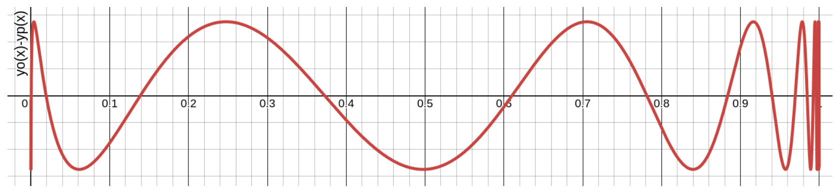

Figure 1: Oscillating quasi-periodic structure of the function $\mathrm{yd}(x) \coloneqq \mathrm{yo}(x) - \mathrm{yp}(x)$.

1.1 Restricted functional equation

If we take into the account the restricted functional equation

$$f(re^{i\varphi}) = re^{i\varphi} - f(r^2 e^{i2\varphi}), \quad r \in \mathbb{R}_{\geq 0}, \quad \varphi \in \mathbb{R}, \tag{4}$$

where we keep track of the value of φ ($e^{i\varphi} \neq e^{i(\varphi+2\pi)}$), i. e. understand f as defined on a Riemann surface with infinitely many Riemann sheets, then, rather obviously, the function h from (2) can be defined arbitrarily in some sector S delimited by

$$re^{i\varphi} \in S \Leftrightarrow \theta \leq \varphi < 2\theta, \quad \theta \neq 0,$$

with θ being fixed and arbitrary . The equation (2) is used to propagate the value to other sectors limited by $2^q\theta$, $q \in \mathbb{Z}$. There are 4 regions which transform independently:

$$\begin{aligned} &\text{A: } 0 \leq r \leq 1 \text{ and } \theta \geq 0,\\ &\text{B: } 0 \leq r \leq 1 \text{ and } \theta \leq 0,\\ &\text{C: } 1 \leq r \leq \infty \text{ and } \theta \geq 0,\\ &\text{D: } 1 \leq r \leq \infty \text{ and } \theta \leq 0.\end{aligned}$$

Nevertheless A and C can share an identical angular sector $\theta \leq \varphi < 2\theta$, the same being true for B and D with θ negative. Thus, one arbitrarily defines h on two such sectors ($\theta \geq 0$ and $\theta \leq 0$) for $0 \leq r \leq \infty$ and gets h defined on the whole Riemann surface, see the figure 2. Adding this h to an existing solution of (4) a new solution of (4) is produced. The solutions are in general non-analytic, but we are not able to exclude the existence of analytic ones (such as $\mathrm{yd}(z)$ for the inside of the unit disk).

1.2 Full functional equation

For the analysis of the unrestricted functional equation we also use the "obvious" approach. But first of all: we are not aware of any solution of the unrestricted equation (1) for $|z| > 1$, thus solving (2) for this region is not interesting. Nevertheless such a solution ought to be fully analogical to the solution inside the unit disk which we are about to present. Here is our reasoning:

