On the sequence: $a(n) = 4\,a(n-1) - 2\,a(n-2) - 4\,a(n-3) - a(n-4)$

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December 17, 2024

ABSTRACT: We provide some formulas related to sequence A026937 in OEIS.

I. Introduction: The sequence A026937 in OEIS

The sequence A026937:

$$a(n) = \{1,\ 3,\ 10,\ 30,\ 87,\ 245,\ 676,\ 1836,\ 4925,\ 13\,079,\ 34\,446,\ 90\,090,\ \ldots\} \tag{1}$$

Generating function:

$$G(x) = \frac{1-x}{\left(1-2\,x-x^2\right)^2} \tag{2}$$

Linear recurrence:

$$a(n) = 4\,a(n-1) - 2\,a(n-1) - 4\,a(n-3) - a(n-4) \tag{3}$$

$$a(0) = 1,\ a(1) = 3,\ a(2) = 10,\ a(3) = 30 \tag{4}$$

Some representations:

$$a(n) = \left(\frac{n+2}{8}\right)\left(\left(\sqrt{2}+2\right)\left(1+\sqrt{2}\right)^n - \left(\sqrt{2}-2\right)\left(1-\sqrt{2}\right)^n\right) \tag{5}$$

$$a(n) = \sum_{k=0}^{n}(k+1)\sum_{m=0}^{k}2^m\binom{k}{m}\binom{n-k}{m} \tag{6}$$

$$a(n) = \sum_{k=0}^{\lfloor n/2\rfloor}(k+1)\,2^k\binom{n+2}{n-2\,k} \tag{7}$$

$$a(n) = \sum_{k=0}^{\lfloor n/2\rfloor}\sum_{m=0}^{n-k}(m+1)\binom{m}{k}\binom{n-k}{n-k-m} \tag{8}$$

$$a(n) = (-1)^n\sum_{k=0}^{\lfloor n/2\rfloor}\sum_{m=0}^{n-2\,k}(-1)^m\,(m+1)\,2^m\binom{m+k}{k} \tag{9}$$

$$a(n) = \begin{cases} 1 & n = 0 \\ b(n) - b(n-1) & n \geq 1 \end{cases} \tag{10}$$

where

$$b(n) = \sum_{k=0}^{\lfloor n/2\rfloor}2^{n-2\,k}\,(n-k+1)\binom{n-k}{k} \tag{11}$$

$$b(n) = \sum_{k=0}^{\lfloor n/2 \rfloor} 2^{n-2k} (k+1) \binom{n-k+1}{n-2k} \tag{12}$$

$$b(n) = \sum_{k=0}^{\lfloor n/2 \rfloor} 2^{k} (k+1) \binom{n+3}{n-2k} \tag{13}$$

$$b(n) = \sum_{k=0}^{\lfloor n/2 \rfloor} 2^{n-2k} (n-2k+1) \binom{n-k+1}{k} \tag{14}$$

$$\sum_{k=0}^{n} a(k) = \sum_{k=0}^{\lfloor n/2 \rfloor} (n-k+1)\, 2^{n-2k} \binom{n-k}{k} \tag{15}$$

$$\sum_{k=0}^{n} a(k) = \frac{1}{16} \left(\left(8 - 5\sqrt{2}\right)\left(1 - \sqrt{2}\right)^n + \left(8 + 5\sqrt{2}\right)\left(1 + \sqrt{2}\right)^n + \left(6 - 4\sqrt{2}\right) n \left(1 - \sqrt{2}\right)^n + \left(6 + 4\sqrt{2}\right) n \left(1 + \sqrt{2}\right)^n \right) \tag{16}$$

Linear recurrence for $A(n) = a(2n)$:

$$A(n) = 12\, A(n-1) - 38\, A(n-2) + 12\, A(n-3) - A(n-4) \tag{17}$$

$$A(0) = 1,\ A(1) = 10,\ A(2) = 87,\ A(3) = 676 \tag{18}$$

II. Pi formulas

Recall that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \ldots \tag{19}$$

we have

$$\frac{\pi \sqrt{2}}{32} + \frac{\sqrt{3}}{16} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \left(\sqrt{3} - \sqrt{2}\right)^{2n+1} a(2n) \tag{20}$$

$$\frac{\pi \sqrt{2}}{48} + \frac{\sqrt{7}}{32} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \left(\sqrt{7} - \sqrt{6}\right)^{2n+1} a(2n) \tag{21}$$

$$\frac{\pi \sqrt{2}}{64} + \frac{\sqrt{7 + 4\sqrt{2}}}{4 + 4\left(7 + 4\sqrt{2}\right)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \left(\sqrt{7 + 4\sqrt{2}} - \sqrt{2} - 2\right)^{2n+1} a(2n) \tag{22}$$

$$\frac{\pi \sqrt{2}}{96} + \frac{\sqrt{9 - 4\sqrt{3}}}{32} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \left(\sqrt{15 + 8\sqrt{3}} - \sqrt{6} - 2\sqrt{2}\right)^{2n+1} a(2n) \tag{23}$$

III. Endnote

for $\alpha = 1 - \sqrt{2 - 2\sqrt{2 - \sqrt{3}}}$ we have

$$\pi = 24 \sum_{n=0}^{\infty} (-1)^n a(n) f(n) \tag{24}$$

where

$$f(n) = \int_0^{\alpha} \frac{x^n (1-x)\left(1 + 2x - x^2\right)}{(1+x)\sqrt{x(2-x)\left(2 + 2x - x^2\right)}} \, dx \tag{25}$$

for $\beta = 1 - \sqrt{4\sqrt{3} - 6}$ we have

$$\frac{1}{8} + \frac{\pi}{24} = \sum_{n=0}^{\infty} (-1)^n a(n) g(n) \tag{26}$$

where

$$g(n) = \int_0^{\beta} \frac{x^n (1-x)}{(1+x)\sqrt{x(2-x)}} \, dx \tag{27}$$

for $s = \left(2 - \sqrt{3}\right)\left(\sqrt{2} - 1\right)$ we have

$$\frac{3\sqrt{2} + \left(2 + \sqrt{2}\right)\pi}{72 + 48\sqrt{2}} = \sum_{n=0}^{\infty} (-1)^n a(n) \left(\sqrt{2} + 1\right)^n s^{2n+1} \left(\frac{1}{2n+1} - \frac{2s^2}{2n+3} + \frac{s^4}{2n+5}\right) \tag{28}$$

If ${}_2F_1$ is the Gauss hypergeometric function then

$$\frac{6 + 3\sqrt{2} - \sqrt{2}\pi}{24} = \sum_{n=0}^{\infty} \frac{a(n)\left(\sqrt{2} + 1\right)^n \left(2 - \sqrt{3}\right)^{2n+1}}{2n+1} \, {}_2F_1\left(-2, n + \frac{1}{2}, n + \frac{3}{2}, \left(\sqrt{2} + 1\right)^2 \left(2 - \sqrt{3}\right)^2\right) \tag{29}$$

IV. References

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