The Geodesic Principle and the Nature of Passive Mass

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Abstract

The geodesic principle represents an essential aspect of general relativity and is the physical manifestation of the space-time manifold but can also be considered as the metric field effect on the passive mass of a freely falling test particle. - The equation of motion is derived from the given stress-energy tensor field of an isolated body with the help of its moments in the near limit case, on the basis of the universal conservation condition. Then the reduced stress-energy tensor that is based on the energy density of the body matter is being used in the context of its local energy balance to get the global solution in the form of the geodesic equation. Finally, the influence of an external force field on such a solution is presented.

I. Introduction

In A. Einstein and N. Rosen "The Particle Problem in the General Theory of Relativity" one can read: "One of the imperfections of the original relativistic theory of gravitation was that as a field theory it was not complete; it introduced the independent postulate that the law of motion of a particle is given by the equation of the geodesic."[1]. This postulate says: *Free massive point particles traverse timelike geodesics.* Einstein tried to remedy that shortcoming without success. "Over the last century numerous ostensible proofs claiming to have derived the geodesic principle from Einstein's field equations have been developed. (…) Grouping these results into three major families, which I refer to as (1) limit operation proofs, (2) 0th-order proofs, and (3) singularity proofs, (…) none of these strategies successfully demonstrates the geodesic principle, canonically interpreted as a dynamical law that massive bodies must actually follow geodesic paths in Einstein's theory." [2] "By reviewing the three major classes of proof, we have seen that would-be geodesic following bodies are forced either (i) to meet unrealistically restrictive special-case conditions, (ii) to have no matter-energy at all (i.e. vanish), (iii) to violate Einstein's field equations, or (iv) to be located on paths that don't just fail to be geodesic but fail to exist in the space-time manifold at all." [2] "Though the geodesic principle can be recovered as theorem in general relativity, it is not a consequence of Einstein's equation (or the conservation principle) alone. Other assumptions are needed to drive the theorems in question." [3]. The following is a proof of the geodesic principle and its consequences for the understanding of passive mass. The proof is not canonical in the sense that it does not directly confirm the solution but only its sufficient convergence, which is linked to the diameter: Ø of a spatial domain that encompasses the body and the current point on the geodesic. The prerequisite here is that the solution at least converges with $O(\text{Ø})$. The proof can be assigned to the family of limit operation proofs. It is not based on the distributions but on density moments. Compared to the Geroch-Jang theorem, it has the advantage of not requiring the "strengthened dominant energy condition" [3] but only the natural condition of the positive minimal body energy: $E_0 = mc^2 > 0$ in the locally inertial (LI) proper frame of reference is applied. It is presumed as well that in the vicinity of the geodesic without the gravitational influence of the body, the given function of metric field is sufficiently smooth. Furthermore, the compatibility with the weak equivalence principle is required. The physically relevant case in which the body density is constrained: $m = O(\text{Ø}^3)$ is analyzed here. It is demonstrated that even for $m = O(\text{Ø})$ the gravity field originating from the body can be sufficiently separated from the foreign gravitational field so that the test body problem can be limited to such an extent that because of the sufficient grade of convergence it has an insignificant share in the overall solution. The question of whether the solution converges to a geodesic at all when the mass is constrained to $O(\text{Ø}^0)$, which would correspond to the canonical account [2], is left open here. - In the first part: (1,2) a suitable stationary LI coordinate system is constructed, in the second part: (3,4) the approximation uncertainties and errors as well as deviations of temporal SE-tensor derivative are estimated, and in the third part: (5) the geodesic principle is confirmed for the SE-tensor and the geodesic equation derived from the reduced SE-tensor. For the sake of simplicity in the following the natural units are used, furthermore to provide better overview, in the summation notation the corresponding indices are additionally crossed out when summing.

II. The physics behind the geodesic principle

1) The locally gauged, stationary locally (in the $\Delta\tau$ span) inertial coordinate system: $\underline{x}^{\hat{\mu}}:\ \mathcal{P} \mapsto x^{\hat{\mu}}(\mathcal{P})$

▷ a) A space-time coordinate system: $\underline{x}^{\mu}$ its base: $\boldsymbol{e}_{\alpha}$ and metric: $g_{\alpha\beta}$. $\eta_{\alpha\beta} \equiv [\mathrm{diag}(-1,1,1,1)]_{\alpha\beta}$

$$\tau \in \mathbb{R}\ ,\ \underline{x}^{\mu}:\ \forall x_{,}{}^{\mu}\big(\mathcal{P}(\tau)\big)\ \exists\, \Lambda_{\mu}^{\mu'}\big(x^{\mu} \to x_{,}{}^{\mu}\big)\ ,\ g_{\alpha\beta} \equiv \boldsymbol{e}_{\alpha}\cdot\boldsymbol{e}_{\beta} = \Lambda_{\alpha}^{\alpha'}\Lambda_{\beta}^{\beta'}\eta_{\alpha'\beta'}\ ; \qquad (1.1)$$

▷ b) For *any* $\underline{x}^{\mu}$, the *stationary locally inertial* (SLI) coordinate system: $\underline{x}^{\hat{\mu}}$ is (implicitly) pre-defined

$$x^{\bar{\bar{0}}} := \tau \qquad \to \qquad \mathcal{P}\big(x^{\bar{\bar{0}}},\, x^{\bar{\bar{n}}} = 0\big) := \mathcal{P}(\tau) \qquad (1.2a,b)$$

$$\mathbf{e}_{\bar{\bar{\alpha}}} \equiv \mathbf{e}_{\bar{\bar{\alpha}}}(\tau) := \boldsymbol{e}_{\bar{\bar{\alpha}}}\big(x^{\bar{\bar{0}}},\ x^{\bar{\bar{n}}} \to 0\big) \quad ; \quad \Lambda_{\bar{\bar{\nu}}}^{\mu} \equiv \Lambda_{\bar{\bar{\nu}}}^{\mu}(\tau) := \Lambda_{\bar{\bar{\nu}}}^{\mu}\big(x^{\bar{\bar{0}}},\ x^{\bar{\bar{n}}} \to 0\big) \qquad (1.3a,b)$$

(1.8a)
(1.9a) $$\boldsymbol{\Delta}\mathcal{P}(\tau) = \boldsymbol{e}_{\alpha}\big(\mathcal{P}(\tau)\big)\Lambda_{\bar{\bar{0}}}^{\alpha}(\tau)\Delta\tau := \mathbf{e}_{\bar{\bar{0}}}(\tau)\Delta\tau\ \big|\ \Delta\tau \to 0 \quad \to \quad \frac{\eth x^{\mu}\big(\mathcal{P}(\tau)\big)}{\eth\tau} = \Lambda_{\bar{\bar{0}}}^{\mu}(\tau) \qquad (1.4a,b)$$
(2.5a)
(2.5b) $$x^{\mu} =:\ x^{\mu}\big(\mathcal{P}(\tau)\big) + \Lambda_{\bar{\bar{n}}}^{\mu}(\tau)x^{\bar{\bar{n}}} + 2^{-1}\Lambda_{\bar{\bar{n}},\bar{\bar{m}}}^{\mu}(\tau)x^{\bar{\bar{n}}}x^{\bar{\bar{m}}}\ \big|\ \big|x^{\hat{k}}\big| \leq \text{Ø}_0 : \text{"small enough"} \qquad (1.5)$$

