Relativistic Corrections to the Quantization of a Classical Spinning Particle with Constant Electric and Magnetic fields

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Abstract

A quantization of classical spinning particle equations is carried out using the Euler angles of the particle. Relativistic corrections are found and compared to the Foldy-Wouthuysen transformation of the Dirac equation. We only consider constant linear electric and magnetic fields, and find agreement up to order $1/c^6$.

I. Introduction

There are many papers on connecting classical spin to the Dirac equation and we will not try to give a comprehensive review of all the methods. For information on the Dirac equation see for example Sakurai [1]. For a review of some of the different classical spin ideas see Rivas [2]. As an example, Grossmann and Peres [3] use a classical particle with 8 internal degrees of freedom to reproduce the properties of the Dirac eq. Rafanelli and Schiller [4] as well as Rubinow and Keller [5] use a WKB expansion of the Dirac equation and compare the result to the BMT spin equation [6].

A number of papers, including some by Silenko [7], look at a classical analogy of the Dirac eq. by taking the Foldy-Wouthuysen transformation [8] and using the Heisenberg equations of motion to find corresponding classical equations. Chen and Chiou [9] derive the classical equations found in Jackson [10] which are based on the BMT spin equation, and set up a spin and orbital Hamiltonian based on those equations and compare the results to the Foldy-

Wouthuysen transformation of the Dirac equation. They find the same relations that we do but use a different method. We give a summary of their method at the end of the paper.

We will use a method similar to that of Bopp and Haag [11]. They use the Euler angles with the classical spin equations to derive the non-relativistic quantum mechanical equation of spin, the Pauli spin equation. Other papers follow their method, for example Bozic and Maric [12], but do not include relativistic corrections. We extend their ideas by including some relativistic corrections based on the translational and spin equations for a particle in constant Electric and Magnetic fields. These equations can be found in Jackson [10]. We will only consider terms linear in the Electric and Magnetic fields and quantize the system in the Schrodinger picture. We then compare our results to those of the Foldy-Wouthuysen transformation for the Dirac equation. We obtain the exact case for a constant magnetic field and find agreement with the Foldy-Wouthuysen terms up to 6th order in 1/c when a constant electric field is also included. We will only consider the spin one half case.

II. Classical equations

From Jackson [10] we have the translational and spin equations for constant electric and magnetic fields

$$m\frac{d\boldsymbol{\beta}}{dt} = \frac{q}{\gamma c}(\mathbf{E} + \boldsymbol{\beta}\times\mathbf{B} - \boldsymbol{\beta}(\boldsymbol{\beta}\cdot\mathbf{E})) \tag{1}$$

$$\frac{d\mathbf{s}}{dt} = \frac{q}{mc}\mathbf{s}\times\{\left(\frac{g}{2} - 1 + \frac{1}{\gamma}\right)\mathbf{B} - (\frac{g}{2} - 1)\frac{\gamma}{\gamma+1}(\boldsymbol{\beta}\cdot\mathbf{B})\boldsymbol{\beta} - (\frac{g}{2} - \frac{\gamma}{\gamma+1})\boldsymbol{\beta}\times\mathbf{E}\} \tag{2}$$

where **s** is the spin angular momentum in the particle's rest frame and **E** and **B** are the electric and magnetic fields in a general frame. In these expressions $\boldsymbol{\beta} = \mathbf{v}/c$ and $\gamma = (1-\beta^2)^{-1/2}$

where **v** is the velocity of the particle in the general frame and c is the speed of light. q is the charge of the particle and m is its rest mass. g is the g-factor which is close to 2 for the electron. We use a bold symbol to indicate a vector.

Setting $\mathbf{s} = \mathrm{I}\boldsymbol{\omega}_0$ where I is the moment of inertia for the particle and $\boldsymbol{\omega}_0$ is the angular velocity in the particle's rest frame, for g = 2 eq. (2) reduces to

$$\frac{d\boldsymbol{\omega}_0}{dt} = \frac{q}{mc}\boldsymbol{\omega}_0 \times \{\frac{1}{\gamma}\mathbf{B} - \frac{1}{\gamma+1}\boldsymbol{\beta} \times \mathbf{E}\} \quad (3)$$

