Solutions of higher degree equations

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Abstract :

In this paper I solve equations of degrees 5,7 and 9 using an equation of degrees 3. I have the method (the counter that allows me to calculate the different values of the parameters: a, b, c, α,β….etc.) to solve an infinity of equations of degrees (2n+1) using equations of lower degrees..

My wish is to share it with the young researchers for deepening

key words: Equations of the fifth, seventh, ninth degree.

Systems of équations, polynomials.

I) Solution of five degree equation

1.1: Either the polynomial:

$$P(x)=x^5-(6c+4d)x^4+(4d^2+24cd+9c^2)x^3+\left(24\alpha\beta-36c^2d-24cd^2+4ab\right)x^2-(72c\alpha\beta-36\beta^2-36c^2d^2+24abc)x+36abc^2..........(A)$$

$6c+4d\neq 0$

1.2: Either the polynomial:

$$Q(x)=x^3-4dx^2-4(\alpha^2-d^2)x+4ab.......(B)$$

The two polynomials (A) et (B) admit zeros in common.

1.3:Noticed In the polynomial (B), there is no parameter β , it is by making the identification with the equation to solved , we find that it is linked to the other parameters: α, ab, c, d

Exemple 1.4 :

$$x^5+6x^4+13x^3-36x^2-32x-24=0\Rightarrow x=2$$

Lest identify :

$$\begin{cases} 6c+4d=-6\Rightarrow 3c+2d=-3.......(1) \\ 4d^2+24cd+9c^2=13\Rightarrow(2d+3c)^2+12cd=13\Rightarrow cd=\dfrac{1}{3}.....(2) \end{cases}$$

We replace $d=\dfrac{1}{3c}$ in (1) $\Rightarrow 9c^2+9c+2=0$, lets take $c=\dfrac{-1}{3}$ so $d=-1$

$$36abc^2=-24\Rightarrow ab=-6........(5)$$

$$24\alpha\beta-36c^2d-24cd^2+4ab=-36\Rightarrow\alpha\beta=-1.......(3)$$

$$72c\alpha\beta-36\beta^2-36c^2d^2+24abc=32....(4)\Rightarrow\beta^2=1$$,si $\beta=1;\alpha=-1$

By solving: $x^3-4dx^2-4(\alpha^2-d^2)x+4ab=0$

$$x^3+4x^2-4(1-1)x-24=0\Rightarrow x^3+4x^2-24=0\Rightarrow x=2$$

Exemple 1.5 :

$$x^5-2x^4-3x^3+48x^2-112x+128=0\Rightarrow x=-4$$

$$6c+4d=2\Rightarrow 3c+2d=1.......(1)$$

$$4d^2+24cd+9c^2=-3\Rightarrow (2d+3c)^2+12cd=-4\Rightarrow cd=-\frac{1}{3}.....(2)$$

We replace $d=\frac{-1}{3c}$ in (1) $\Rightarrow 9c^2-9c-2=0$, lets take $c=\frac{2}{3}$; $d=\frac{-1}{2}$

So : $36abc^2=128\Rightarrow ab=8........(5)$

$$24\alpha\beta-36c^2d-24cd^2+4ab=48\Rightarrow \alpha\beta=\frac{1}{2}.......(3)$$

$$72c\alpha\beta-36\beta^2-36c^2d^2+24abc=32....(4)\Rightarrow \beta^2=1$$

lets take : $\alpha=\frac{1}{2},\beta=1$

By solving : $x^3-4dx^2-4(\alpha^2-d^2)x+4ab=0$

$$x^3+2x^2-4(\frac{1}{4}-\frac{1}{4})x+32=0\Rightarrow x^3+2x^2+32=0\Rightarrow x=-4$$

Exemple 1.6 :

$$x^5-10x^4+33x^3-72x^2+152x+40=0\Rightarrow x=5$$

$$6c+4d=10\Rightarrow 3c+2d=5.......(1)$$

$$4d^2+24cd+9c^2=33\Rightarrow (2d+3c)^2+12cd=33\Rightarrow cd=\frac{2}{3}.....(2)$$

We replace $d=\frac{2}{3c}$ dans (1) $\Rightarrow 9c^2-15c+4=0$, $c=\frac{1}{3}$; so $d=2$

$$36abc^2=40\Rightarrow ab=10........(5)$$

$$\begin{cases} 24\alpha\beta - 36c^2d - 24cd^2 + 4ab = -36 \Rightarrow \alpha\beta = -3.......(3) \\ 72c\alpha\beta - 36\beta^2 - 36c^2d^2 + 24abc = -152....(4) \Rightarrow \beta^2 = 4 \end{cases}$$