▷ c) Conditions for the SLI base in the (infinitesimal) proximity: $x^{\bar{\bar{n}}} \to 0$; of any point: $\mathcal{P}(\tau)$ of the trajectory following the geodesic (a kind of situation like inside a freely moving non-rotating spaceship)

$$\mathrm{g}_{\bar{\bar{\alpha}}\bar{\bar{\beta}}}(\tau) := g_{\bar{\bar{\alpha}}\bar{\bar{\beta}}}\big(x^{\bar{\bar{0}}},\ x^{\bar{\bar{n}}} \to 0\big) \equiv \mathbf{e}_{\bar{\bar{\alpha}}}\cdot\mathbf{e}_{\bar{\bar{\beta}}} := \eta_{\bar{\bar{\alpha}}\bar{\bar{\beta}}} \quad \to \quad \mathbf{e}_{\bar{\bar{0}}}\cdot\mathbf{e}_{\bar{\bar{0}}} = -1 \qquad (1.6a,b)$$

(1.6b)
(2.5b) $$\mathbf{e}_{\tau}(\tau) := \mathbf{e}_{\bar{\bar{0}}}(\tau) \quad : \quad \frac{\eth\mathbf{e}_{\tau}}{\eth\tau} \equiv \frac{\partial\mathbf{e}_{\bar{\bar{0}}}}{\partial x^{\bar{\bar{0}}}} = 0 \quad \to \quad \Gamma^{\bar{\bar{\gamma}}}{}_{\bar{\bar{0}}\bar{\bar{0}}}(\tau) := \Gamma^{\bar{\bar{\gamma}}}{}_{\bar{\bar{0}}\bar{\bar{0}}}\big(x^{\bar{\bar{0}}},\ x^{\bar{\bar{n}}} \to 0\big) = 0 \qquad (1.7a,b)$$

(1.6a) $$\frac{\partial\Lambda_{\bar{\bar{\nu}}}^{\mu}}{\partial x^{\bar{\bar{0}}}} \equiv \frac{\eth\Lambda_{\bar{\bar{\nu}}}^{\mu}}{\eth\tau} \quad : \quad \frac{\partial\mathbf{e}_{\bar{\bar{\alpha}}}}{\partial x^{\bar{\bar{0}}}} \equiv \frac{\partial\mathbf{e}_{\bar{\bar{0}}}}{\partial x^{\bar{\bar{\alpha}}}} = 0 \quad \to \quad \Gamma^{\bar{\bar{\gamma}}}{}_{\bar{\bar{\alpha}}\bar{\bar{0}}}(\tau) := \Gamma^{\bar{\bar{\gamma}}}{}_{\bar{\bar{\alpha}}\bar{\bar{0}}}\big(x^{\bar{\bar{0}}},\ x^{\bar{\bar{n}}} \to 0\big) = 0 \qquad (1.8a,b)$$

$$\Lambda_{\bar{\bar{m}},\bar{\bar{n}}}^{\mu} \equiv \Lambda_{\bar{\bar{n}},\bar{\bar{m}}}^{\mu} := \frac{\partial\Lambda_{\bar{\bar{n}}}^{\mu}}{\partial x^{\bar{\bar{m}}}} \quad : \quad \frac{\partial\mathbf{e}_{\bar{\bar{\alpha}}}}{\partial x^{\bar{\bar{m}}}} \equiv \frac{\partial\mathbf{e}_{\bar{\bar{m}}}}{\partial x^{\bar{\bar{\alpha}}}} = 0 \quad \to \quad \Gamma^{\bar{\bar{\gamma}}}{}_{\bar{\bar{\alpha}}\bar{\bar{\beta}}}(\tau) := \Gamma^{\bar{\bar{\gamma}}}{}_{\bar{\bar{\alpha}}\bar{\bar{\beta}}}\big(x^{\bar{\bar{0}}},\ x^{\bar{\bar{n}}} \to 0\big) = 0 \qquad (1.9a,b)$$

▷ d) The SLI gauge transformation of the $\underline{x}^{\bar{\bar{n}}}$ SLI coordinates and the SLI Lorenz gauge as its example

$$\big|x^{\hat{k}}\big| \leq \text{Ø}_0 \quad \Rightarrow \quad x^{\hat{\mu}} := x^{\bar{\bar{\mu}}} + \hat{\Delta}x^{\bar{\bar{\mu}}}\ \big|\ \hat{\Delta}x^{\bar{\bar{\mu}}}(\tau,0) = 0\ ,\ \hat{\Delta}x^{\bar{\bar{\mu}}}{}_{,\bar{\bar{\nu}}}(\tau,0) = 0\ ,\ \hat{\Delta}x^{\bar{\bar{\mu}}}{}_{,\bar{\bar{\nu}},\bar{\bar{\kappa}}}(\tau,0) = 0 \qquad (1.10)$$

(4.1) $$\hat{\Delta}x^{\bar{\bar{\mu}}}(\tau):\ \bar{h}^{\hat{\alpha}\hat{\nu}}{}_{,\hat{\nu}} = 0 \qquad (1.11)$$

2) General definitions in the context of the body stress-energy (SE-)tensor field: $T^{\mu\nu}(x^{\mu}) \equiv T^{\nu\mu}(x^{\mu})$

▷ a) The convex spatial domain: $\underline{V}(\tau)$ of the *minimal* diameter: Ø, containing the whole body *and* $\mathcal{P}(\tau)$

$$\underline{V} := \underline{V}(\tau):\ \underline{V} \cup \partial\underline{V} = \underline{V} \in \underline{\boldsymbol{V}}\quad ,\ \mathcal{P}\big(x^{\hat{\mu}}\big) \in \underline{V}(\tau) \ \Rightarrow\ \mathcal{P}\big(x^{\hat{\mu}}\big) \equiv \mathcal{P}\big(\tau, x^{\hat{n}}\big)\ ; \qquad (2.1)$$

$$\underline{V}(\tau):\ \mathcal{P}\big(\tau, x^{\hat{n}}\big) \in \Big(\partial\underline{V} \cup \big(\sim\underline{V}\big)\Big) \ \Rightarrow\ T^{\alpha\beta}\big(\tau, x^{\hat{n}}\big) = 0\ ; \qquad (2.2)$$

The body diameter: $$\mathrm{d} := \mathrm{d}(\tau) :\leq \text{Ø} := \text{Ø}(\tau) := \text{Ø}\Big(\underline{V}(\tau)\Big) :\leq \text{Ø}_0 \qquad (2.3)$$

▷ b) The notation of a spatial integral on the volume: $\underline{V}$ that is embedded in its space-time domain: $\underline{\boldsymbol{V}}$

$$< f > := \int_{\underline{V}} f\ \big|\eth\underline{V}\big| \qquad (2.4)$$

▷ c) Synchronizing (initial) condition for $\underline{x}^{\hat{\mu}}\big(x^{\bar{\bar{0}}} = \tau_0\big)$, which codetermine the matrix: $\Lambda_{\bar{\bar{\nu}}}^{\mu}$ at $\tau = \tau_0$

(1.4,5) $$\begin{cases} \mathcal{P}(\tau_0)\ :\ < x^{\hat{n}}T^{\hat{0}\hat{0}}\big(\tau_0, x^{\hat{n}}\big) > = 0 & (2.5a) \\ \mathbf{e}_{\bar{\bar{0}}}(\tau_0):\ < T^{\hat{n}\hat{0}}\big(\tau_0, x^{\hat{n}}\big) > \equiv < T^{\hat{0}\hat{n}}\big(\tau_0, x^{\hat{n}}\big) > = 0 & (2.5b) \end{cases}$$

If the SLI coordinate system fulfills this condition at $\tau = \tau_0$, it can on $\underline{V}(\tau \to \tau_0)$ be referred to as the (locally inertial momentarily comoving) *proper frame* (of reference) and after that, as long as $\text{Ø} \leq \text{Ø}_0$ is satisfied, as the *locally inertial* (LI) *comoving frame* (of reference). The parameter: τ is the proper time.