We wish to find a Lagrangian for eqs. (1) and (3) using the Euler angles ϕ, θ, and ψ and the particle position as the degrees of freedom. $\boldsymbol{\omega}_0$ can be expressed in terms of the Euler angles and their time derivatives, however these derivatives are with respect to the proper time τ, that is the time in the rest frame of the particle and in the variation of the Lagrangian we want to use the time t in the general frame. From Goldstein [13] we have

$$\boldsymbol{\omega}_0 = \left(\cos(\phi)\frac{d\theta}{d\tau} + \sin(\phi)\sin(\theta)\frac{d\psi}{d\tau}\right)\mathbf{x} + \left(\sin(\phi)\frac{d\theta}{d\tau} - \cos(\phi)\sin(\theta)\frac{d\psi}{d\tau}\right)\mathbf{y}$$

$$+ \left(\frac{d\phi}{d\tau} + \cos(\theta)\frac{d\psi}{d\tau}\right)\mathbf{z} \quad (4)$$

where **x**, **y** and **z** are unit vectors in the x, y, and z directions, and using τ as the time in the rest frame. Now define $\boldsymbol{\omega}$ to the same form as $\boldsymbol{\omega}_0$ except for $\frac{d}{d\tau}$ replaced by $\frac{d}{dt}$ and since $\frac{d}{d\tau} = \frac{dt}{d\tau}\frac{d}{dt} = \gamma\frac{d}{dt}$ we have $\boldsymbol{\omega}_0 = \gamma\boldsymbol{\omega}$. Thus eq. (3) takes the form

$$\frac{d}{dt}(\gamma\boldsymbol{\omega}) = \frac{q}{mc}\boldsymbol{\omega} \times \mathbf{B}' \quad (5)$$

where we have set $\mathbf{B}' = \mathbf{B} - \frac{\gamma}{\gamma+1}\boldsymbol{\beta}\times\mathbf{E}$.

III. Variational principles

Now look at finding a Lagrangian. If we take a function L($\boldsymbol{\omega}$) and apply the Euler-Lagrange equations to it using the Euler angles as variables we find the equations

$$\frac{d}{dt}\boldsymbol{\nabla}_{\boldsymbol{\omega}}L = \boldsymbol{\omega}\times\boldsymbol{\nabla}_{\boldsymbol{\omega}}L \tag{6}$$

where $\boldsymbol{\nabla}_{\boldsymbol{\omega}}L$ represents the gradient of L with respect to $\boldsymbol{\omega}$. We also have the usefull relation

$$\boldsymbol{\omega}\cdot\boldsymbol{\nabla}_{\boldsymbol{\omega}}L = \frac{d\phi}{dt}\frac{\partial L}{\partial\frac{d\phi}{dt}} + \frac{d\theta}{dt}\frac{\partial L}{\partial\frac{d\theta}{dt}} + \frac{d\psi}{dt}\frac{\partial L}{\partial\frac{d\psi}{dt}} \tag{7}$$

Now define $m_0 = m - \frac{1}{2c^2}I\gamma^2\omega^2$. m and m_0 are both constants of the motion since $\frac{d}{dt}(\gamma^2\omega^2) = 0$ as can be seen from eq. (5). m_0 can be thought of as the non-rotating rest mass.

For a Lagrangian try

$$L = -m_0c^2\gamma^{-1} + q\left(\frac{1}{c}\mathbf{v}\cdot\mathbf{A} - \Phi\right) + \frac{1}{2}I\gamma\omega^2 + \frac{Iq}{mc}\boldsymbol{\omega}\cdot\mathbf{B}' \tag{8}$$

where **A** is the vector potential and Φ is the scalar potential.

Using the Euler-Lagrange equations for the particle position which are given by

$$\boldsymbol{\nabla}L - \frac{d}{dt}\boldsymbol{\nabla}_{\mathbf{v}}L = 0 \tag{9}$$

we find that if we only keep linear **E** and **B** fields we obtain the translational eq. (1) by using the relations $\mathbf{B} = \nabla \times \mathbf{A}$ and $\mathbf{E} = -\nabla\Phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}$. $\nabla_{\mathbf{v}}L$ represents the gradient of L with respect to the velocity **v**. For the rotational part the Euler-Lagrange equation for the Euler angles, eq. (6), reduces to the rotational equation (5) if we only keep linear **E** and **B** fields.

