A new continued fraction approximation and inequalities for the Lugo's constant

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ABSTRACT: In this paper, we provide a new continued fraction approximation for the Lugo's constant. Then, we derive the inequalities concerning the Lugo's constant. Finally, we give some numerical computations to demonstrate the superiority of our new results.

Keywords: Lugo's constant, Continued fraction, Inequality, Rate of convergence

1. Introduction

Mathematical constants play a key role in several areas of mathematics such as number theory, special functions, analysis or probability. As you know, π and e are the important constants which are widely used in mathematics and engineering. Besides these constants, there are many mathematical constants such as the Euler-Mascheroni constant, generalized Euler-Mascheroni constant, Lugo's constant and Somos' quadratic recurrent constant, etc..

The Euler-Mascheroni constant γ, now universally known as gamma, was introduced by the Swiss mathematician Leonhard Euler (1707-1783) in 1734, which is defined as the limit of the sequence

$$D_n = \sum_{k=1}^{n} \frac{1}{k} - \ln n. \tag{1.1}$$

The constant γ is closely related to the gamma function $\Gamma(z)$ by means of the familiar Weierstrass formula [1, p. 255, Equation (6.1.3)] (see also [12, Chapter 1, Section 1.1]):

$$\frac{1}{\Gamma(z)} = ze^{\gamma z} \prod_{n=1}^{\infty} \left[\left(1 + \frac{z}{n}\right) e^{-z/n} \right] \quad \left(|z| < \infty\right). \tag{1.2}$$

Lugo[4] considered the sequence L_n, which is essentially an interesting analogue of the sequence D_n, defined by

$$L_n = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{1}{i+j} - (2\ln 2)n + \ln n \ . \tag{1.3}$$

We can easily find that

$$L = \lim_{n\to\infty} L_n = -\frac{1}{2} - \gamma + \ln 2 \ , \tag{1.4}$$

where L is called Lugo's constant.

As you can see, the Lugo and Euler-Mascheroni constants are related to each other.

In the study of mathematical constants, the remarkable trend is to find more accurate approximations for them, so during the past several decades, many mathematicians and scientists have

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worked on this subject.

Up to now, many researchers have made great efforts in this area of establishing more accurate approximations for the Lugo and Euler-Mascheroni constant and had lots of inspiring results.

Lugo[10] proved the following asymptotic formula:

$$L_n \sim -\frac{1}{2}-\gamma+\ln 2-\frac{5}{8n}+\frac{7}{48n^2}+o\left(n^{-3}\right) \quad (n\to\infty). \tag{1.5}$$

Chen and Srivastava[6] established new analytical representations for the Euler Mascheroni constant γ:

$$\gamma=-\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{i+j}+\ln 2-1+\left(n+\frac{1}{2}\right)\psi\left(n+\frac{1}{2}\right)-\left(n+\frac{3}{2}\right)\psi(n)+(2\ln 2)n-\frac{3}{2n},\ (n\in\mathbb{N}) \tag{1.6}$$

in terms of the psi (or digamma) function defined by $\psi(z)$ defined by

$$\psi(z)=\frac{\Gamma'(z)}{\Gamma(z)} \text{ or } \ln\Gamma(z)=\int_1^z\psi(t)dt. \tag{1.7}$$

Recently, authors have focused on continued fractions in order to obtain new approximations. [11,13-15]

For example, Lu[7] provided the faster sequence convergent to γ as follows.

$$r_{n,s}=H_n-\ln n-\cfrac{a_1}{n+\cfrac{a_2 n}{n+\cfrac{a_3 n}{n+\cfrac{a_4 n}{n+\ddots}}}}, \tag{1.8}$$

where

$$H_n=\sum_{k=1}^{n}\frac{1}{k}$$

is the n^{th} harmonic number and

$$a_1=\frac{1}{2},a_2=\frac{1}{6},a_4=\frac{3}{5},a_6=\frac{79}{126},a_8=\frac{7230}{6241},a_{10}=\frac{4146631}{3833346},\cdots.$$
$$a_{2k+1}=-a_{2k}\ (1\le k\le 6)$$