3) The locally inertial coordinates: the vector integration and the SE-tensor divergence spatial integral

▷ a) The flat base approximation and factoring of the local coordinates base out of the spatial integral
The norm: $|x^{\bar{n}}|$ is defined as the *spatial* distance (shortest length) from $\mathcal{P}(x^{\bar{\mu}})$ to the $\mathcal{P}(x^{\bar{0}}, x^{\bar{n}} = 0)$.

$$\mathbf{e}_\alpha := \boldsymbol{e}_\alpha\big(x^\alpha \to x_,{}^\alpha(\mathcal{P}_0)\big) \quad \to \quad \frac{\partial \mathbf{e}_\alpha}{\partial x^\beta} \equiv \Gamma^{\gamma}{}_{\alpha\beta}\mathbf{e}_\gamma \equiv \Gamma^{\gamma}{}_{\beta\alpha}\mathbf{e}_\gamma := \Gamma^{\gamma}{}_{\alpha\beta}\big(x^\alpha \to x_,{}^\alpha(\mathcal{P}_0)\big)\,\mathbf{e}_\gamma \tag{3.1a,b}$$

$$\boldsymbol{e}_\alpha = \mathbf{e}_\alpha + \Gamma^{\mu}{}_{\alpha\nu}\mathbf{e}_\mu x^\nu + \widetilde{\boldsymbol{O}}\left(2^{-1}\left|\Gamma^{\mu}{}_{\alpha\kappa,\gamma} + \Gamma^{\beta}{}_{\alpha\kappa}\Gamma^{\mu}{}_{\beta\gamma}\right|\mathbf{e}_\mu|x^\kappa x^\gamma|\right) \tag{3.2}$$

$$\underline{x}^\mu \to \underline{x}^{\bar{\mu}}(\mathcal{P}_0) : x^{\bar{\mu}}(\mathcal{P} = \mathcal{P}_0) = 0 \;,\; \Gamma^{\bar{\gamma}}{}_{\bar{\alpha}\bar{\beta}} = 0 \;; \tag{3.3}$$

$$\boldsymbol{e}_{\bar{\alpha}} = \mathbf{e}_{\bar{\alpha}} + \widetilde{\boldsymbol{O}}\big(2^{-2}\big|(\Gamma^{\bar{\mu}}{}_{\bar{\alpha}\bar{\kappa},\bar{\gamma}} + \Gamma^{\bar{\mu}}{}_{\bar{\alpha}\bar{\gamma},\bar{\kappa}}) - \mathrm{R}^{\bar{\mu}}{}_{\bar{\alpha}\bar{\kappa}\bar{\gamma}}\big||x^\kappa||x^\gamma|\mathbf{e}_{\bar{\mu}}\big) = \mathbf{e}_{\bar{\alpha}} + \tilde{O}^{\bar{\mu}}{}_{\bar{\alpha}}\big(|x^{\bar{0}}|^2 + |x^{\bar{n}}|^2\big)\mathbf{e}_{\bar{\mu}} \tag{3.4}$$

$$\big(x^{\bar{0}} \leftarrow 0\big) \to \boldsymbol{e}_{\bar{\alpha}} = \mathbf{e}_{\bar{\alpha}} + \tilde{O}^{\bar{\mu}}{}_{\bar{\alpha}}(|x^{\bar{n}}|^2)\mathbf{e}_{\bar{\mu}} \tag{3.5}$$

$$Y^\alpha :\in \mathbb{R}^4 \to \;< Y^{\bar{\alpha}}\boldsymbol{e}_{\bar{\alpha}} > = < Y^{\bar{\alpha}}\mathbf{e}_{\bar{\alpha}} > + < Y^{\bar{\alpha}}\,\widetilde{\boldsymbol{O}}_{\bar{\alpha}}(|x^{\bar{n}}|^2) > = < Y^{\bar{\alpha}} > \mathbf{e}_{\bar{\alpha}} + \|Y^{\bar{\alpha}}\|\,\widetilde{\boldsymbol{O}}_{\bar{\alpha}}(\emptyset^2) \tag{3.6}$$

(1.9b) (3.3)
$$\underline{\underline{x^{\hat{\mu}} - x_,{}^{\hat{\mu}}\big(\mathcal{P}(\tau)\big)}} \subset \underline{x^{\bar{\mu}}\big(\mathcal{P}(\tau)\big)} \tag{3.7}$$

This means that (3.6) is valid for $\underline{x}^{\hat{\mu}}$ too. For the rest the LI coordinates components can be separated:

(3.6)
$$\forall\hat{\alpha} \qquad < Y^{\hat{\alpha}}\boldsymbol{e}_{\hat{\alpha}} > = \Big(< Y^{\hat{\alpha}} > + \tilde{O}^{\hat{\alpha}}(\|Y\|\emptyset^2)\Big)\mathbf{e}_{\hat{\alpha}} \tag{3.8}$$

The uncertainty: $\tilde{O}^{\hat{\alpha}}(*\,\emptyset^2)$ is the price for making the vector integration on curved domain reasonable.

▷ b) The total neutrality of spatial divergence for the body SE-tensor field in the LI comoving frame

(2.2) Gauss
$$T^{\hat{\alpha}\hat{\beta}}\big(\mathcal{P} \in \underline{S} := \partial\underline{V}\big) \equiv 0 \to\; < T^{\hat{\mu}\hat{n}}{}_{,\hat{n}}\big(\tau, x^{\hat{k}}\big) > + \tilde{O}^{\hat{\mu}}(\mathrm{m}\emptyset^2) = \oint_{\underline{S}} T^{\hat{\mu}\hat{n}}\big(\tau, x^{\hat{k}}\big)\, n_{\hat{n}}\big|\partial\underline{S}\big| \equiv 0 \tag{3.9}$$

4) Finding the $O(\emptyset^2)$ approximation of the four-momentum temporal partial derivative for a small body

▷ a) The test body problem: the cross effect of gravity fields, its convergence upper bound estimation
Based on the equation for weak gravitational field and its source given by a hypothetical scalable massive body with a density limit (no singularities), which on closer inspection can be treated as the superposition of infinitesimally small dispersed point masses with their $\partial h_{\hat{\mu}\hat{\nu}} = O_{\hat{\mu}\hat{\nu}}(\partial \mathrm{m}/r)$ partial fields; by negligible external reflections, the following basic estimates for the metric deviation and its derivatives can be found out in the originally (without the active body mass as the field source) LI comoving frame.