The conjugate momentum and Hamiltonian H take the form

$$\mathbf{p}_{\mathbf{v}} = \nabla_{\mathbf{v}}L = \left(m_0c^2 + \frac{1}{2}I\gamma^2\omega^2\right)\frac{1}{c}\gamma\boldsymbol{\beta} + \frac{q}{c}\mathbf{A} + \frac{Iq}{mc^2}\left\{\frac{\gamma}{\gamma+1}\boldsymbol{\omega} \times \mathbf{E} + \frac{\gamma^3}{(\gamma+1)^2}\big(\boldsymbol{\beta}\cdot(\boldsymbol{\omega} \times \mathbf{E})\big)\boldsymbol{\beta}\right\} \qquad (10)$$

$$\mathbf{p}_{\boldsymbol{\omega}} = \nabla_{\boldsymbol{\omega}}L = I\gamma\boldsymbol{\omega} + \frac{Iq}{mc}\mathbf{B}' \qquad (11)$$

$$H = \mathbf{v}\cdot\mathbf{p}_{\mathbf{v}} + \frac{d\phi}{dt}\frac{\partial L}{\partial\frac{d\phi}{dt}} + \frac{d\theta}{dt}\frac{\partial L}{\partial\frac{d\theta}{dt}} + \frac{d\psi}{dt}\frac{\partial L}{\partial\frac{d\psi}{dt}} - L = \mathbf{v}\cdot\mathbf{p}_{\mathbf{v}} + \boldsymbol{\omega}\cdot\nabla_{\boldsymbol{\omega}}L - L$$

$$= \left(m_0c^2 + \frac{1}{2}I\gamma^2\omega^2\right)\gamma + \frac{Iq}{mc}\frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}\cdot(\boldsymbol{\omega} \times \mathbf{E}) + q\Phi \qquad (12)$$

using the relation $\gamma^2 - 1 = \gamma^2\beta^2$, and eq. (7). We want to express the Hamiltonian in terms of the $\mathbf{p}_{\mathbf{v}}$ and $\mathbf{p}_{\boldsymbol{\omega}}$ so that we can quantize the system.

First consider the case of **E** = Φ = 0 in which case the conjugate momentum relations (10) and (11) can be arranged so that

$$\gamma\boldsymbol{\omega} = \frac{1}{I}\mathbf{p}_{\boldsymbol{\omega}} - \frac{q}{mc}\mathbf{B} \qquad (13)$$

$$\boldsymbol{\beta} = c\{(m'c^2 - \frac{q}{mc}\mathbf{p}_{\boldsymbol{\omega}}\cdot\mathbf{B})^2 + c^2\pi^2\}^{-1/2}\boldsymbol{\pi} \qquad (14)$$

again only keeping linear fields.   We have set $\boldsymbol{\pi} = \mathbf{p_v} - \frac{q}{c}\mathbf{A}$ and $m' = m_0 + \frac{1}{2Ic^2}{p_{\boldsymbol{\omega}}}^2$.

Using eq. (12-14) the Hamiltonian takes the form

$$H = \{m'^2c^4 + c^2\pi^2 - \frac{2q}{m}m'c\mathbf{p_\omega} \cdot \mathbf{B}\}^{\frac{1}{2}} = m'c^2(1 + \Pi)^{1/2} \qquad (15)$$

again only keeping linear fields and we have set $\Pi = \frac{1}{m'^2c^2}(\pi^2 - \frac{2q}{mc}m'\mathbf{p_\omega} \cdot \mathbf{B})$.

Assuming that Π is small compared to one we can expand eq. (15) to the form

$$H = m'c^2(1 + \frac{1}{2}\Pi - \frac{1}{8}\Pi^2 + \frac{1}{16}\Pi^3 - \frac{5}{128}\Pi^4)$$

$$= m'c^2\left(1 + \frac{1}{2}\frac{\pi^2}{m'^2c^2} - \frac{1}{8}\frac{\pi^4}{m'^4c^4} + \frac{1}{16}\frac{\pi^6}{m'^6c^6} - \frac{5}{128}\frac{\pi^8}{m'^8c^8}\right)$$

$$+\frac{q}{mc}(-1 + \frac{1}{2}\frac{\pi^2}{m'^2c^2} - \frac{3}{8}\frac{\pi^4}{m'^4c^4} + \frac{5}{16}\frac{\pi^6}{m'^6c^6})\mathbf{p_\omega} \cdot \mathbf{B} \qquad (16)$$

plus higher order terms. We have only kept linear **B** field terms. Eq. (16) will be useful in the next section.

Next consider the case of including the electric field and making an expansion. In this case eqs. (10) and (11) can be combined to yield

$$\boldsymbol{\pi} = \left(m'c - \frac{q}{mc^2}\mathbf{p_\omega} \cdot \mathbf{B}'\right)\gamma\boldsymbol{\beta} + \frac{q}{mc^2}(\frac{1}{\gamma+1}\mathbf{p_\omega} \times \mathbf{E} + \frac{\gamma^2}{(\gamma+1)^2}(\boldsymbol{\beta} \cdot (\mathbf{p_\omega} \times \mathbf{E}))\boldsymbol{\beta}) \qquad (17)$$

again only keeping linear fields.   We can put this into a more convenient form by defining the terms $\mathbf{E}_2 = -\frac{q}{mc^2}\mathbf{p_\omega} \times \mathbf{E}$ and $m_2 = m'c - \frac{q}{mc^2}\mathbf{p_\omega} \cdot \mathbf{B}$ so that eq. (17) takes the form

$$\boldsymbol{\pi} + \frac{1}{\gamma+1}\mathbf{E}_2 = (\mathrm{m}_2 + \frac{\gamma^2}{(\gamma+1)^2}\boldsymbol{\beta}\cdot\mathbf{E}_2)\gamma\boldsymbol{\beta} \tag{18}$$