Moreover, he used continued fraction approximation to consider new classes of sequences for the Euler–Mascheroni constant as follows.[8]

$$L_{r,n}=H_{n-1}+\frac{1}{rn}-\ln n-\cfrac{a_1}{n+\cfrac{a_2 n}{n+\cfrac{a_3 n}{n+\cfrac{a_4 n}{n+\ddots}}}}\quad (r\neq 2),$$
$$L_{2,n}=H_{n-1}+\frac{1}{2n}-\ln n-\frac{1}{n}\cfrac{b_1}{n+\cfrac{b_2}{n+\cfrac{b_3}{n+\cfrac{b_4}{n+\ddots}}}} \tag{1.9}$$

where

$$a_1 = \frac{2-r}{2r}, a_2 = \frac{r}{6(2-r)}, a_3 = \frac{r}{6(r-2)}, a_4 = \frac{3(2-r)}{5r}, \cdots$$
$$b_1 = -\frac{1}{12}, b_2 = \frac{1}{10}, b_3 = \frac{79}{210}, b_4 = \frac{1205}{1659}, \cdots$$

In [9], he introduced new classes of sequences.

$$L_{n,k} = H_n - \ln n - \frac{1}{k}\ln\left(1 + \cfrac{a_1}{n + \cfrac{a_2 n}{n + \cfrac{a_3 n}{n + \cfrac{a_4 n}{n + \ddots}}}}\right), \tag{1.10}$$

where

$$a_1 = \frac{k}{2}, a_2 = \frac{2-3k}{12}, a_3 = \frac{3k^2+4}{12(3k-2)}, a_4 = -\frac{15k^4 - 30k^3 + 60k^2 - 104k + 96}{20(3k-2)(3k^2+4)}, \cdots.$$

In [3], a sequence concerning the Lugo's constant is given as,

$$v_n = L - L_n - \cfrac{\frac{5}{8}}{n + a + \cfrac{b}{n + c + \cfrac{p}{n+q}}}, \tag{1.11}$$

where

$$a = \frac{7}{30}, b = \frac{53}{1800}, c = \frac{1339}{1590}, p = \frac{15975}{22472}, q = -\frac{6528287}{59267250}.$$

In this paper, we provide a new continued fraction approximation for the Lugo's constant. Then, we derive the inequalities concerning the Lugo's constant. Finally, we give some numerical computations to demonstrate the superiority of our new results.

The rest of this paper is arranged as follows.

In Sect. 2, some useful lemmas and a new continued fraction approximation for the Lugo's constant are given. In Sect. 3, inequalities for the Lugo's constant are provided. In the last section, some numerical computations are given.

2. A new continued fraction approximation for the Lugo's constant

In this section, some useful lemmas and a new continued fraction approximation for the Lugo's constant are given.

Lemma 2.1. We have the following power series representation of $\cfrac{A_1}{n + \cfrac{A_2 n}{n + \cfrac{A_3 n}{n + \ddots}}}$ in $1/n\,(n \to \infty)$.

Proof. We make the following substitution.

$$a_i = \frac{A_i}{1+\frac{a_{i+1}}{n}}, (i=1,2,\cdots), a_s = A_s. \tag{2.1}$$

Then, we can easily get that

$$\cfrac{A_1}{n+\cfrac{A_2 n}{n+\cfrac{A_3 n}{n+\ddots}}} = \frac{a_1}{n}. \tag{2.2}$$

Using the following power series expansion,

$$a_i = \frac{A_i}{1+\frac{a_{i+1}}{n}} = A_i\left(1-\frac{a_{i+1}}{n}+\frac{a_{i+1}^2}{n^2}-\frac{a_{i+1}^3}{n^3}+\cdots\right), \tag{2.3}$$

we can obtain coefficients M_k in $\cfrac{A_1}{n+\cfrac{A_2 n}{n+\cfrac{A_3 n}{n+\ddots}}} = \sum_{k=1}^{\infty} M_k \left(\frac{1}{n}\right)^k$.

We show the first few terms as follows:

$M_1 = A_1$

$M_2 = -A_1 A_2$

$M_3 = A_1 A_2 A_3 + A_1 A_2{}^2$

$M_4 = -A_1 A_2 A_3 A_4 - A_1 A_2^3 - -2 A_1 A_2^2 A_3 - A_1 A_2 A_3^2$

$M_5 = A_1 A_2 A_3 A_4 A_5 + A_1 A_2^4 + +2 A_1 A_2 A_3^2 A_4 + A_1 A_2 A_3^3 + A_1 A_2 A_3 A_4^2 +$

$+2 A_1 A_2^2 A_3 A_4 + 3 A_1 A_2^3 A_3 + 3 A_1 A_2^2 A_3^2$

$\cdots \cdots$

We can easily see that $M_k = (-1)^{k+1} A_1 \cdots A_k + f(A_1, \cdots, A_{k-1})$.