[5] (1.11)
$$\eta^{\hat{\mu}\hat{\nu}}\,\bar{h}^{\hat{\alpha}\hat{\beta}}{}_{,\hat{\mu},\hat{\nu}} \equiv \eta^{\hat{\mu}\hat{\nu}}\left(h^{\hat{\alpha}\hat{\beta}} - \frac{1}{2}\eta^{\hat{\alpha}\hat{\beta}}h^{\hat{\gamma}}{}_{\hat{\gamma}}\right)_{,\hat{\mu},\hat{\nu}} = -16\pi T^{\hat{\alpha}\hat{\beta}} \tag{4.1}$$

(5.1)
$$\mathrm{m} := \mathrm{m}_0\emptyset_0^{-3}\emptyset^3 = O(\emptyset^3) \quad\to\quad T^{\hat{\alpha}\hat{\beta}} := O^{\hat{\alpha}\hat{\beta}}(\emptyset^0) \in \mathbb{R}^{4\times4} \tag{4.2a,b}$$

$$g_{\hat{\alpha}\hat{\beta}} := \eta_{\hat{\alpha}\hat{\beta}} + \Delta g_{(ex)\hat{\alpha}\hat{\beta}} + \Delta g_{(in)\hat{\alpha}\hat{\beta}} := \eta_{\hat{\alpha}\hat{\beta}} + h_{(ex)\hat{\alpha}\hat{\beta}} + h_{(in)\hat{\alpha}\hat{\beta}} = \eta_{\hat{\alpha}\hat{\beta}} + O_{\hat{\alpha}\hat{\beta}}\big(\emptyset^2 + \mathrm{m}\emptyset^{-1}\big) \tag{4.3}$$

$$h_{(ex)\hat{\alpha}\hat{\beta}} \;\hat{=}\; O(\emptyset^2) \;\to\; h_{(ex)\hat{\alpha}\hat{\beta},\hat{\gamma}} \;\hat{=}\; O(\emptyset) \;\to\; h_{(ex)\hat{\alpha}\hat{\beta},\hat{\gamma},\hat{\varepsilon}} \;\hat{=}\; O(\emptyset^0) \;\to\; h_{(ex)\hat{\alpha}\hat{\beta},\hat{\gamma},\hat{\varepsilon},\hat{\kappa}} \;\hat{=}\; O(\emptyset^0) \tag{4.4a..d}$$

$$h_{(in)\hat{\alpha}\hat{\beta}} \;\hat{=}\; O(\mathrm{m}\emptyset^{-1}) \to h_{(in)\hat{\alpha}\hat{\beta},\hat{\gamma}} \;\hat{=}\; O(\mathrm{m}\emptyset^{-2}) \to h_{(in)\hat{\alpha}\hat{\beta},\hat{\gamma},\hat{\varepsilon}} \;\hat{=}\; O(\mathrm{m}\emptyset^{-3}) \to h_{(in)\hat{\alpha}\hat{\beta},\hat{\gamma},\hat{\varepsilon},\hat{\kappa}} \;\hat{=}\; O(\mathrm{m}\emptyset^{-4}) \tag{4.5a..d}$$

This two fields: of the body (internal) and of the externally imposed curvature, are superimposed resulting in two partial Christoffel symbol fields and creating additionally the cross-term: $\Delta_{(x)}$ in which range the separation between the two fields is no longer definite. The degree of convergence of this term and its derivatives can be determined. Since (due to Newton's first law) the internal field itself in an inertial system can have no effect on the overall motion, it is arbitrary omitted but the cross-term remains valid.

$$\Gamma^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta}} := \Gamma_{(in)}{}^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta}} + \Gamma_{(ex)}{}^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta}} + \Delta_{(x)}{}^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta}} \tag{4.6}$$

(4.7)
$$\Gamma^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta}} \approx \frac{\{\eta_{\hat{\mu}\hat{\nu}} - h_{(in)\hat{\mu}\hat{\nu}} - h_{(ex)\hat{\mu}\hat{\nu}}\}^{\hat{\mu}\hat{\nu}}}{2}\Big(h_{(in)\hat{\nu}\hat{\alpha},\hat{\beta}} + h_{(in)\hat{\nu}\hat{\beta},\hat{\alpha}} - h_{(in)\hat{\alpha}\hat{\beta},\hat{\nu}} + h_{(ex)\hat{\nu}\hat{\alpha},\hat{\beta}} + h_{(ex)\hat{\nu}\hat{\beta},\hat{\alpha}} - h_{(ex)\hat{\alpha}\hat{\beta},\hat{\nu}}\Big)$$

$$\Delta_{(x)}{}^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta}} \approx \frac{\{h_{(in)\dots}\}^{\hat{\mu}\hat{\nu}}}{2}\left(h_{(ex)\hat{\alpha}\hat{\beta},\hat{\nu}} - h_{(ex)\hat{\nu}\hat{\alpha},\hat{\beta}} - h_{(ex)\hat{\nu}\hat{\beta},\hat{\alpha}}\right) + \frac{\{h_{(ex)\dots}\}^{\hat{\mu}\hat{\nu}}}{2}\left(h_{(in)\hat{\alpha}\hat{\beta},\hat{\nu}} - h_{(in)\hat{\nu}\hat{\alpha},\hat{\beta}} - h_{(in)\hat{\nu}\hat{\beta},\hat{\alpha}}\right)$$

(4.4,5,6) $$\left(\Gamma_{(in)}{}^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta}} \leftarrow 0\right) \quad \rightarrow \quad \Gamma^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta}} = \Gamma_{(ex)}{}^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta}} + \Delta_{(x)}{}^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta}} = \Gamma_{(ex)}{}^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta}} + O^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta}}(\mathrm{m}\emptyset^{0})$$ ☝(4.8) (4.9a)

$$\rightarrow \quad \Gamma^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta},\hat{\gamma}} = \Gamma_{(ex)}{}^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta},\hat{\gamma}} + O^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta}\hat{\gamma}}(\mathrm{m}\emptyset^{-1}) \quad \rightarrow \quad \Gamma^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta},\hat{\gamma},\hat{\varepsilon}} = \Gamma_{(ex)}{}^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta},\hat{\gamma},\hat{\varepsilon}} + O^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\varepsilon}}(\mathrm{m}\emptyset^{-2}) \quad (4.9b,c)$$

Based on (4.9) it can be shown, that in (4.19) the resulting cross-connection error would converge one degree faster than the approximation error there and it is also worth noting that even for m = $O(\emptyset)$ the solutions (5.8,21) would converge with $O(\emptyset)$. Consequently, the term: $\Delta_{(x)}{}^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta}}$ is neglected from here on.