Expanding γ in terms of β^2 and keeping terms in $\boldsymbol{\beta}$ up to the 6th order, eq. (18) can be written as

$$\mathbf{p}_2 = \frac{1}{8}\beta^2\left(1+\frac{1}{2}\beta^2+\frac{5}{16}\beta^4\right)\mathbf{E}_2 + \{\left(1+\frac{1}{2}\beta^2+\frac{3}{8}\beta^4\right)\mathrm{m}_2 + \frac{1}{4}\left(1+\beta^2+\frac{15}{16}\beta^4\right)\boldsymbol{\beta}\cdot\mathbf{E}_2\}\boldsymbol{\beta} \tag{19}$$

where $\mathbf{p}_2 = \boldsymbol{\pi} + \frac{1}{2}\mathbf{E}_2$.

Now expand $\boldsymbol{\beta}$ in a power series of $\mathbf{p}_2$ to the 6th order and require that eq. (19) be obeyed. By equating powers and only keeping linear terms in the fields, we find the expansion

$$\boldsymbol{\beta} = \left\{\frac{1}{\mathrm{m}_2}\left(1-\frac{1}{2}\frac{\mathrm{p}_2{}^2}{\mathrm{m}_2{}^2}+\frac{3}{8}\frac{\mathrm{p}_2{}^4}{\mathrm{m}_2{}^4}\right) - \frac{1}{4}\frac{1}{\mathrm{m}_2{}^3}\left(1-2\frac{\mathrm{p}_2{}^2}{\mathrm{m}_2{}^2}+\frac{45}{16}\frac{\mathrm{p}_2{}^4}{\mathrm{m}_2{}^4}\right)\mathbf{p}_2\cdot\mathbf{E}_2\right\}\mathbf{p}_2$$

$$-\frac{1}{8}\frac{\mathrm{p}_2{}^2}{\mathrm{m}_2{}^3}(1-\frac{\mathrm{p}_2{}^2}{\mathrm{m}_2{}^2}+\frac{15}{16}\frac{\mathrm{p}_2{}^4}{\mathrm{m}_2{}^4})\mathbf{E}_2 \tag{20}$$

We now want to put eq. (20) into the Hamiltonian.

Expanding γ in terms of β^2 and keeping terms in $\boldsymbol{\beta}$ up to the 6th order, eq. (12) for the Hamiltonian takes the form

$$\mathrm{H} = \left(1+\frac{1}{2}\beta^2+\frac{3}{8}\beta^4+\frac{5}{16}\beta^6\right)\left(\mathrm{m}'\mathrm{c}^2-\frac{\mathrm{q}}{\mathrm{mc}}\mathbf{p}_{\boldsymbol{\omega}}\cdot\mathbf{B}\right) - \frac{1}{4}\beta^2(1+\beta^2)\frac{\mathrm{q}}{\mathrm{mc}}\boldsymbol{\beta}\cdot(\mathbf{p}_{\boldsymbol{\omega}}\times\mathbf{E}) + \mathrm{q}\Phi \tag{21}$$

where we have also used eq. (11). Now use eq. (20) in eq. (21) and using the relations for $\mathbf{p}_2$, $\mathbf{E}_2$ and m_2 we find

$$H = m'c^2\left(1 + \frac{1}{2}\frac{\pi^2}{m'^2c^2} - \frac{1}{8}\frac{\pi^4}{m'^4c^4} + \frac{1}{16}\frac{\pi^6}{m'^6c^6}\right) + \frac{q}{mc}\left(-1 + \frac{1}{2}\frac{\pi^2}{m'^2c^2} - \frac{3}{8}\frac{\pi^4}{m'^4c^4} + \frac{5}{16}\frac{\pi^6}{m'^6c^6}\right)\mathbf{p_\omega}\cdot\mathbf{B}$$

$$-\frac{1}{2}\frac{q}{mm'c^2}\left(1 - \frac{3}{4}\frac{\pi^2}{m'^2c^2} + \frac{5}{8}\frac{\pi^4}{m'^4c^4}\right)\boldsymbol{\pi}\cdot(\mathbf{p_\omega}\times\mathbf{E}) + q\Phi \tag{22}$$

