Laplace limit constant and Lambert W function

Edgar Valdebenito
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Abstract: We estimate the Laplace constant using Lambert's W function.

I. Introduction

The Laplace limit is the unique real solution of the equation:

$$\frac{x\, e^{\sqrt{1+x^2}}}{1+\sqrt{1+x^2}} = 1 \tag{1}$$

It is approximately

$$\lambda = 0.66274341934918158097474209710925290705 81 \ldots \tag{2}$$

II. Lambert W function

The Lambert W function $W(z)$ is defined as

$$W(z)\, e^{W(z)} = z \;,\; z \in \mathbb{C} \tag{3}$$

The Lambert W function is a multivalued function with infinitely many branches. The branches are indexed by the integer k $(W(k, z))$.

The principal branch is $W(0, z) = W(z)$. For details see [1].

III. Laplace limit via Lambert W function

We have

$$\lambda = \frac{1}{2}\sqrt{\alpha\,(4+\alpha)} = 0.66274341934918158097474209710925290705 81 \ldots \tag{4}$$

where

$$\alpha = W\left(4\, e^{-2} + e^{-2}\, W\left(4\, e^{-2} + e^{-2}\, W\left(4\, e^{-2} + e^{-2}\, W\left(4\, e^{-2} + \ldots\right)\right)\right)\right) \tag{5}$$

$$\alpha = 0.39935728051546766783273969728228388852 29 \ldots \tag{6}$$

Remark 1:

$$\alpha_1 = 0 \;,\; \alpha_{n+1} = W\left(4\, e^{-2} + e^{-2}\, \alpha_n\right) \Longrightarrow \alpha_n \longrightarrow \alpha \tag{7}$$

$$| \alpha_{n+1} - \alpha | \sim 0.06487 \, | \alpha_n - \alpha | \tag{8}$$

$$\lambda_n = \frac{1}{2} \sqrt{\alpha_n (4 + \alpha_n)} \;, \; \lambda_n \longrightarrow \lambda \tag{9}$$

Remark 2: e is the Euler number

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = 2.718281 \ldots \tag{10}$$

Remark 3: some estimates

n	α_n	λ_n	$\lvert\lambda - \lambda_n\rvert$
1	0.37285636 ...	0.63844485 ...	$2.42 \ldots \times 10^{-2}$
2	0.39763562 ...	0.66118390 ...	$1.55 \ldots \times 10^{-3}$
3	0.39924558 ...	0.66264232 ...	$1.01 \ldots \times 10^{-4}$
4	0.39935003 ...	0.66273686 ...	$6.55 \ldots \times 10^{-6}$
5	0.39935681 ...	0.66274299 ...	$4.25 \ldots \times 10^{-7}$
6	0.39935725 ...	0.66274339 ...	$2.75 \ldots \times 10^{-8}$
7	0.39935727 ...	0.66274341 ...	$1.79 \ldots \times 10^{-9}$

Remark 4: for $n = 20$ we have

$$\lambda_{20} = 0.66274341934918158097474 14\,522 \ldots \tag{11}$$

$$| \lambda_{20} - \lambda | = 6.448 \ldots \times 10^{-25} \tag{12}$$

IV. References

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