▷ b) The approximate factoring of the Christoffel symbol out of the spatial integral on the volume: $\underline{V}$

(1.6a) $$\Gamma^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta}}(\tau, x^{\hat{n}}) = \Gamma^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta}}(\mathrm{g}_{\hat{*}\hat{*}}, \mathrm{g}_{\hat{*}\hat{*},\hat{*}}) + \Gamma^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta},\hat{n}} x^{\hat{n}} + 2^{-1} O\left(\left|\Gamma^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta},\hat{k},\hat{l}}\right| |x^{\hat{k}} x^{\hat{l}}|\right) := \Gamma_{(ex)}{}^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta}} \quad (4.10)$$

$$< \Gamma^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta}} T^{\hat{\gamma}\hat{\nu}} > = \Gamma^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta}} < T^{\hat{\gamma}\hat{\nu}} > + \Gamma^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta},\hat{n}} < x^{\hat{n}} T^{\hat{\gamma}\hat{\nu}} > + O^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta}}(\|T^{\hat{\gamma}\hat{\nu}}\|\emptyset^2) \quad (4.11)$$

$$\mathrm{T}^{\hat{\gamma}\hat{\nu}} := < T^{\hat{\gamma}\hat{\nu}} > \quad \rightarrow \quad < \Gamma^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta}} T^{\hat{\gamma}\hat{\nu}} > = \Gamma^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta}} \mathrm{T}^{\hat{\gamma}\hat{\nu}} + \Gamma^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta},\hat{n}} \mathrm{T}^{\hat{\gamma}\hat{\nu}\langle\hat{n}\rangle} + O^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta}}(\|T^{\hat{\gamma}\hat{\nu}}\|\emptyset^2) \quad (4.12a,b)$$

With (1.9b) it leads to the upper bound estimation of deviation of the temporal partial derivative (4.19).

(4.11) (1.9b) $$< \Gamma^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta}} T^{\hat{\gamma}\hat{\nu}} > = O\left(\left|\Gamma^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta},\hat{n}}\right| \|T^{\hat{\gamma}\hat{\nu}}\|\emptyset\right) + O^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta}}(\|T^{\hat{\gamma}\hat{\nu}}\|\emptyset^2) = O^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta}}(\|T^{\hat{\gamma}\hat{\nu}}\|\emptyset) \quad (4.13)$$

(4.2) $$< \Gamma^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta}} T^{\hat{\alpha}\hat{\beta}} > = O^{\hat{\mu}}(\mathrm{m}\emptyset) \quad ; \quad < \Gamma^{\hat{\beta}}{}_{\hat{\alpha}\hat{\beta}} T^{\hat{\mu}\hat{\alpha}} > = O^{\hat{\mu}}(\mathrm{m}\emptyset) \quad (4.14a,b)$$

▷ c) The temporal partial derivative of the four-momentum, obtained from the conservation condition

$$T^{\mu\beta}{}_{;\beta} \equiv T^{\mu\beta}{}_{,\beta} + \Gamma^{\mu}{}_{\alpha\beta} T^{\alpha\beta} + \Gamma^{\beta}{}_{\alpha\beta} T^{\mu\alpha} := 0 \quad (4.15)$$

$$T^{\mu\beta}{}_{,\beta} = -\Gamma^{\mu}{}_{\alpha\beta} T^{\alpha\beta} - \Gamma^{\beta}{}_{\alpha\beta} T^{\mu\alpha} \quad (4.16)$$

(3.8) $$< T^{\hat{\mu}\hat{\beta}}{}_{,\hat{\beta}} > = -< \Gamma^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta}} T^{\hat{\alpha}\hat{\beta}} > - < \Gamma^{\hat{\beta}}{}_{\hat{\alpha}\hat{\beta}} T^{\hat{\mu}\hat{\alpha}} > + \tilde{O}^{\hat{\mu}}(\mathrm{m}\emptyset^2) \quad (4.17)$$

(3.9) $$< T^{\hat{\mu}\hat{0}}{}_{,\hat{0}} > = -< \Gamma^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta}} T^{\hat{\alpha}\hat{\beta}} > - < \Gamma^{\hat{\beta}}{}_{\hat{\alpha}\hat{\beta}} T^{\hat{\mu}\hat{\alpha}} > + \tilde{O}^{\hat{\mu}}(\mathrm{m}\emptyset^2) \quad (4.18)$$

(4.11,14) (1.9b) $$< T^{\hat{\mu}\hat{0}}{}_{,\hat{0}} > = -\Gamma^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta},\hat{n}} < x^{\hat{n}} T^{\hat{\alpha}\hat{\beta}} > - \Gamma^{\hat{\beta}}{}_{\hat{\alpha}\hat{\beta},\hat{n}} < x^{\hat{n}} T^{\hat{\mu}\hat{\alpha}} > + O^{\hat{\mu}}(\mathrm{m}\emptyset^2) = O^{\hat{\mu}}(\mathrm{m}\emptyset) \quad (4.19)$$

5) The freely falling small body and its, founded on the conservation conditions, near geodesic solutions

Since $\mathcal{P}(\tau_0)$ can be any given point on the geodesic, in the following is assumed that the body is currently situated in the proper or at least in the LI comoving frame of reference, the behavior of the body in the vicinity of the spatial coordinates origin: $\mathcal{P}(\tau)$ is analyzed, and if the result follows the geodesic in the limit case (for $\mathrm{d} \rightarrow 0$), so it must also follow it inside the $2\emptyset_0$ tube for $\mathrm{d} > 0$ in a certain timespan.

▷ a) The body- (rest) mass: m , four-position: $\mathrm{x}^{\hat{\alpha}}$, four-velocity: $\mathrm{U}^{\hat{\alpha}}$ and its minimal or rest energy: E_0

$$\mathrm{m}(\tau = \tau_0) := < T^{\hat{0}\hat{0}}(\tau_0, x^{\hat{n}}) > + \tilde{O}(\mathrm{md}^2) \quad (5.1)$$

$$\mathrm{x}^{\hat{0}}(\tau) := \tau \quad ; \quad \mathrm{x}^{\hat{n}}(\tau) := < T^{\hat{0}\hat{0}}(\tau, x^{\hat{n}}) >^{-1} < x^{\hat{n}} T^{\hat{0}\hat{0}}(\tau, x^{\hat{n}}) > + \tilde{O}^{\hat{n}}(\emptyset^2 \mathrm{d}) \quad (5.2a,b)$$

(2.5a) $$\emptyset(\tau_0) \equiv \mathrm{d}(\tau_0) \quad \rightarrow \quad \mathrm{x}^{\hat{0}}(\tau_0) = \tau_0 \ , \quad \mathrm{x}^{\hat{n}}(\tau_0) = \tilde{O}^{\hat{n}}(\mathrm{d}^3) \quad (5.3a,b)$$

(1.2a) $$\mathrm{U}^{\hat{\mu}} := \frac{\mathrm{d}\mathrm{x}^{\hat{\mu}}}{\mathrm{d}\tau} \equiv \frac{\partial \mathrm{x}^{\hat{\mu}}}{\partial x^{\hat{0}}} \quad \Big| \ |\mathrm{U}^{\hat{n}}| \ll 1 \quad \leftarrow \quad \tilde{O}^{\hat{\alpha}}{}_{,\hat{0}}(\emptyset^2 \mathrm{d}) = \tilde{O}^{\hat{\alpha}}(\emptyset^2 \mathrm{d}) \quad (5.4a,b)$$