where we have ignored non-linear field terms and used the relation

$\frac{1}{{m_2}^n} = \frac{1}{m'^n c^n}\left(1 + \frac{nq}{mm'c^3}\mathbf{p_\omega}\cdot\mathbf{B}\right)$ where n is a positive integer. Notice that if **E** = Φ = 0 then eq. (22) agrees with the power series expansion of the exact solution, eq. (16). It is interesting that in eq. (22) the **E** and **B** expansions have similar terms, and because the expansion for the **B** part can be calculated to any precision it is possible to guess what the expansion of the **E** part might be to any precision.

IV. Quantization

For quantization we replace $\boldsymbol{\pi}$ by $\hat{\boldsymbol{\pi}} = -i\hbar\boldsymbol{\nabla} - \frac{q}{c}\mathbf{A}$ and, following Bopp and Haag [11], $\mathbf{p_\omega}$ by $-i\hbar\hat{\mathbf{D}}_{\boldsymbol{\omega}}$ where

$$\hat{\mathbf{D}}_{\boldsymbol{\omega}} = \mathbf{x}\left\{\cos(\phi)\frac{\partial}{\partial\theta} + \frac{\sin(\phi)}{\sin(\theta)}\left(\frac{\partial}{\partial\psi} - \cos(\theta)\frac{\partial}{\partial\phi}\right)\right\} + \mathbf{y}\left\{\sin(\phi)\frac{\partial}{\partial\theta} - \frac{\cos(\phi)}{\sin(\theta)}\left(\frac{\partial}{\partial\psi} - \cos(\theta)\frac{\partial}{\partial\phi}\right)\right\}$$

$$+\mathbf{z}\frac{\partial}{\partial\phi} \tag{23}$$

and a hat indicates an operator.

The Hamiltonians we are dealing with have the form

$$H = \sum_{n=0}^{\infty}\left(a_n {p_{\boldsymbol{\omega}}}^{2n} + (\mathbf{b}_n\cdot\mathbf{p_\omega}){p_{\boldsymbol{\omega}}}^{2n}\right) \tag{24}$$

where the terms a_n and $\mathbf{b}_n$ depend upon $\boldsymbol{\pi}$.

We have the Schrodinger type equation

$$i\hbar\frac{\partial\Psi}{\partial t} = \widehat{H}\Psi \tag{25}$$

where $\widehat{H}$ is H turned into an operator. The wavefunction Ψ is a function of the particle position, time and the Euler angles, and for spin 1/2 can be written in the form

$$\Psi = \sum_{m_1=1}^{2}\sum_{m_2=1}^{2}\chi_{m_1m_2}\Psi^{m_1m_2} \tag{26}$$

where the $\chi_{m_1m_2}$ are elements of the matrix χ given by

$$\chi = \frac{1}{\pi}\begin{bmatrix} e^{i(\phi+\psi)/2}\cos(\theta/2) & ie^{i(-\phi+\psi)/2}\sin(\theta/2) \\ ie^{i(\phi-\psi)/2}\sin(\theta/2) & e^{-i(\phi+\psi)/2}\cos(\theta/2) \end{bmatrix} \tag{27}$$

and $\Psi^{m_1m_2}$ is a function of the particle position and time. We also have the relations

$$\widehat{\mathbf{D}}_{\boldsymbol{\omega}}\chi_{m_1m_2} = \frac{1}{2}i\sum_{p=1}^{2}\chi_{m_1p}\,\boldsymbol{\sigma}^{p}{}_{m_2} \tag{28}$$

and

$$\widehat{D}_{\boldsymbol{\omega}}{}^{2}\chi_{m_1m_2} = \widehat{\mathbf{D}}_{\boldsymbol{\omega}}\cdot\widehat{\mathbf{D}}_{\boldsymbol{\omega}}\chi_{m_1m_2} = -\frac{3}{4}\chi_{m_1m_2} \tag{29}$$

where the $\boldsymbol{\sigma}^{p}{}_{m_2}$ represent the elements of the Pauli spin matrices $\boldsymbol{\sigma}$ in vector form. Using eq. (26), eq.(28) and eq. (29) in eq. (24) and eq. (25) we have

$$i\hbar \sum_{m_1=1}^{2} \sum_{m_2=1}^{2} \chi_{m_1 m_2} \frac{\partial}{\partial t} \Psi^{m_1 m_2} = \sum_{n=0}^{\infty} \sum_{m_1=1}^{2} \sum_{m_2=1}^{2} \{a_n (-\hbar^2 \hat{D}_{\boldsymbol{\omega}}{}^2)^n$$