Lemma 2.2. The psi function ψ has the asymptotic formulas as follows:

$$\psi(x) \sim \ln x - \frac{1}{2x} - \sum_{i=1}^{\infty} \frac{B_{2i}}{2i x^{2i}} \quad , \quad x \to \infty \tag{2.4}$$

$$\psi\left(x+\frac{1}{2}\right) \sim \ln x - \sum_{i=1}^{\infty} \frac{B_{2i}(1/2)}{2i x^{2i}} = \ln x + \sum_{i=1}^{\infty} \frac{B_{2i}(1-2^{1-2i})}{2i x^{2i}} \quad , \quad x \to \infty \tag{2.5}$$

where B_n ($n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$) denotes the Bernoulli numbers defined by the generating formula

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!}, \ |z| < 2\pi, \tag{2.6}$$

then the first few terms of B_n are as follows.

$$B_{2n+1}=0, n\geq 1,$$
$$B_0=1, B_1=-\frac{1}{2}, B_2=\frac{1}{6}, B_4=-\frac{1}{30}, B_6=\frac{1}{42}, B_8=-\frac{1}{30}, B_{10}=\frac{5}{66}, \cdots.$$

We can find expressions above by differentiating expressions (3.14) and (5.4) in [4].

Theorem 2.1. For any integer $s\geq 0$ and $r\in\mathbb{R}$, we have the following sequence convergent to the Lugo's constant.

$$L_{n,r}^{s}=\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{i+j}-(2\ln 2)n+\ln(n+r)-\cfrac{A_1}{n+\cfrac{A_2 n}{n+\cfrac{A_3 n}{n+\ddots+A_s}}}\quad , \tag{2.7}$$

where

$$A_1=\frac{8r-5}{8},\ A_2=\frac{24r^2-7}{6(8r-5)},\ A_3=\frac{384r^4-960r^3+672r^2-72r-53}{12(8r-5)(24r^2-7)}\cdots.$$

For any fixed s and r, we can obtain sequences with coefficients of which rate of convergence is faster than $n^{-(s+1)}$.

Furthermore, for any integer s, let

$$\lim_{n\to\infty} n^{s+1}(L_{n,r}^{s}-L)=C_s \quad ,$$

where

$$C_0=\frac{8r-5}{8}, C_1=-\frac{24r^2-7}{48}, C_2=\frac{384r^4-960r^3+672r^2-72r-53}{576(8r-5)},$$

$$C_3=-\frac{2560r^6-6720r^4+1920r^3+1488r^2-479}{3840(24r^2-7)}.$$

Then, we have

$$C_s=(-1)^s\prod_{i=1}^{s+1}A_i \ . \tag{2.8}$$

Proof. We need to find the value of parameters which produce the best approximation.
From [2], we can obtain

$$L-L_{n,r}^{s}=\frac{1}{2}-(n+\frac{1}{2})\psi(n+\frac{1}{2})+(n+\frac{3}{2})\psi(n)-\ln(n+r)+\frac{3}{2n}+\cfrac{A_1}{n+\cfrac{A_2 n}{n+\ddots}}. \tag{2.9}$$

Using the expression above and Lemma 2.1 and Lemma 2.2, we can obtain

$$L-L_{n,r}^{s}=\sum_{k=1}^{\infty}\left[-\frac{(1-2^{-(k+1)})B_{2\left[\frac{k+1}{2}\right]}}{\left[\frac{k+1}{2}\right]}+\frac{(-r)^k}{k}+M_k\right]\frac{1}{n^k}+\frac{3}{4n}. \tag{2.10}$$

Let $V_k = -\dfrac{(1-2^{-(k+1)})B_{2\left[\frac{k+1}{2}\right]}}{\left[\dfrac{k+1}{2}\right]} + \dfrac{(-r)^k}{k} + M_k!$.

If we make $V_1 = -\dfrac{3}{4}, V_k = 0(k = \overline{2,s})$, we can obtain sequences of which rate of convergence is $n^{-(s+1)}$.

So we can obtain $M_1 = \dfrac{8r-5}{8}$, $M_2 = \dfrac{7-24r^2}{48}$, $M_3 = \dfrac{64r^3-3}{192}\cdots$.

Consequently using Lemma 2.1, we can obtain

$$A_1 = \frac{8r-5}{8},\ A_2 = \frac{24r^2-7}{6(8r-5)},\ A_3 = \frac{384r^4-960r^3+672r^2-72r-53}{12(8r-5)(24r^2-7)}\cdots.$$

In other words, let

$$\lim_{n\to\infty} n^{s+1}(L_{n,r}^s - L) = C_s .$$

Using Lemma 2.1, we can obtain

$$L_{n,r}^{s+1} - L = C_s n^{-(s+1)} - (-1)^{s+2}\prod_{k=1}^{s+1} A_i \cdot n^{-(s+1)} + o(n^{-(s+2)}) = C_{s+1} n^{-(s+2)} + o(n^{-(s+3)}). \qquad (2.11)$$

So we can see that

$$C_s = (-1)^s \prod_{k=1}^{s+1} A_i .$$

The proof of Theorem 2.1 is complete.