(2.5b) (5.3a) $$\mathrm{E}_0 = \mathrm{E}(\tau_0) := \min\left(\mathrm{m}(\tau_0) / \sqrt{1 - \mathrm{U}^{\hat{n}}(\tau_0) \mathrm{U}_{\hat{n}}(\tau_0)}\right) \quad \Rightarrow \quad \mathrm{U}^{\hat{n}}(\tau_0) = \tilde{O}^{\hat{n}}(\mathrm{d}^3) \quad (5.5a,b)$$

$$\mathrm{U}^{\hat{0}}(\tau_0) = 1 \ , \quad \mathrm{U}^{\hat{n}}(\tau_0) = \tilde{O}^{\hat{n}}(\mathrm{d}^3) \quad \rightarrow \quad \mathrm{U}^{\hat{0}}{}_{,\hat{0}}(\tau_0) = 0 \quad (5.6a,b)$$

▷ b) The solution based on the stress-energy tensor, in the locally inertial comoving frame of reference
The point-idealization: $p^{\bar{\mu}}$ of a four-momentum field can be defined as the "T4- momentum": $< T^{\bar{\mu}0} >$ or as the four-velocity based "U4-momentum": $mU^{\bar{\mu}}$. Given their equivalence, it follows for $|U^{\hat{n}}| \ll 1$:

(5.2b) (4.19) $$\frac{dp^{\hat{\mu}}}{d\tau} = -\Gamma^{\hat{\mu}}_{\hat{\alpha}\hat{\beta},\hat{n}} p^{\hat{\alpha}} U^{\hat{\beta}} x^{\hat{n}} - \Gamma^{\hat{\mu}}_{\hat{\alpha}\hat{\beta},\hat{n}} < x^{\hat{n}} T^{\hat{\alpha}\hat{\beta}} > - \Gamma^{\hat{\beta}}_{\hat{\alpha}\hat{\beta},\hat{n}} < x^{\hat{n}} T^{\hat{\mu}\hat{\alpha}} > + \tilde{O}^{\hat{\mu}}(m\emptyset^2) \quad (5.7)$$

(4.12) (4.14) $$\frac{dp^{\hat{\mu}}}{d\tau} = O^{\hat{\mu}}(m|x^{\hat{n}}|) - \Gamma^{\hat{\mu}}_{\hat{\alpha}\hat{\beta}} T^{\hat{\alpha}\hat{\beta}} - \Gamma^{\hat{\beta}}_{\hat{\alpha}\hat{\beta}} T^{\hat{\mu}\hat{\alpha}} - \Gamma^{\hat{\mu}}_{\hat{\alpha}\hat{\beta},\hat{n}} T^{\hat{\alpha}\hat{\beta}\langle\hat{n}\rangle} - \Gamma^{\hat{\beta}}_{\hat{\alpha}\hat{\beta},\hat{n}} T^{\hat{\mu}\hat{\alpha}\langle\hat{n}\rangle} = O^{\hat{\mu}}(m\emptyset) \quad (5.8)$$

(5.1) (5.4) $$mU^{\hat{\mu}} = < T^{\hat{\mu}\hat{0}} > + \tilde{O}^{\hat{\mu}}(m\emptyset^2) \;\Rightarrow\; \frac{dU^{\hat{\mu}}}{d\tau} \approx \frac{d(p^{\hat{\mu}}/m)}{d\tau} = \frac{1}{m}\left(\frac{dp^{\hat{\mu}}}{d\tau} - \frac{p^{\hat{\mu}}}{m}\frac{dm}{d\tau}\right) = O^{\hat{\mu}}(\emptyset) \quad (5.9a)\,(5.9b)$$

Since the origin of $\underline{x}^{\hat{n}}$ follows the geodesic, the equation (5.9b) proves at the limit that the body follows the geodesic too. The tidal forces in (5.7) die out on the flat domain if a suitable gauge (1.10) is applied. The critical physical problem is that (5.8) is not coordinate-invariant and the particular thing about this is that the body must freely levitate in the $\underline{V}$ domain proximal to the spatial origin and the LI comoving frame makes it possible since the gravity almost disappears there. The reason for this effect are the local translation symmetries: (t-xyz) on $\underline{V}$ in $\underline{x}^{\hat{\mu}}$, yet these symmetries are not perfect since on $\underline{V}$ the derivative: $g_{\hat{\mu}\hat{\nu},\hat{\alpha}} = O_{\hat{\mu}\hat{\nu}\hat{\alpha}}(\emptyset) \neq 0$ and that is in this case the limiting factor for the convergence of the solution.

▷ c) The body mass density, the (proper) energy (E-)tensor of the body, and its *local integral* divergence
The state of overall motion is defined by the body velocity, thus to find the tensor equation of motion the SE-tensor component that incorporates only this velocity ought to be used and this for a small body is its E-tensor: $T_E^{\mu\nu}$ reflecting the convective flux of the body energy. Because "the proper SLI gauge" hasn't been found, the E-tensor is here inevitably undetermined in the $O(m|x^{\hat{n}}|^2)$ range. Accordingly, in order to avoid the problem resulting from the equation (5.8) it needs to be postulated that gravitation does not act on the whole SE-tensor but exclusively on its stress (but not necessarily divergence) -free component: the E-tensor. In concrete terms, this means that the LHS of (5.13) has to vanish, and indeed due to the time symmetry of (1.8b) and because the first moment of T_E is nullified at τ_0 thanks to (2.5a), the local integral conservation condition for $< T_E^{\hat{\mu}\hat{\nu}} >$ reaches $O(md^2)$ convergence in the proper frame.

(1.2a,10) $$\rho(x^{\hat{\alpha}}) := T^{\hat{0}\hat{0}}(\tau, x^{\hat{n}}) + O(m|x^{\hat{n}}|^2) \quad ; \quad U^{\hat{\mu}}(x^{\hat{\alpha}}) := U^{\hat{\mu}}(\tau) + O^{\hat{k}}(m|x^{\hat{n}}|^2) \quad (5.10a,b)$$

$$T_E^{\hat{\mu}\hat{\nu}} := \rho\, U^{\hat{\mu}} U^{\hat{\nu}} = T^{\hat{0}\hat{0}}\, U^{\hat{\mu}} U^{\hat{\nu}} + O^{\hat{\mu}\hat{\nu}}(m|x^{\hat{n}}|^2) \quad (5.11)$$

t'⊥x'y'z' ; *t'*=τ ; (5.6a) $$T_S^{\hat{\mu}\hat{\nu}} := T^{\hat{\mu}\hat{\nu}} - T_E^{\hat{\mu}\hat{\nu}} \;\Big|\; T_E^{\mu\nu} T_{S\mu\nu} = \tilde{O}(m^2\emptyset^2 d) \quad (5.12)$$