$$- i\hbar (\mathbf{b}_n \cdot \hat{\mathbf{D}}_{\boldsymbol{\omega}})(-\hbar^2 \hat{D}_{\boldsymbol{\omega}}{}^2)^n\} \chi_{m_1 m_2} \Psi^{m_1 m_2}$$

$$= \sum_{n=0}^{\infty} \sum_{m_1=1}^{2} \sum_{m_2=1}^{2} \chi_{m_1 m_2} \{a_n (\tfrac{3}{4}\hbar^2)^n \Psi^{m_1 m_2} + \tfrac{1}{2}\hbar (\tfrac{3}{4}\hbar^2)^n \mathbf{b}_n \cdot \sum_{p=1}^{2} \boldsymbol{\sigma}^{m_2}{}_p \Psi^{m_1 p}\} \quad (30)$$

The $\chi_{m_1 m_2}$ are independent so these equations represent two identical equations for $m_1 = 1$ and 2. If we drop the m_1 idici then Ψ^{m_2} represents the components of a 2x1 matrix which we will represent by Ψ, so that eq. (30) becomes

$$i\hbar \frac{\partial \Psi}{\partial t} = \sum_{n=0}^{\infty} \{a_n (\tfrac{3}{4}\hbar^2)^n + \tfrac{1}{2}\hbar (\tfrac{3}{4}\hbar^2)^n \mathbf{b}_n \cdot \boldsymbol{\sigma}\} \Psi \quad (31)$$

Now we have terms in the Hamiltonian of the form

$$m'^{-n} = (m_0 + \frac{1}{2Ic^2} p_{\boldsymbol{\omega}}{}^2)^{-n} = m_0{}^{-n} (1 + \frac{1}{2Im_0 c^2} p_{\boldsymbol{\omega}}{}^2)^{-n} \quad (32)$$

for some positive integer n. If we assume that $p_{\boldsymbol{\omega}}{}^2$ is much smaller than $2Im_0c^2$ then we can make an expansion of eq. (32) so that

$$m'^{-n} = m_0{}^{-n} \sum_{p=0}^{\infty} c_p \left(\frac{1}{2Im_0 c^2}\right)^p p_{\boldsymbol{\omega}}{}^{2p} \quad (33)$$

for some constants c_p. When we make an operator out of this and put it in the wave equation we will obtain the expression

$$m'^{-n} = m_0{}^{-n}\sum_{p=0}^{\infty} c_p\left(\frac{1}{2Im_0c^2}\right)^p(\tfrac{3}{4}\hbar^2)^p = m_0{}^{-n}(1+\frac{3\hbar^2}{8Im_0c^2})^{-n} = (m_0+\frac{3\hbar^2}{8Ic^2})^{-n} \quad (34)$$

Now consider the exact case with only a magnetic field. In that case from eq. (15) we have

$$H = m'c^2(1+\Pi)^{1/2} = m'c^2\sum_{p=0}^{\infty} d_p\Pi^p$$

$$= m'c^2\sum_{p=0}^{\infty} d_p\frac{1}{m'^{2p}c^{2p}}(\pi^{2p} - \frac{2qp}{mc}\pi^{2(p-1)}m'\mathbf{p_\omega}\cdot\mathbf{B}) \quad (35)$$

for some constants d_p. We are also only keeping linear fields. Using eq. (34) in eq. (35), the Schrodinger like equation (25) then becomes

$$i\hbar\frac{\partial\Psi}{\partial t} = c^2\sum_{p=0}^{\infty} d_p\left(m_0+\frac{3\hbar^2}{8Ic^2}\right)^{1-2p}c^{-2p}(\hat{\pi}^{2p} - \frac{qp}{mc}\hat{\pi}^{2(p-1)}(m_0+\frac{3\hbar^2}{8Ic^2})\hbar\boldsymbol{\sigma}\cdot\mathbf{B})\,\Psi$$

$$= (m_0+\frac{3\hbar^2}{8Ic^2})c^2\sum_{p=0}^{\infty} d_p\left((m_0+\frac{3\hbar^2}{8Ic^2})c\right)^{-2p}\{\hat{\pi}^2 - \frac{q}{mc}(m_0+\frac{3\hbar^2}{8Ic^2})\hbar\boldsymbol{\sigma}\cdot\mathbf{B}\}^p\,\Psi$$

$$= (m_0+\frac{3\hbar^2}{8Ic^2})c^2\{1+(m_0+\frac{3\hbar^2}{8Ic^2})^{-2}c^{-2}(\hat{\pi}^2 - \frac{q}{mc}(m_0+\frac{3\hbar^2}{8Ic^2})\hbar\boldsymbol{\sigma}\cdot\mathbf{B})\}^{1/2}\Psi$$

$$= \{(m_0+\frac{3\hbar^2}{8Ic^2})^2c^4 + \hat{\pi}^2c^2 - \frac{q}{m}c(m_0+\frac{3\hbar^2}{8Ic^2})\hbar\boldsymbol{\sigma}\cdot\mathbf{B})\}^{1/2}\Psi \quad (36)$$

keeping only linear fields.