Remark 2.1. It is easy to see that $L_{n,0}^0 = L_n$.

We can verify $L - L_{n,0}^6 = v_n$ [3], though the types of expressions are different.

For the convenience of readers, we show

$$L_{n,0}^6 = \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{i+j} - (2\ln 2)n + \ln n + \cfrac{5/8}{n + \cfrac{an}{n + \cfrac{bn}{n + \cfrac{cn}{n + \cfrac{pn}{n+q}}}}}, \qquad (2.12)$$

where

$$a = \frac{7}{30}, b = -\frac{53}{420}, c = \frac{1437}{1484}, p = -\frac{37275}{50774}, q = \frac{167116367}{267820875}.$$

3. Inequalities associated with the Lugo's constant

In this section, we provide some useful lemmas and inequalities for the Lugo's constant.

In our investigation, we need the following lemmas.

Lemma 3.1. (see [5] Lemma 2) Let $k \ge 1$ and $n \ge 0$ be integers. Then for all real numbers $x > 0$:

$$S_k(2n;x) < (-1)^{k+1}\psi^{(k)}(x) < S_k(2n+1;x) \quad , \tag{3.1}$$

where

$$S_k(p;x) = \frac{(k-1)!}{x^k} + \frac{k!}{2x^{k+1}} + \sum_{i=1}^{p}\left[B_{2i}\prod_{j=1}^{k-1}(2i+j)\right]\frac{1}{x^{2i+k}}. \tag{3.2}$$

Lemma 3.2. (see [2] Lemma 2) For $x > \frac{1}{2}$ and N=0, 1, 2…,

$$\log(x-\frac{1}{2}) - \sum_{k=1}^{2N}\frac{B_{2k}(1/2)}{2k(x-\frac{1}{2})^{2k}} < \psi(x) < \log(x-\frac{1}{2}) - \sum_{k=1}^{2N+1}\frac{B_{2k}(1/2)}{2k(x-\frac{1}{2})^{2k}} \tag{3.3}$$

and

$$\frac{(n-1)!}{(x-\frac{1}{2})^n} + \sum_{k=1}^{2N+1}\frac{B_{2k}(1/2)}{(2k)!}\frac{(n+2k-1)!}{(x-\frac{1}{2})^{n+2k}} < (-1)^{n+1}\psi^{n}(x) < \frac{(n-1)!}{(x-\frac{1}{2})^n} + \sum_{k=1}^{2N}\frac{B_{2k}(1/2)}{(2k)!}\frac{(n+2k-1)!}{(x-\frac{1}{2})^{n+2k}} \tag{3.4}$$

Lemma 3.3. Let $v(x) = \frac{384}{19}(\frac{1}{2} - (x+\frac{1}{2})\psi(x+\frac{1}{2}) + (x+\frac{3}{2})\psi(x) - \ln(x+\frac{5}{8}) + \frac{3}{2x}).$

Then, for $x \ge 3$,

$$-v'(x) < 2(v(x))^{\frac{3}{2}}. \tag{3.5}$$

Proof. First, we prove that for $x \ge 3$,

$$v(x) > M(x) = \frac{1}{x^2} - \frac{101}{76x^3} + \frac{13343}{12160x^4} - \frac{2643}{4864x^5} + \frac{68029}{1634304x^6} - \frac{234375}{2179072x^7} > 0, \tag{3.6}$$

$$-v'(x) < N(x) = \frac{2}{x^3} - \frac{303}{76x^4} + \frac{13343}{3040x^5} - \frac{13215}{4864x^6} + \frac{68029}{272384x^7} + \frac{51}{38x^8} + \frac{511}{380x^9}. \tag{3.7}$$

Define the function $S(x)$ by

$$S(x) = -\ln(1+\frac{5}{8x}) - (-\frac{5}{8x} + \frac{25}{128x^2} - \frac{125}{1536x^3} + \frac{625}{16384x^4} - \frac{625}{32768x^5} + \frac{15625}{1572864x^6} - \frac{78125}{14680064x^7}). \tag{3.8}$$