(3.8,9) $$< T_E{}^{\hat{\mu}\hat{\nu}}{}_{;\hat{\nu}} > = < T_E{}^{\hat{\mu}\hat{0}}{}_{,\hat{0}} > + < \Gamma^{\hat{\mu}}_{\hat{\alpha}\hat{\nu}} T_E^{\hat{\alpha}\hat{\nu}} > + < \Gamma^{\hat{\nu}}_{\hat{\alpha}\hat{\nu}} T_E^{\hat{\mu}\hat{\alpha}} > + \tilde{O}^{\hat{\mu}}(m\emptyset^2) \quad (5.13); \;(5.14a,b)$$

$$O^{\hat{\mu}}{}_{,\hat{0}}(md^2) = O^{\hat{\mu}}(md^2) \;\rightarrow\; < T_E{}^{\hat{\mu}\hat{0}}{}_{,\hat{0}} > + < \Gamma^{\hat{\mu}}_{\hat{\alpha}\hat{\nu}} T_E^{\hat{\alpha}\hat{\nu}} > + < \Gamma^{\hat{\nu}}_{\hat{\alpha}\hat{\nu}} T_E^{\hat{\mu}\hat{\alpha}} > = O^{\hat{\mu}}(md^2) \;\Big|\; \tau \rightarrow \tau_0$$

▷ d) The coordinate-invariant solution based on the energy tensor, in the (LI) proper frame of reference
Consequently, (5.14b) corresponds directly to (4.18) and because the body four-position on the world line is defined in the same way for the E-tensor as for the SE-tensor, that's why in the limit case the following prove of the geodesic solution for the E-tensor confirms the result (5.9b) for the SE-tensor as well. The below equations are studied for $\tau \rightarrow \tau_0$, therefore (2.5a) makes the offset: $x^{\hat{n}}$ (5.2b) negligible.

(4.11) (2.5a) $$< T_E^{\hat{\mu}\hat{0}} >_{,\hat{0}} = -\Gamma^{\hat{\mu}}_{\hat{\alpha}\hat{\nu}} < T_E^{\hat{\alpha}\hat{\nu}} > - \Gamma^{\hat{\nu}}_{\hat{\alpha}\hat{\nu}} < T_E^{\hat{\mu}\hat{\alpha}} > + O^{\hat{\mu}}(md^2) \quad (5.15)$$

(5.11) $$(< \rho > U^{\hat{\mu}} U^{\hat{0}})_{,\hat{0}} = -< \rho > \Gamma^{\hat{\mu}}_{\hat{\alpha}\hat{\beta}} U^{\hat{\alpha}} U^{\hat{\beta}} - < \rho > \Gamma^{\hat{\beta}}_{\hat{\alpha}\hat{\beta}} U^{\hat{\mu}} U^{\hat{\alpha}} + O^{\hat{\mu}}(md^2) \quad (5.16)$$

(5.1,6b) (5.10a) $$(mU^{\hat{\mu}} U^{\hat{0}})_{,\hat{0}} = mU^{\hat{0}} U^{\hat{\mu}}{}_{,\hat{0}} + U^{\hat{\mu}} U^{\hat{0}} m_{,\hat{0}} = -m\Gamma^{\hat{\mu}}_{\hat{\alpha}\hat{\beta}} U^{\hat{\alpha}} U^{\hat{\beta}} - m\Gamma^{\hat{\beta}}_{\hat{\alpha}\hat{\beta}} U^{\hat{\mu}} U^{\hat{\alpha}} + O^{\hat{\mu}}(md^2) \quad (5.17)$$

Even though here $\Gamma^{\hat{\alpha}}_{\hat{\nu}\hat{k}} = 0$, it is the $\Gamma^{\hat{\alpha}}_{\hat{\nu}\hat{k}}$ that carries the key information about the origin of this zero. Since $U^{\hat{0}}$ is a constant and $U^{\hat{n}} \rightarrow 0$, hence (5.17) can be decomposed into the system of two equations:

(5.6) (5.1) $$\begin{cases} mU^{\hat{\mu}}{}_{,\hat{0}} = -m\Gamma^{\hat{\mu}}_{\hat{\alpha}\hat{\beta}} U^{\hat{\alpha}} U^{\hat{\beta}} + O^{\hat{\mu}}(md^2) \;\Big|\; O^{\hat{0}} = 0 & (5.18a) \\ U^{\hat{\mu}} m_{,\hat{0}} = -m\Gamma^{\hat{\beta}}_{\hat{\alpha}\hat{\beta}} U^{\hat{\mu}} U^{\hat{\alpha}} + O^{\hat{\mu}}(md^2) \;\Big|\; O^{\hat{n}} = O^{\hat{n}}(md^5) & (5.18b) \end{cases}$$

(5.1,11) $\mathrm{m}^{\hat\mu\hat\nu} := -\mathrm{m}U^{\hat\mu}U^{\hat\nu} = -T_{\mathrm{E}}^{\hat\mu\hat\nu} = -< T_{\mathrm{E}}^{\hat\mu\hat\nu} > \quad \rightarrow \quad \mathrm{m} = \eta_{\hat\mu\hat\nu}\,\mathrm{m}^{\hat\mu\hat\nu} = \mathrm{m}^{\mu}{}_{\mu}$ (5.19a,b)

(5.3b) $\tau \rightarrow \tau_0$
$$\begin{cases} \mathrm{m}\dfrac{dU^{\hat\mu}}{d\tau} = -\mathrm{m}U^{\hat\alpha}U^{\hat\beta}\Gamma^{\hat\mu}{}_{\hat\alpha\hat\beta} + O^{\hat\mu}(\mathrm{md}^2) - \mathrm{m}\Gamma^{\hat\mu}{}_{\hat\alpha\hat\beta,\hat\kappa}U^{\hat\alpha}U^{\hat\beta}\tilde{O}^{\hat\kappa}(\mathrm{d}^3) & (5.20a) \\ \dfrac{d\mathrm{m}}{d\tau} = -\mathrm{m}\Gamma^{\hat\beta}{}_{\hat0\hat\beta}U^{\hat0} + O^{\hat0}(\mathrm{md}^2) = O(\mathrm{md}^2) & (5.20b) \end{cases}$$

The −E-tensor integral: $\mathrm{m}^{\hat\mu\hat\nu}$, which invariant trace equals the body mass, can be called the mass tensor. The equations turn out to be coordinate-invariant since the first one is the four-momentum form of the geodesic equation and the second one is a scalar equation, which is always coordinate-invariant, hence

(1.8b) $\forall\tau$
$$\begin{cases} \dot{\mathrm{p}}^{\mu} \approx \mathrm{m}\dot{U}^{\mu} := \mathrm{m}\dfrac{dU^{\mu}}{d\tau} = -\mathrm{m}U^{\alpha}U^{\beta}\Gamma^{\mu}{}_{\alpha\beta} + O^{\mu}(\mathrm{md}^2) \quad \rightarrow \quad \dfrac{DU^{\mu}}{d\tau} = O^{\mu}(\mathrm{d}^2) & \textbf{(5.21a)}\ \textbf{(5.21b)} \\ \dot{\mathrm{m}} := \dfrac{d\mathrm{m}}{d\tau} = O(\mathrm{md}^2) & (5.21c) \end{cases}$$

▷ e) The limit case turns out to be the (rest) mass conservation law and the standard geodesic equation

