Identification of Conformable fractional order system with input delay

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ABSTRACT

Classical fractional derivative does not reflect hysteresis characteristic, the unique characteristic of fractional derivative. In order to overcome it, identification approach for the system with input delay expressed by conformable fractional order derivative is proposed. Simulation results shows that the proposed approach infers systematic parameters with high accuracy.

Keyword: fractional calculus, conformable fractional derivative, system identification-

1. Introduction

Fractional calculus is now being widely used in many aspects such as automatic engineering, signal processing, mechanics and biology [1]. It is because it reflects characteristics of many objects with hysteresis characteristic more accurately than integer calculus [3]. A number of approaches on this subject have been proposed up to now.

Because of its uncertainty in its physical meaning, fractional identification has become the most primitive process in its application. Riemann-Liouville and Caputo, Grunwald-Letnikov's fractional calculus is called as classical fractional calculus[9,12-17].

But this is unable to reflect completely its past history of its physical process because weight function of integrand has power characteristic [4-8, 10-11]. For this, conformable fractional calculus is proposed [2]. This paper proposes identification approach in case the system expressed by conformable fractional derivative with input delay. Input delay gives a big effect on output property.

Thus, input delay should be considered in systemization. This paper consists of 5 parts. Part 2 covers fundamental conception, 3 proposes conformable fractional identification approach, 4 gives simulation results, and finally, 5 gives conclusion.

2. Background knowledge

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This chapter covers definitions and properties of classical fractional calculus and conformable fractional calculus in sense of Liouville-Caputo.

2.1 classical fractional calculus [18]

Definition 2.1 When $\Omega = [a,b]$ is a finite interval on real axis, Riemann-Liouville's $\alpha \in R(\alpha > 0)$ order fractional integral $I_{a+}^{\alpha} f$ is defined as follows;

$$\left(I_{a+}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)dt}{(x-t)^{1-\alpha}}, (x > a, \alpha > 0)$$

where $\Gamma(\cdot)$ is gamma function.

Definition 2.2 Riemann-Liouville's $\alpha \in R^+$ order fractional derivative $D_{a+}^{\alpha} y$ is defined as follows;

$$\left(D_{a+}^{\alpha} y\right)(x) = \left(\frac{d}{dx}\right)^n \left(I_{a+}^{n-\alpha} y\right)(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int_a^x \frac{y(t)dt}{(x-t)^{\alpha-n+1}}, (n = [\alpha]+1, x > a)$$

Definition 2.3 Caputo's fractional derivative $\left({}^C D_{a+}^{\alpha} y\right)(x)$ defined on $[a,b] \subset R$ is defined as follows based on $\left(D_{a+}^{\alpha} y\right)(x)$;

$$\left({}^C D_{a+}^{\alpha} y\right)(x) = \left(D_{a+}^{\alpha}\left[y(x) - \sum_{k=0}^{n-1} \frac{y^{(k)}(a)}{k!}(x-a)^k\right]\right), \ n = [\alpha]+1$$

Lemma 2.1 When $\alpha > 0$, $y(x) \in L_2(a,b)$ or $y(x) \in C[a,b]$, the following equation is sustained;

$$\left(I_{a+}^{\alpha} \, {}^C D_{a+}^{\alpha} y\right)(x) = y(x) - \sum_{k=0}^{n-1} \frac{y^{(k)}(a)}{k!}(x-a)^k$$

2.2 conformable fractional calculus in sense of Liouville-Caputo [2]

Definition 2.4 Conformable fractional integral in sense of L-C for $0 < \alpha \le 1$ is defined as follows;

$${}_a^{C\beta} I_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\frac{(t-a)^{\beta} - (\tau-a)^{\beta}}{\beta}\right)^{\alpha-1} \frac{f(t)}{(\tau-a)^{1-\beta}} d\tau, (\alpha > 0, \beta > 0) \quad (2.1)$$

Definition 2.5 Conformable fractional integral in sense of L-C is defined as follows;

$${}_a^{C\beta} D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \left(\frac{(t-a)^{\alpha} - (\tau-a)^{\alpha}}{\alpha}\right)^{n-\beta-1} \frac{{}_a^n D_{\tau}^{\alpha} f(\tau)}{(\tau-a)^{1-\alpha}} d\tau \quad (2.2)$$

Lemma 2.2 Conformable fractional integral in sense of L-C is equal to R-L fractional integral, when $\beta = 1$.

Lemma 2.3 Assume that $f(t) \in C^n_{\alpha,a}([a,b])$. Then,

$$ {}_a^{C\beta}I_t^{\alpha}\left({}_a^{C\beta}D_t^{\alpha} f(t)\right) = f(t) - \sum_{k=0}^{n-1} \frac{{}_a^k D_t^{\alpha} f(a)(t-a)^{\alpha k}}{\alpha^k k!}. $$

3. Main results

3.1 Conformable fractional operational matrix in sense of L-C

Definition 3.1 [19] The block pulse functions defined in half-closed interval $[0,T)$ is defined as follows;

$$ b_n(t) = \begin{cases} 1, & (\mathrm{n}-1)h \le t < nh \\ 0, & otherwise \end{cases} \tag{3.1} $$

where $n = 1,\dots,N$. N is the number of Block Pulse Function(BPF)s, $h = T/N$. $B_N(t) = [b_1(t),\dots,b_N(t)]^T$ is block pulse vector.