We can easily get that $\lim_{x\to\infty} S(x) = 0.$

Differentiation yields

$$S'(x) = -\frac{390625}{2097152x^8(5+8x)} < 0. \tag{3.9}$$

So we have $S(x) > \lim_{x\to\infty} S(x) = 0.$

Using Lemma 3.1 and Lemma 3.2, we have

$$v(x) > \frac{1}{2} - (x+\frac{1}{2})(\frac{1}{24x^2} - \frac{7}{960x^4} + \frac{31}{8064x^6}) + (x+\frac{3}{2})(-\frac{1}{2x} - \frac{1}{12x^2} + \frac{1}{120x^4} - \frac{1}{252x^6}) + $$
$$+\frac{3}{2n} + (-\frac{5}{8x} + \frac{25}{128x^2} - \frac{125}{1536x^3} + \frac{625}{16384x^4} - \frac{625}{32768x^5} + \frac{15625}{1572864x^6} - \frac{78125}{14680064x^7}) \tag{3.10}$$
$$= \frac{1}{x^2} - \frac{101}{76x^3} + \frac{13343}{12160x^4} - \frac{2643}{4864x^5} + \frac{68029}{1634304x^6} - \frac{234375}{2179072x^7} > 0.$$

This proves the first expression. The proof of second is similar. Consequently,

$$4(M(x))^3 - (N(x))^2 = \frac{P(x)}{8730287113846652928000\,x^{21}} > 0 \quad , \tag{3.11}$$

where

$$P(x) = 29317457913987462327053426 7 + 4013690159781453735679088556(x-3) +$$
$$1346774529184121267469525268 8(x-3)^2 + 2316065381130634868589209158 4(x-3)^3 +$$
$$2502761431252249756013038387 2(x-3)^4 + 1861011584013460198538902118 4(x-3)^5 +$$
$$997547907753904849712042803 2(x-3)^6 + 394366655927320444433144217 6(x-3)^7 +$$
$$1157472560387497274064764928(x-3)^8 + 250196070149322217623650304\,(x-3)^9 +$$
$$38831499895157589509406720(x-3)^{10} + 4106320166417102662533120(x-3)^{11} +$$
$$265540436401969535385600(x-3)^{12} + 7937215498778207846400\,(x-3)^{13}.$$

Lemma 3.4. Let

$$v(x) = \frac{1728}{14\sqrt{42}-27}\left(-\frac{1}{2} + (x+\frac{1}{2})\psi(x+\frac{1}{2}) - (x+\frac{3}{2})\psi(x) + \ln(x+\sqrt{\frac{7}{24}}) - \frac{3}{2x} - \frac{-5+2\sqrt{\frac{14}{3}}}{8x}\right).$$

Then, for $x \geq 5$,

$$-v'(x) < 3(v(x))^{\frac{4}{3}}. \tag{3.12}$$

Proof. First, we prove that for $x \geq 5$,

$$v(x) > M(x) = \frac{1}{x^3} - \frac{1293}{20(14\sqrt{42}-27)x^4} + \frac{270+49\sqrt{42}}{20(14\sqrt{42}-27)x^5} + \frac{2171}{336(14\sqrt{42}-27)x^6}$$
$$- \frac{459}{32(14\sqrt{42}-27)x^7} - \frac{4599}{320(14\sqrt{42}-27)x^8} > 0, \tag{3.13}$$

$$-v'(x) < N(x) = \frac{3}{x^4} - \frac{1293}{5(14\sqrt{42}-27)x^5} + \frac{270+49\sqrt{42}}{4(14\sqrt{42}-27)x^6} + \frac{2171}{56(14\sqrt{42}-27)x^7}$$
$$+ \frac{459}{32(14\sqrt{42}-27)x^8}. \tag{3.14}$$

Define the function S(x) by

$$S(x) = \ln(1+\sqrt{\frac{7}{24}}\frac{1}{x}) - \frac{-5+2\sqrt{\frac{14}{3}}}{8x} - (\frac{5}{8x} - \frac{7}{48x^2} + \frac{7\sqrt{\frac{7}{6}}}{144x^3} - \frac{49}{2304x^4} + \frac{49\sqrt{\frac{7}{6}}}{5760x^5} - \frac{343}{82944x^6}). \tag{3.15}$$

We can easily get that $\lim_{x\to\infty} S(x) = 0$.

Differentiation yields

$$S'(x) = -\frac{343\sqrt{\frac{7}{6}}}{2304x^7(12x+\sqrt{42})} < 0\,. \tag{3.16}$$

So we have $S(x) > \lim_{x\to\infty} S(x) = 0$.