(5.4a) $$\lambda := a\tau + b\ ,\quad \mathrm{d} \rightarrow 0 \quad \Rightarrow \quad \frac{d^2x^{\mu}}{d\lambda^2} = -\Gamma^{\mu}{}_{\alpha\beta}\frac{dx^{\alpha}}{d\lambda}\frac{dx^{\beta}}{d\lambda} \qquad ■ \qquad (5.22)$$

III. Summary

The behavior of a massive body located in the gravity field, free of other influences and with a *negligible radiation*, has been analyzed here. The body is defined on the basis of its stress-energy (SE-)tensor field, which together with the geodesic determine the spatial domain on which it exists. Therefore, based on the general space-time coordinates, the stationary locally inertial (SLI) coordinate system has been constructed in the area of validity. The SLI coordinates make it possible at least for a certain amount time, to conveniently describe the movement of the body. If their spatial origin matches the comoving body at a certain τ, they are referred to there as the proper frame. This situation has been achieved by selecting the suitable initial conditions for the SLI coordinate system itself. The proper frame is locally inertial (LI) in the neighborhood of its origin. The requested SLI gauge aims to limit the fictitious tidal forces especially on the flat domain but can be modified if necessary to any other gauge that complies with the spatial origin conditions of the SLI coordinates without violating the convergence estimations. The spatial origin of the SLI coordinate system follows a geodesic and forms in the space-time a kind of geodesic tunnel, but it doesn't mean that this by itself can somehow affect the tensor solution. Furthermore, the local integral divergence of the SE-tensor on the spatial domain is essential here. If the body is sufficiently small, after flattening of the coordinate base four separate conservation equations of energy and momentum arise in the LI comoving frame from the SE-tensor zero-divergence due to the (t-xyz) symmetry that in curved space is only locally possible using approximation. Therefore, it is important here to be able to estimate the convergence grade of occurring uncertainties deviations and errors. For that upper bound estimations, the big *O*-notation has been used. Because of the limited spatial extent of the SE-tensor, it is possible to reduce its local integral divergence to the temporal derivative. This is crucial because it makes possible to derive the body equation of motion from the conservation condition of the SE-tensor. To guarantee that the test body problem is not critical here, the cross effect of gravity fields was proven to be negligible up to m = *O*(Ø). Based on the SE-tensor the body geodesic trajectory has been found only in the SLI coordinates. This in turn shows that the SE-tensor as a basis for the geodesic equation, which is a tensor equation, is suitable only to a limited extent. To solve this problem, the SE-tensor has been reduced by setting all its subcomponents except the energy density *in the proper frame* arbitrary at zero, thus defining its component in the form of the energy (E-)tensor, which point-idealization is the mass tensor: $\mathrm{m}^{\mu\nu}$; that depends on the energy density but not on the stress seen here as an internal four-momentum flux, shear stress and pressure, that form the stress tensor: T_{S} . The mass tensor is together with mass and four-momentum, just another quantity associated with the body. Clearly, by replacing the body SE-tensor with its E-tensor the dependence on the stress disappears, which is directly visible for the body as an internal observer in its proper frame and the external observer perceives only the final tensor result. Then the energy flux balance equation is solved by splitting it into two separate equations: the first one relates to the body acceleration, the second one to its mass change. Ultimately, these equations can be represented by tensor equations where in the limit case the first one becomes the geodesic- and the second one the mass conservation equation for a freely falling body in the space-time.

IV. Conclusions

There are two testable possibilities: For a small body diameter: d the effective gravitational tidal forces *A*: *can* influence the body trajectory at the *O*(d) level *B*: *cannot* because its stress tensor field vanishes or on the average does not interact with gravity. - It has been shown that even if no coordinate-invariant solution on the body SE-tensor basis has been found, an associated with its SE-tensor free "point" body traverse timelike geodesic. Basically, merely the conservation condition based on the SE-tensor is necessary to determine it if the isolated from all "conventional" forces free body is also apparently isolated from gravity according to the local symmetries that emerge in the LI comoving frame in the SLI system. - Simultaneously this statement can be extended analogously to the solution with the body energy (E-) tensor that is the stress-free component of its SE-tensor, but in this case the direct coordinate-invariant solution is the geodesic equation; yet should "*A*" be true, a trajectory error at the *O*(d) level might occur. Consequently, this E-tensor provides the pathway allowing the gravitation to influence the massive body, and that with the inherent condition of the body mass being conserved. By applying the SE-tensor instead of the E-tensor in a tensor equation of motion the weak equivalence principle could be disregarded, but since there are good reasons to conclude that the weak equivalence principle remains valid, the solution on the E-tensor base is preferable to rule out the theoretic dependence on the intrinsic stress field of the body matter. Therefore, the following thesis should be proposed: The total gravitational influence upon a sufficiently small freely falling massive body or particle is equal to this influence on its E-tensor field. Formulated in such a way, the above thesis in the near limit case leads not only to the geodesic principle and the weak equivalence principle but also to the mass conservation law for an isolated body and to the explicit consequence that the rest mass and the passive mass must always have the same value; all this with the additional advantage of not having the form of a quasi-mathematical axiom but containing the mechanism based on the local integral conservation condition for the E-tensor field in the proper frame.

Since the geodesic principle hasn't been derived here from the geometrical approach but from the conservation condition, it is natural not to limit oneself to freely falling bodies and the external influences, such as that of the Lorenz force, can be taken into account in order to get the equation of motion like this

(5.9a,21a,19) $$\mathrm{m}\dot{\mathrm{U}}^{\alpha} = -\mathrm{m}\mathrm{U}^{\mu}\mathrm{U}^{\nu}\Gamma^{\alpha}_{\ \mu\nu} + \mathrm{q}\mathrm{U}^{\mu}\mathrm{F}^{\alpha}_{\ \mu} \equiv \mathrm{m}^{\mu\nu}\Gamma^{\alpha}_{\ \mu\nu} + \mathrm{q}\mathrm{U}^{\mu}\mathrm{F}^{\alpha}_{\ \mu} \tag{6.1}$$

In the above equation the mass: m is not just a result of "adjusting" the geodesic equation to the Lorenz force, but was already there in (5.21a). It is also clear that m on the LHS expresses the inertial mass and m on the RHS the passive mass having the same value. Moreover, it is noteworthy that the inertial mass "hides" in the body-bound coordinates ($\dot{\mathrm{U}}^{\alpha} = 0$) and the passive mass "hides" in the free-falling coordinates ($\Gamma^{\alpha}_{\ \mu\nu} = 0$), as in the famous free-falling elevator thought experiment. This equation shows also that the thesis above contains the basis for the weak equivalence principle for extended bodies. This is because in a uniform gravitational field for a body in its proper frame the superposition of the E-tensor fields of all its particles results in the SE-tensor macro-field whose mass tensor: $\mathrm{m}^{\mu\nu}$ carrying there the rest energy equivalent to the body mass, entirely determines the influence of gravity on the world line of the body and all other body properties are irrelevant in this respect. - That is why it seems reasonable to generalize the thesis, which has been proposed above, in the postulate: The gravitational influence upon a massive object results only from the gravity interaction with the energy tensor fields of all its particles. However, from a certain level a definition of the E-tensor for quantum objects would be necessary here.

V. References

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