Now consider the case of no magnetic field so that **B** = **A** = 0. In that case eq. (36) becomes

$$i\hbar\frac{\partial\Psi}{\partial t} = \{(m_0 + \frac{3\hbar^2}{8Ic^2})^2c^4 - \hbar^2c^2\nabla^2\}^{1/2}\Psi \qquad (37)$$

A solution to eq. (37) is $\Psi = \exp(i(\mathbf{p}\cdot\mathbf{x} - Et)/\hbar)$ where **p** is the momentum, **x** the particle position, and

$$E^2 = (m_0 + \frac{3\hbar^2}{8Ic^2})^2c^4 + p^2c^2 \qquad (38)$$

In the case of **p** = 0, E can be taken as the rest energy and $m_0 + \frac{3\hbar^2}{8Ic^2}$ as the rest mass, so set $m = m_0 + \frac{3\hbar^2}{8Ic^2}$. In this case eq. (36) becomes

$$i\hbar\frac{\partial\Psi}{\partial t} = \{m^2c^4 + \hat{\pi}^2c^2 - qc\hbar\boldsymbol{\sigma}\cdot\mathbf{B}\}^{1/2}\Psi \qquad (39)$$

which is the equation for a constant **B** field derived by the Foldy-Wouthuysen method from the Dirac eq. See Case [14] and for example Silenko [15] .

Now consider the case of including a non-zero electric field and using an expansion. For quantization m' gets replaced by m, $\boldsymbol{\pi}$ by $\hat{\boldsymbol{\pi}}$, and $\mathbf{p}_{\boldsymbol{\omega}}$ by $\frac{1}{2}\hbar\boldsymbol{\sigma}$ in the Hamiltonian given by eq. (22). So the corresponding quantum equation is

$$i\hbar\frac{\partial\Psi}{\partial t} = [mc^2\left(1+\frac{1}{2}\frac{\hat{\pi}^2}{m^2c^2}-\frac{1}{8}\frac{\hat{\pi}^4}{m^4c^4}+\frac{1}{16}\frac{\hat{\pi}^6}{m^6c^6}\right)$$

$$+\frac{q}{2mc}\left(-1+\frac{1}{2}\frac{\hat{\pi}^2}{m^2c^2}-\frac{3}{8}\frac{\hat{\pi}^4}{m^4c^4}+\frac{5}{16}\frac{\hat{\pi}^6}{m^6c^6}\right)\hbar\boldsymbol{\sigma}\cdot\mathbf{B}$$

$$-\frac{1}{4}\frac{q}{m^2c^2}\left(1-\frac{3}{4}\frac{\hat{\pi}^2}{m^2c^2}+\frac{5}{8}\frac{\hat{\pi}^4}{m^4c^4}\right)\hbar\boldsymbol{\pi}\cdot(\boldsymbol{\sigma}\times\mathbf{E})+q\Phi]\Psi \qquad (40)$$

Now compare eq. (40) to the Foldy-Wouthuysen transformation of the Dirac equation. Jentschura [16] carries the transformation out to the 8th order and if we take his result for linear constant fields we obtain

$$\hat{H}_{FW} = m+\frac{\hat{\pi}^2}{2m}-\frac{\hat{\pi}^4}{8m^3}+\frac{\hat{\pi}^6}{16m^5}-\frac{5\hat{\pi}^8}{128m^7}+\left(-\frac{1}{2m}+\frac{1}{4}\frac{\hat{\pi}^2}{m^3}-\frac{3}{16}\frac{\hat{\pi}^4}{m^5}+\frac{5}{32}\frac{\hat{\pi}^6}{m^7}\right)e\boldsymbol{\sigma}\cdot\mathbf{B}$$

$$-\left(\frac{1}{4m^2}-\frac{3}{16}\frac{\hat{\pi}^2}{m^4}+\frac{5}{32}\frac{\hat{\pi}^4}{m^6}\right)e\boldsymbol{\pi}\cdot(\boldsymbol{\sigma}\times\mathbf{E})+V \qquad (41)$$

Note that he has set $\hbar = c = 1$ and has used e for q and V for $q\Phi$. We have also only written out the top part of the Foldy-Wouthuysen transformation. As can be seen we get agreement to the orders which have been calculated.