Using Lemma 3.1 and Lemma 3.2, we have

$$\begin{aligned} v(x) &> -\frac{1}{2} + (x+\frac{1}{2})(\frac{1}{24x^2} - \frac{7}{960x^4} + \frac{31}{8064x^6} - \frac{127}{30720x^8}) \\ &\quad -(x+\frac{3}{2})(-\frac{1}{2x} - \frac{1}{12x^2} + \frac{1}{120x^4} - \frac{1}{252x^6} + \frac{1}{240x^8}) - \frac{3}{2n} \\ &\quad +(\frac{5}{8x} - \frac{7}{48x^2} + \frac{7\sqrt{\frac{7}{6}}}{144x^3} - \frac{49}{2304x^4} + \frac{49\sqrt{\frac{7}{6}}}{5760x^5} - \frac{343}{82944x^6}) \\ &= \frac{1}{x^3} - \frac{1293}{20(14\sqrt{42}-27)x^4} + \frac{270+49\sqrt{42}}{20(14\sqrt{42}-27)x^5} + \frac{2171}{336(14\sqrt{42}-27)x^6} \\ &\quad - \frac{459}{32(14\sqrt{42}-27)x^7} - \frac{4599}{320(14\sqrt{42}-27)x^7} > 0. \end{aligned} \tag{3.17}$$

This proves the first expression. The proof of second is similar. Consequently,

$$27(M(x))^4 - (N(x))^3 = \frac{P(x)}{75528929280000(14\sqrt{42}-27)^4\, x^{32}} > 0\ , \tag{3.18}$$

where

$$\begin{aligned} P(x) = {} & 87002316710865504081 + 347328231096214536840x + 363514738470092443080\, x^2 \\ & -(449382624039165337920 + 59325780649113768384\sqrt{42})\, x^3 \\ & +(310996905832363928688 - 177629049693041224320\sqrt{42})\, x^4 \\ & +(4860135710596603543680 - 436270822199355870720\sqrt{42})\, x^5 \\ & +(6250492784745109901952 - 747104043775227747840\sqrt{42})\, x^6 \\ & -(3597288163999522836480 + 978980312165892148224\sqrt{42})\, x^7 \\ & +(4850648719796253017344 - 424294991498538316800\sqrt{42})\, x^8 \\ & +(38410952598735739545600 + 3562531070308500369408\sqrt{42})\, x^9 \\ & +(54518489822789306431488 - 9257186136905197424640\sqrt{42})\, x^{10} \\ & +(10311296239896605614080 - 8500774806487000547328\sqrt{42})\, x^{11} \\ & +(38049640801459109093376 + 4431355918986391879680\sqrt{42})\, x^{12} \\ & +(86879093306113284341760 - 15186094760897704820736\sqrt{42})\, x^{13} \\ & +(234662217550023955120128 - 42152707194661164810240\sqrt{42})\, x^{14} \\ & +(289543175267951786065920 - 44182464915439831744512\sqrt{42})\, x^{15} \\ & +(186567854727350072180736 - 21910035930934023290880\sqrt{42})\, x^{16} \end{aligned}$$

$$+ \ (49926943797639260405760 - 14350634479704625643520\sqrt{42})\ x^{17}$$
$$+ \ (39208984091192092262400 - 4881747293211879014400\sqrt{42})\ x^{18}. \tag{3.19}$$

Theorem 3.1. For all natural numbers *n*,

$$\frac{19}{384}\frac{1}{(n+\alpha_1)^2} \le L - L^0_{n,\frac{5}{8}} < \frac{19}{384}\frac{1}{(n+\beta_1)^2} \tag{3.20}$$

with the best possible constants

$$\alpha_1 = \frac{1}{\sqrt{\frac{384}{19}(-1-\gamma+\ln\frac{64}{13})}} - 1 = 0.720356\cdots \text{ and } \ \beta_1 = \frac{101}{152}.$$

Proof .The inequality is equivalent to the following inequality.

$$\beta_1 < f(n) = \frac{1}{\sqrt{\frac{384}{19}(L - L^0_{n,\frac{5}{8}})}} - n \le \alpha_1 \tag{3.21}$$

In order to prove, we define the function by

$$f(x) = v(x)^{-1/2} - x, \tag{3.22}$$

where $v(x)$ is the same as in Lemma 3.3.
Differentiating $f(x)$,we obtain

$$f'(x) = -\frac{1}{2}v(x)^{-3/2}\cdot v'(x) - 1. \tag{3.23}$$

Using Lemma 3.3, we can obtain, for $x \ge 3$, $f'(x) < 0$.