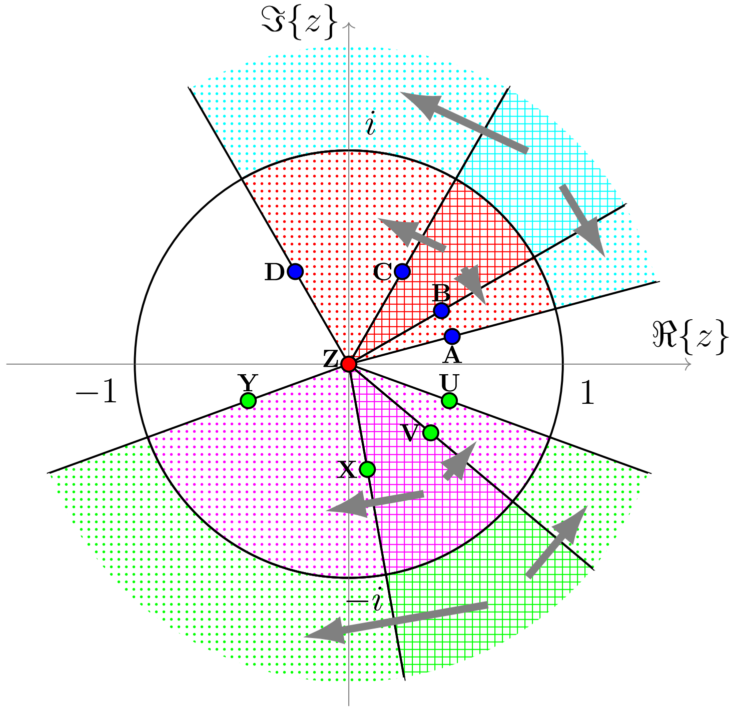

Figure 2: Defining the function arbitrarily in two angular sectors (areas with a grid pattern) such that $\angle 1ZC = 2 \times \angle 1ZB$, $\angle BZC = 2 \times \angle AZB$, $CZD = 2 \times \angle BZC$, $\angle 1ZX = 2 \times \angle 1ZV$,$\angle VZX = 2 \times \angle UZV$ and $XZY = 2 \times \angle VZX$, the function values can be propagated (gray arrows) using (2) to neighboring sectors (dotted areas) and, by repetition, to the whole Riemann surface.

- Let us imagine that the function h reaches some value h_0 at the point $z_0 = r_0 e^{i\varphi_0}$, $0 \leq r \leq 1$. If we understand z_0 as being a square root, then it has a unique daughter $z_{-1} = z_0^2$ situated at a lower orbit $r_{-1} = r_0^2 < r_0$. The function value there is $h_{-1} = -h_0$.
- Now: the point z_{-1} has two parents (square roots), one of them being z_0. The other is $-z_0$. What is the function value there? The answer is

$$h(-z_0) = -h\left[(-z_0)^2\right] = -h\left(z_0^2\right) = h(z_0). \tag{5}$$

- Next consider the following: We start again at some z_0 and go “down” two orbits (squaring two times) which is unique:

$$z_0 \overset{(.)^2}{\to} z_{-1} \overset{(.)^2}{\to} z_{-2}.$$

  Then we come back: z_{-2} has two parents, namely z_{-1} and $-z_{-1}$ at the orbit -1. Each of them has two parents at the orbit 0, i.e. z_{-2} has four

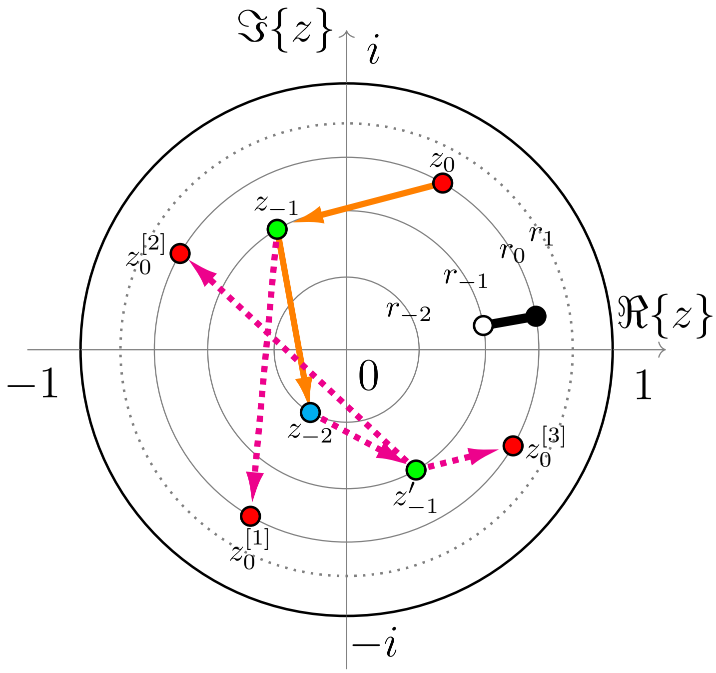

Figure 3: The value at z_0 can be propagated using (2) first by squaring the argument (orange arrows). Next, coming back by taking square roots (dotted magenta arrows), one increases the number of points, possibly defining the values on a dense subset of the r_0 orbit. From a dense subset on r_0, it is easy to propagate the values to a dense subset of the higher orbit r_1 (dotted) and other higher orbits. Between two orbits the function values cannot be defined in this way, one is therefore allowed to define the function values arbitrarily on a segment joining two neighboring orbits (black bolt line), thus defining the values on the dense subset of the whole unit disk.