V. Alternate method

This method follows the ideas of Chen and Chiou [9] and gives the same results that we obtain following Bopp and Haag [11]. For g = 2 we can write eq. (2) as

$$\frac{d\mathbf{s}}{dt} = \frac{q}{mc}\mathbf{s}\times\{\frac{1}{\gamma}\mathbf{B}-\frac{1}{\gamma+1}\boldsymbol{\beta}\times\mathbf{E}\} \qquad (42)$$

and following Jackson [10] we can associate an energy U with eq. (42)

$$U = -\frac{q}{mc}\mathbf{s}\cdot\{\frac{1}{\gamma}\mathbf{B} - \frac{1}{\gamma+1}\boldsymbol{\beta}\times\mathbf{E}\} \tag{43}$$

When we include the translational energy and the potential energy associated with the scalar potential Φ we have for the total energy

$$E = m\gamma c^2 - \frac{q}{mc}\mathbf{s}\cdot\left\{\frac{1}{\gamma}\mathbf{B} - \frac{1}{\gamma+1}\boldsymbol{\beta}\times\mathbf{E}\right\} + q\Phi \tag{44}$$

Now define a conjugate momentum **p** based on only the translational Lagrangian which, based on eq. (10) with no spin, takes the form

$$\mathbf{p} = mc\gamma\boldsymbol{\beta} + \frac{q}{c}\mathbf{A} \tag{45}$$

Then setting $\boldsymbol{\pi} = \mathbf{p} - \frac{q}{c}\mathbf{A}$ we have the relationship

$$\gamma = (1-\beta^2)^{-1/2} = (1 - \frac{\pi^2}{m^2c^2\gamma^2})^{-1/2} \tag{46}$$

Solving for γ we find

$$\gamma = \gamma_\pi = (1 + \frac{\pi^2}{m^2c^2})^{1/2} \tag{47}$$

where we have defined γ_π by the eq. (47). The total energy then takes the form

$$E = m\gamma_\pi c^2 - \frac{q}{mc}\mathbf{s}\cdot\left\{\frac{1}{\gamma_\pi}\mathbf{B} - \frac{1}{mc}\frac{1}{\gamma_\pi+1}\frac{1}{\gamma_\pi}\boldsymbol{\pi}\times\mathbf{E}\right\} + q\Phi \tag{48}$$

using $mc\gamma\boldsymbol{\beta} = \boldsymbol{\pi}$.

To quantize the system replace $\boldsymbol{\pi}$ by $\hat{\boldsymbol{\pi}} = -i\hbar\boldsymbol{\nabla} - \frac{q}{c}\mathbf{A}$ and $\mathbf{s}$ by $\frac{1}{2}\hbar\boldsymbol{\sigma}$, so that using the energy E for the Hamiltonian we obtain the quantum equation

$$i\hbar\frac{\partial\Psi}{\partial t} = E\Psi = [m\hat{\gamma}_\pi c^2 - \frac{q\hbar}{2mc}\frac{1}{\hat{\gamma}_\pi}\boldsymbol{\sigma}\cdot\mathbf{B} - \frac{q\hbar}{2m^2c^2}\frac{1}{\hat{\gamma}_\pi+1}\frac{1}{\hat{\gamma}_\pi}\hat{\boldsymbol{\pi}}\cdot(\boldsymbol{\sigma}\times\mathbf{E}) + q\Phi]\Psi \tag{49}$$

where $\hat{\gamma}_\pi = (1 + \frac{\hat{\pi}^2}{m^2c^2})^{1/2}$. If we expand eq. (49) in a power series in $\frac{\hat{\pi}^2}{m^2c^2}$ using our relation for $\hat{\gamma}_\pi$ we obtain eq. (40) plus higher order terms. It is interesting that we get the same relativistic corrections by two different methods.

Conclusion

We have only considered the spin 1/2 case and compared our results to the Foldy-Wouthuysen transformation of the Dirac equation. In principle higher order spins could be considered and compared to non-relativistic expansions of higher order relativistic spin equations. It would be interesting if this method could also be extended to non-constant fields.

The fact that we get agreement with the Foldy-Wouthuysen transformation to the 6th order in 1/c indicates that, at least for constant linear fields, the Dirac equation is equivalent to the canonical quantization of a classical spinning charge to that order of approximation. It would be interesting to expand this to higher orders to see if we continue to get agreement with the higher order Foldy-Wouthuysen terms.

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