Using $f(k)(k=\overline{1,3})$ calculated by mathematica, we conclude that the sequence *f(n)* is strictly decreasing. This leads to

$$\lim_{n\to\infty} f(n) < f(n) \le f(1) = \frac{1}{\sqrt{\frac{384}{19}(\frac{1}{2}-\frac{3}{2}\psi(\frac{3}{2})+\frac{5}{2}\psi(1)-\ln(1+\frac{5}{8})+\frac{3}{2})}} - 1 =$$
$$= \frac{1}{\sqrt{\frac{384}{19}(-1-\gamma+\ln\frac{64}{13})}} - 1 = 0.720356\cdots. \tag{3.24}$$

It is easy to prove that $\lim_{n\to\infty} f(n) = \frac{101}{152}$.

$$\beta_1 = \lim_{n\to\infty} f(n) = \lim_{n\to\infty}(\frac{1}{n^2} - \frac{101}{76n^3} + 0(\frac{1}{n^4}))^{-1/2} - n =$$
$$= \lim_{n\to\infty} n[(1 - \frac{101}{76n} + 0(\frac{1}{n^2}))^{-1/2} - 1] = \frac{101}{152}$$

The proof of Theorem 3.1 is complete.

Theorem 3.2. For all natural numbers *n*,

$$\frac{14\sqrt{42}-27}{1728}\frac{1}{(n+\alpha_2)^3} \le L^1_{n,\sqrt{\frac{7}{24}}} - L < \frac{14\sqrt{42}-27}{1728}\frac{1}{(n+\beta_2)^3} \tag{3.25}$$

with the best possible constants

$$\alpha_2 = \frac{1}{\sqrt[3]{\frac{1728}{14\sqrt{42}-27}(\gamma + \ln\frac{1+\sqrt{\frac{7}{24}}}{8} + \frac{13-2\sqrt{\frac{14}{3}}}{8})}} - 1 = 0.363945\cdots \text{ and } \beta_2 = \frac{431}{20(14\sqrt{42}-27)}.$$

Proof. The inequality is equivalent to the following inequality.

$$\beta_2 < f(n) = \frac{1}{\sqrt[3]{\frac{1728}{14\sqrt{42}-27}(L^1_{n,\sqrt{\frac{7}{24}}} - L)}} - n \le \alpha_2 \tag{3.26}$$

In order to prove, we define the function by

$$f(x) = v(x)^{-1/3} - x, \tag{3.27}$$

where $v(x)$ is the same as in Lemma 3.4.
Differentiating $f(x)$, we obtain

$$f'(x) = -\frac{1}{3}v(x)^{-4/3}\cdot v'(x) - 1. \tag{3.28}$$

Using Lemma 3.4, we can obtain, for $x \ge 5$, $f'(x) < 0$.

Using $f(k)(k=\overline{1,5})$ calculated by mathematica, we conclude that the sequence *f(n)* is strictly decreasing. This leads to

$$\begin{aligned}
&\lim_{n\to\infty} f(n) < f(n) \le f(1) = \\
&= \frac{1}{\sqrt[3]{\frac{1728}{14\sqrt{42}-27}(-\frac{1}{2}+\frac{3}{2}\psi(\frac{3}{2})-\frac{5}{2}\psi(1)+\ln(1+\sqrt{\frac{7}{24}})-\frac{3}{2}-\frac{-5+2\sqrt{\frac{14}{3}}}{8})}} - 1 = \\
&= \frac{1}{\sqrt[3]{\frac{1728}{14\sqrt{42}-27}(\gamma+\ln\frac{1+\sqrt{\frac{7}{24}}}{8}+\frac{13-2\sqrt{\frac{14}{3}}}{8})}} - 1 \\
&= 0.363945\cdots.
\end{aligned} \tag{3.29}$$

It is easy to prove that $\lim_{n\to\infty} f(n) = \frac{431}{20(14\sqrt{42}-27)}$.

$$\begin{aligned}
\beta_2 &= \lim_{n\to\infty} f(n) = \lim_{n\to\infty}(\frac{1}{n^3} - \frac{1293}{20(14\sqrt{42}-27)n^4} + 0(\frac{1}{n^5}))^{-1/3} - n = \\
&= \lim_{n\to\infty} n[(1-\frac{1293}{20(14\sqrt{42}-27)n} + 0(\frac{1}{n^2}))^{-1/3} - 1] = \frac{431}{20(14\sqrt{42}-27)}
\end{aligned} \tag{3.30}$$

The proof of Theorem 3.2 is complete.

4. Numerical Computation

In this section, we give two tables to demonstrate the superiority of our new sequences $L^0_{n,\frac{5}{8}}$, $L^1_{n,\sqrt{\frac{7}{24}}}$, $L^6_{n,\frac{13}{10}}$ over sequences given in [3].