Prenons : $\alpha = \frac{3}{2}, \beta = -2$

By solving : $x^3 - 4dx^2 - 4(\alpha^2 - d^2)x + 4ab = 0$

$$x^3 - 8x^2 - 4(\frac{9}{4} - 4)x + 40 = 0 \Rightarrow x^3 - 8x^2 + 7x + 40 = 0 \Rightarrow x = 5$$

Exemple 1.7 : $x^5 - 7x^4 + \frac{73}{4}x^3 - \frac{13}{2}x^2 - \frac{19}{2}x + \frac{9}{2} = 0 \Rightarrow x = \frac{1}{2}$

$$ab = \frac{1}{2}; c = \frac{1}{2}; d = 1; \alpha = \frac{5}{4}; \beta = \frac{5}{12}$$

By solving : $x^3 - 4dx^2 - 4(\alpha^2 - d^2)x + 4ab = 0$

$$x^3 - 4x^2 - 4(\frac{25}{16} - 1)x + 2 = 0 \Rightarrow x^3 - 4x^2 - \frac{9}{4}x + 2 = 0 \Rightarrow x = \frac{1}{2}$$

Exemple 1.8 :

$$x^5 + 16x^4 - 32x^3 - \frac{2536}{3}x^2 + 1792x - 1920 = 0 \Rightarrow x = 6$$

$$a = \frac{-1}{3}; b = 10; c = -4; d = 2; \alpha = \frac{2}{3}; \beta = -4$$

By solving: $x^3 - 4dx^2 - 4(\alpha^2 - d^2)x + 4ab = 0$

$$x^3 - 8x^2 + \frac{128}{9}x - \frac{40}{3} = 0 \Rightarrow x = 6$$

Exemple 1.9 :

$$x^5-26x^4+229x^3-816x^2+1116x+648=0 \Rightarrow x=9$$

$$a=2; b=9; c=1; d=5; \alpha=\frac{3}{2}; \beta=-3$$

$$x^3-20x^2-4(\frac{9}{4}-25)x+72=0 \Rightarrow x^3-20x^2+91x+72=0 \Rightarrow x=9$$

Example 1.10:

$$x^5-30x^4+313x^3-968x^2-5808x+42592=0 \Rightarrow x=11$$

$$a=1; b=88; d=2; c=\frac{11}{3}; \alpha=\frac{9}{2}; \beta=0$$

By solving: $x^3-8x^2-65x+352=0 \Rightarrow x=11$

Example 1.11

$$x^5-(6\sqrt{2}+12)x^4+(54+72\sqrt{2})x^3-(74\sqrt{2}+192)x^2-(144\sqrt{2}-96)x+432-36\sqrt{2}=0$$
$$\Rightarrow x=\sqrt{2}$$

By solving: $x^3-12x^2+24-2\sqrt{2}=0 \Rightarrow x=\sqrt{2}$

Example 1.12

$$x^5-(4-4i)x^4-16ix^3+(48+16i)x^2-(16+128i)x+128=0 \Rightarrow x=2i$$

$$a=2; b=-4; d=1; c=\frac{-2}{3}i; \alpha=i+1; \beta=\frac{4}{3}(1-i)$$

By solving: $x^3-4dx^2-4(\alpha^2-d^2)x+4ab=0$

$$x^3-4x^2-4(2i-1)x-32=0 \Rightarrow x=2i$$

II) Solution of equation of degree 7

2.1 : $x^7 + 4abx^4 - (40\alpha\beta - 25c^2 - 40c\alpha^2)x^3 - (200\alpha\beta c - 100\beta^2)x + 100abc^2 = 0$

This equation admits at least one solution in common with the equation :

2.2: $x^3 - 4\alpha^2 x + 4ab = 0$. After identification with the equation to solve.