0-orbit grand parents, namely

$$z_{-2} = r_0^4 e^{i4\varphi_0} \overset{\sqrt{\cdot}\sqrt{\cdot}}{\rightarrow} z_0^{[n]}|_{n=0,1,2,3} = r_0 e^{i(\varphi_0 + n\times 2\pi/4)}.$$

By (5) we see that at a given orbit the function value is constant.

- This is how we can densely populate the orbit zero: we go "down" arbitrarily deep, many orbits, which is unique and then come back, many orbits, which is not unique: Going up one orbit means that the number of parents doubles. The takeaway is: considering a fixed radius r_0, it is enough to define the value of h at one single point of this radius and we are able to densely populate the same orbit (radius), the values are constant and equal to h_0, see the figure 3.

- Having one densely populated orbit r_0, it is obvious that one can densely populate higher orbits. Indeed, each point on r_0 has two r_1 parents, so one can trivially populate all higher orbits at radii $r_1 = \sqrt{r_0}$, $r_2 = \sqrt{r_1}$ etc... What is also obvious is that in total one populates only a discrete set of orbits r_0, $r_{\pm 1}$, $r_{\pm 2}$, ... From this we draw our conclusion:

- It is sufficient to arbitrarily define the value of h at a segment between two nonzero arbitrary points with $r_1 = r_0$ and $r_2 = r_0^2$ at some fixed φ (why not $\varphi = 0$?) and one can densely populate the whole inside of the unit disk by using the equation (2). One gets a function which is constant on circles and quasi-repeats itself as a function of r.

The construction covers only a countable number of points, i. e. there are many "holes" with undefined value inside the unit disk. Nevertheless, considering a single orbit with a dense subset on which the value is constant, we naturally consider (e. g. by requiring the function to be continuous on the circle) the solutions where *all* points of a circle have the same constant value.

One may wonder whether our procedure of assigning a value is unambiguous: Is it not possible to get two different values for a given point? We believe it is impossible. First, choosing two radii r_a and r_b such that $r_b^2 < r_a < r_b < 1$, the sets populated by starting at r_a and r_b (and some fixed angle φ) are independent. Thus, to answer the question, one can consider only one discrete set of orbits indexed by integers as done previously. Then the value-assigning procedure is equivalent to an infinite tree-graph, where each node has two parents and a single offspring, the starting node being arbitrary. It is not possible in such a graph to come to some node in two ways, there are no loops and thus no ambiguous way to assign a value. Although we believe our answer is funded, a more rigorous formulation may be useful.

The result also shows that solutions g one gets by adding h to an existing analytic solution f, formula (3), are not analytic meaning that yo is the only analytic solution to the unrestricted functional equation. For g to be different from f one has to have non-zero h. The equation $h(z) = -h(z^2)$ implies h is not constant. Let us study h at some z_0 where h is changing in some direction. Then the derivative in this direction is non-zero. Yet, if we take the derivative of h at z_0 in a direction which is tangent to the circle centered at $z = 0$, then it is zero, the function is constant on circles. This implies h is not analytic because for an analytic function the complex derivative does not depend of the direction, i. e. is the same for all directions.

2 Desperate attempt

First observation: when h is the solution of (2), then also $H = \lambda h$ is

$$H(z) = \lambda h(z) = -\lambda h(z^2) = -H(z^2)$$

for some real λ. Another interesting feature of (2) is the possibility to reflect its solution with respect to the unit circle. Having some solution of (2) inside the unit disk, we can extend it to the outside by

$$h(z) \coloneqq h\left(\frac{1}{z}\right), \quad |z| > 1. \tag{6}$$

Indeed, for $|z| > 1$ one gets

$$h(z) \coloneqq h\left(\frac{1}{z}\right) = -h\left(\frac{1}{z^2}\right) =: -h\left(z^2\right),$$

because $1 < |z| \Rightarrow 1 < \left|z^2\right|$.