For the convenience of readers, we show $L^6_{n,\frac{13}{10}}$.

$$L^6_{n,\frac{13}{10}} = \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{i+j} - (2\ln 2)n + \ln(n+\frac{13}{10}) - \cfrac{27/40}{n+\cfrac{a\cdot n}{n+\cfrac{b\cdot n}{n+\cfrac{c\cdot n}{n+\cfrac{p\cdot n}{n+q}}}}}\quad , \tag{4.1}$$

where

$$a=\frac{839}{810}, b=\frac{14561}{1359180}, c=\frac{147420243}{244333580}, p=-\frac{2717239967721}{1325053184150},$$
$$q=\frac{116366971023091051\,87}{3094549545661936425}.$$

Combining Theorem 2.1 and Theorem 3.1, 3.2, we have Tables 4.1 and 4.2.

Table 4.1.

N	$L_n - L$	$L^0_{n,\frac{5}{8}} - L$	$L^1_{n,\sqrt{\frac{7}{24}}} - L$
10	-6.10588×10^{-2}	-4.341993×10^{-4}	3.33127×10^{-5}
25	-2.47677×10^{-2}	-7.509458×10^{-5}	2.26636×10^{-6}
50	-1.24418×10^{-2}	-1.927423×10^{-5}	2.89117×10^{-7}
100	-6.23543×10^{-3}	-4.882702×10^{-6}	3.65086×10^{-8}
250	-2.49767×10^{-3}	-7.874722×10^{-7}	2.35082×10^{-9}
1000	-6.24854×10^{-4}	-4.941347×10^{-8}	3.68436×10^{-11}

Table 4.2.

N	v_n	$L - L^6_{n,\frac{13}{10}}$
10	-2.328984×10^{-10}	2.54673×10^{-11}
25	-4.836829×10^{-13}	5.30753×10^{-14}
50	-4.070918×10^{-15}	4.34522×10^{-16}
100	-3.298011×10^{-17}	3.44074×10^{-18}
250	-5.520799×10^{-20}	5.66064×10^{-21}
1000	-3.405723×10^{-24}	3.45776×10^{-25}

References

[1] Abramowitz, M., Stegun, I.A.: Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing, Applied Mathematics Series, vol. 55, Nation Bureau of Standards, Dover, New York (1972).

[2] Chen, C. P.: Continued fraction estimates for the psi function, Applied Mathematics and Computation. 219, 9865-9871 (2013)

[3] Chen , C. P.: Inequalities for the Lugo and Euler-Mascheroni constants, Appl. Math. Lett. 25, 787-792 (2012)

[4] Chen, C. P.: On the asymptotic expansions of the gamma function related to the Nemes, Gosper and Burnside formulas, Applied Mathematics and Computation. 276, 417-431 (2016)

[5] Chen, C. P., Mortici, C.: New sequence converging towards the Euler-Mascheroni constant, Comput. Math. Appl. 64, 391-398 (2012)

[6] Chen, C.-P., Srivastava, H.M.: New representations for the Lugo and Euler–Mascheroni constants, Appl. Math. Lett. 24 (7), 1239–1244 (2011)

[7] Lu, D.: A new quicker sequence convergent to Euler's constant, J. Number Theory 136, 320–329 (2014)

[8] Lu, D.: Some new improved classes of convergence towards Euler's Constant, Applied Mathematics and Computation 243, 24–32 (2014)

[9] Lu, D., Song, L.X., Yu, Y.: Some new continued fraction approximation of Euler's Constant, J. Number Theory 147, 69–80 (2015)

[10] Lugo, I.: An Euler-Maclaurin summation. Available at: http://www.math.upenn.edu/~mlugo /114-07C/euler-maclaurin.pdf.

[11] Mortici, C.: A continued fraction approximation of the gamma function, J. Math. Anal. Appl. 402, 405-410 (2013)

[12] Srivastava, H.M., Choi, J.: Series Associated with the Zeta and Related Functions, Kluwer Academic Publishers, Dordrecht, Boston and London, 2001

[13] Wang, H. Z., Zhang, Q. L., Lu, D.: A quicker approximation of the gamma function towards the Windschitl's formula by continued fraction, Ramanujan J. (2018). https://doi.org/10.1007/s11139-017-9974-6.

[14] You, Xu.: Continued Fraction Approximation and Inequality of the Gamma Function, Results Math. 73, 37 (2018)

[15] You, Xu., Chen, D.R.: Some sharp continued fraction inequalities for the EulerMascheroni constant, J. Inequal. Appl. 2015, 308 (2015)