Because we must find the relationship between α and β

Example 2.1 :

$$x^7 + 120x^4 - 487x^3 - 576x + 3000 = 0$$

$$x = 2; a = 3; b = 10; c = -1, \alpha = 4; \beta = \frac{-5}{4}$$

$$4ab = 120 \Rightarrow ab = 30$$

$$100abc^2 = 3000 \Rightarrow abc^2 = 30 \Rightarrow c^2 = 1 \Rightarrow c = -1$$

$$\begin{cases} 40\alpha\beta + 40\alpha^2 = 512 ...(1) \\ -200\alpha\beta - 100\beta^2 = 576 ......(2) \end{cases} \qquad \begin{cases} 200\alpha\beta + 200\alpha^2 = 2560 ....(1) \\ -200\alpha\beta - 100\beta^2 = 576 .....(2) \end{cases}$$

$200\alpha^2 - 100\beta^2 = 3136$ on multiplie par α^2

$200\alpha^4 - 100\alpha^2\beta^2 = 3136\alpha^2$ de (1) : $\alpha\beta = \frac{-5\alpha^2 + 64}{5}$

$$200\alpha^4 - 100\left(\frac{-5\alpha^2 + 64}{5}\right)^2 = 3136\alpha^2$$

$$100\alpha^4 - 576\alpha^2 - 16384 = 0 \Rightarrow \alpha^2 = \frac{576 \pm 2624}{200} \Rightarrow \alpha^2 = 16$$

Of the equation of degree 3 :

$$x^3 - 4\alpha^2 x + 4ab = 0 \Rightarrow x^3 - 64x + 120 = 0 \Rightarrow x = 2$$

Example2.2 :

$$x^7 + 24x^4 + 207x^3 - 144x + 5400 = 0$$

$$x = -3; a = 1; b = 6, c = -3; \alpha = \frac{-1}{2}, \beta = \frac{3}{5}$$

by identification : $ab = 6; abc^2 = 54 \Rightarrow c^2 = 9 \Rightarrow c = -3$

$$\begin{cases} 40\alpha\beta - 25c^2 - 40c\alpha^2 = -207 \\ 200\alpha\beta c - 100\beta^2 = 144 \end{cases} \longrightarrow \begin{cases} 40\alpha\beta + 120\alpha^2 = 18 \ldots\ldots\ldots \alpha = \frac{-1}{2} \\ 600\alpha\beta + 100\beta^2 = -144 .. \beta = \frac{3}{5} \end{cases}$$

Example 2.3 :

$$x^7 - 48x^4 - 28x^3 + 624x - 4800 = 0$$

$$x = 4; a = -3; b = 4; c = -2; \alpha = -1; \beta = \frac{-6}{5}$$

Example2. 4 :

$$x^7 - 120x^4 - 25x^3 + 600x - 3000 = 0$$

$$x = 5; a = -3; b = 10; c = 1, \alpha = \frac{-1}{2}; \beta = -3$$

Example2. 5 :

$$x^7 - 24x^4 - 127x^3 + 240x - 24 = 0$$

$$x = -2; a = 3; b = -2; c = \frac{1}{5}; \alpha = 2; \beta = 2$$

(III) Solution of equation of degree 9

3.1 : $$x^9-(16a^2b^2-49c^2)x^3+\left[(224\alpha^3\beta+16\alpha^4(4ab-14c)\right]x^2$$
$$-\left[392c\alpha\beta-196\beta^2+224ab\alpha\beta+16ab\alpha^2(4ab-14c)\right]x+196abc^2=0$$

3.2 : $x^3-4\alpha^2x+4ab=0........(B)$

Equations **3.1** and **3.2** admit at least 3 roots in common

Note: I chose this form of equation because the calculations are not long. I have the method of solving whatever the coefficients of the polynomial, but the calculations are quite long.

Example3. 1 : solving the equation :

$$x^9-14399x^3+425984x^2-794688x+120=0$$
$$x=2;a=3;b=10;c=\frac{1}{7};\alpha=4;\beta=-4$$

has the same solutions as the equation: $x^3-64x+120=0$

Exemple3. 2 : solving the equation :

$$x^9-1359x^3+62208x^2+104220x-403280=0$$
$$x=-4;a=5;b=-4;c=\frac{-71}{7};\alpha=3;\beta=-3$$

has the same solutions as the equation: $x^3-36x-80=0$

Exemple3. 3 : solving the equation :

$$x^9-1863x^3-3750x^2+14400x-21168=0$$
$$x=3;a=-4;b=-3;c=3;\alpha=\frac{-5}{2};\beta=\frac{15}{7}$$

Exemple3. 4 : solving the equation :

$$x^9-5175x^3-26730x^2-108000x-98000=0$$
$$x=5;a=-4;b=5;c=5;\alpha=\frac{-3}{2};\beta=\frac{135}{7}$$

<u>Exemple3. 5 :</u> solving the equation

$$x^9 - 14350x^3 - 425984x^2 - 797760x - 5880 = 0$$

$$x = -2; a = -3; b = 10; c = -1; \alpha = -4; \beta = \frac{-4}{7}$$