We can apply this to $\mathrm{yd}(z)$ and make our last desperate attempt to solve the non-restricted functional equation (1) trying

$$\mathrm{yo}(z) \coloneqq \mathrm{yp}(z) + \lambda \mathrm{yd}(z), \quad |z| > 1, \tag{7}$$

with the "outside" definition of $\mathrm{yd}(z)$ given by (6), i. e. $\mathrm{yd}(z)_{|z|>1} \coloneqq \mathrm{yd}(1/z)$. Inside the unit disk the above equation is valid for $\lambda = 1$. Outside we profit from the freedom of tuning λ and tune it so as to fulfill (1) at two specific points, namely we chose

$$f(2) = 2 - f(4). \tag{8}$$

Explicitly, combining (1) and (7) with (8) we get

$$\begin{aligned} \mathrm{yo}(2) &= \mathrm{yp}(2) + \lambda \mathrm{yd}(2) \\ \mathrm{yo}(2) &= 2 - \mathrm{yo}(4) = 2 - \mathrm{yp}(4) - \lambda \mathrm{yd}(4) \end{aligned}$$

which gives after the subtraction

$$\begin{aligned} 0 &= \mathrm{yp}(2) + \lambda \mathrm{yd}(2) - \left[2 - \mathrm{yp}(4) - \lambda \mathrm{yd}(4)\right], \\ &\Rightarrow \\ \lambda &= \frac{2 - [\mathrm{yp}(4) + \mathrm{yp}(2)]}{\mathrm{yd}(2) + \mathrm{yd}(4)} = -1.104\ldots. \end{aligned}$$

With this λ we check at

$$z_0 = 2e^{i\pi/4} \quad \text{and} \quad z_1 = 2e^{-i3\pi/4},$$

both satisfying (disregarding the phase)

$$z_{0,1}^2 = 4i.$$

Replacing the functional equation by the numbers in a numerical evaluation one gets (approximate values are shown)

$$\begin{aligned} & \mathrm{yo}(z) = z - \mathrm{yo}(z^2) \\ z_0: \quad & [0.678 + 0.313i] = [1.414 + 1.414i] - [0.736 + 1.100i] = [0.678 + 0.313i], \\ z_1: \quad & [0.523 - 1.129i] \neq [-1.414 - 1.414i] - [0.736 + 1.100i] = [-2.150 - 3.515i]. \end{aligned}$$

As expected, the restricted equation is respected but the full equation is not, the "trick" does not work.

3 Proof at last

During the work on this *Addendum* a rather simple proof, or at least a serious argument, was found implying that an analytic solution to the unrestricted equation (1) does not exist outside the unit circle. Let us denote an analytic function as Γ and the complex variable as α, $|\alpha| > 1$. One aims to solve

$$\Gamma(\alpha) = \alpha - \Gamma(\alpha^2).$$

Let us define a function f as $f(\alpha) = \Gamma(1/\alpha)$, it is analytic. The functional equation becomes

$$f\left(\frac{1}{\alpha}\right) = \alpha - f\left(\frac{1}{\alpha^2}\right).$$

Next we make a substitution and change the variable $z = 1/\alpha$

$$f(z) = \frac{1}{z} - f(z^2), \quad |z| < 1. \tag{9}$$

In this way we have moved the problem to a finite domain inside the unit circle. We search for a function f analytic at all $|z| < 1$, $z \neq 0$. The domain is a disk (or annulus if $z = 0$ is removed) and therefore f, if exists, can be written as a Laurent series

$$f(z) = \sum_{n=-\infty}^{+\infty} a_n z^n.$$

There is only one such series which formally solves (9), namely

$$f(z) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{z^{(2^n)}}. \tag{10}$$

For the sake of "easy-to-see" let us write it expanded

$$f(z) = \frac{1}{z} - \frac{1}{z^2} + \frac{1}{z^4} - \frac{1}{z^8} + \ldots, \quad f(z^2) = \frac{1}{z^2} - \frac{1}{z^4} + \frac{1}{z^8} - \ldots$$

$$\frac{1}{z} - f(z^2) = \frac{1}{z} - \left(\frac{1}{z^2} - \frac{1}{z^4} + \frac{1}{z^8} - \ldots\right) = \frac{1}{z} - \frac{1}{z^2} + \frac{1}{z^4} - \frac{1}{z^8} + \cdots = f(z).$$

The series (10) is rapidly divergent and therefore an analytic solution does not exist. There might by a non-analytic solution: one may define $\mathrm{yo}(x)$ in an arbitrary way on the interval such as $(2, 4]$ and propagate the values to the complex plane using (1). We do not know how the outcome looks like and stop here. Even if some meaningful function is created, being non-analytic, one would not consider it as a continuation of yo (or would one?).

References

[1] Sidharth Ghoshal (https://math.stackexchange.com/users/58294/sidharth ghoshal). Extending lacunary series beyond their disks. Mathematics Stack Exchange. URL:https://math.stackexchange.com/q/2955892 (version: 2019-12-02).