When every component of $\mathbf{f} = [f_1,\dots,f_N]^T$ is defined as

$$ f_n = \frac{M}{T}\int_{\frac{n-1}{N}T}^{\frac{n}{N}T} f(t)dt \approx f((n-1)h) $$

any absolutely integrable function in $[0,T)$ is expressed by BPFs as follows;

$$ f(t) = \sum_{n=1}^{N}\left[\frac{M}{T}\int_{\frac{n-1}{N}T}^{\frac{n}{N}T} f(t)dt\right]b_n(t) \approx \sum_{n=1}^{N} f_n b_n(t) = \boldsymbol{f}^T B_N(t) \tag{3.2} $$

Definition 3.2 Assume that $F(\alpha,\beta)$ is conformable fractional operational matrix in sense of L-C with $\alpha,\beta \in (0,1]$. Then, an element of this matrix is obtained as follows;

$$ F(\alpha,\beta) = \frac{h^{\alpha\beta}}{\Gamma(\alpha+2)} \begin{pmatrix} \eta_{11} & \eta_{12} & \eta_{13} & \cdots & \eta_{1N} \\ 0 & \eta_{22} & \eta_{23} & \cdots & \eta_{2N} \\ 0 & 0 & \eta_{33} & \cdots & \eta_{3N} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \eta_{NN} \end{pmatrix} \tag{3.3} $$

where

$$
\eta_{nm} = \begin{cases} \dfrac{1}{\beta^{\alpha}} \dfrac{\alpha+1}{\alpha\beta+1} \quad , \quad (n=m=1) \\ \dfrac{1}{(n-1)^{(1-\beta)\alpha}} \quad , \quad (n=m \geq 2) \\ \dfrac{(m-1)^{1-\beta}}{\beta^{\alpha+1}} \Big\{ \left[(m-1+\beta)(m-1)^{\beta-1} - (n-1)^{\beta} \right]^{\alpha+1} - \left[(m-1)^{\beta} - (n-1)^{\beta} \right]^{\alpha+1} \\ \qquad - \left[(m-1+\beta)(m-1)^{\beta-1} - n^{\beta} \right]^{\alpha+1} + \left[(m-1)^{\beta} - n^{\beta} \right]^{\alpha+1} \Big\}, (m > n) \end{cases} \tag{3.4}
$$

Proof Case 1

When $t<(n-1)h$, ${}_0^{C\beta}I_t^{\alpha}(b_n(t))=0$. Thus, Matrix $F(\alpha,\beta)$ is upper-triangle matrix.

$f_{ij}(\alpha,\beta)=0,(i>j)$

where, when $(n-1)h \leq t < nh$,

$$
{}_0^{C\beta}I_t^{\alpha}(b_n(t)) = \frac{1}{\Gamma(\alpha)} \int_{(n-1)h}^{t} \left(\frac{t^{\beta}-\tau^{\beta}}{\beta} \right)^{\alpha-1} \frac{1}{\tau^{1-\beta}} d\tau = \frac{1}{\Gamma(\alpha+1)} \frac{\left[t^{\beta} - ((n-1)h)^{\beta} \right]^{\alpha}}{\beta^{\alpha}}.
$$

Diagonal element $f_{nn}(\alpha,\beta),(1 \leq n \leq N)$ of Matrix $F(\alpha,\beta)$ is determined by average value in $[(n-1)h,nh]$ as in Equation (3,2). i.e.

$$
\begin{aligned}
f_{nn}(\alpha,\beta) &= \frac{1}{h} \frac{1}{\Gamma(\alpha+1)} \int_{(n-1)h}^{nh} \frac{\left[t^{\beta} - ((n-1)h)^{\beta} \right]^{\alpha}}{\beta^{\alpha}} dt \\
&\overset{\frac{t-(n-1)h}{(n-1)h}=x}{=} \frac{h^{\alpha\beta}}{\Gamma(\alpha+1)\beta^{\alpha}} (n-1)^{\alpha\beta+1} \int_0^{\frac{1}{n-1}} \left((1+x)^{\beta} - 1 \right)^{\alpha} dx \\
&\overset{(1+x)^{\beta} \approx 1+\beta x}{\approx} \frac{h^{\alpha\beta}}{\Gamma(\alpha+2)} (n-1)^{\alpha\beta-\alpha} \quad ,(n \geq 2)
\end{aligned}
$$

When $n=1$, $f_{11}(\alpha,\beta) = \dfrac{h^{\alpha\beta}}{\Gamma(\alpha+2)} \dfrac{1}{\beta^{\alpha}} \dfrac{\alpha+1}{\alpha\beta+1}$.

Case 3

When $(m-1)h \leq t < mh$,$(m>n)$,

$$
\begin{aligned}
{}_0^{C\beta}I_t^{\alpha}(b_n(t)) &= \frac{1}{\Gamma(\alpha)} \int_{(n-1)h}^{nh} \left(\frac{t^{\beta}-\tau^{\beta}}{\beta} \right)^{\alpha-1} \frac{1}{\tau^{1-\beta}} d\tau \\
&= \frac{1}{\Gamma(\alpha+1)\beta^{\alpha}} \left[\left(t^{\beta} - ((n-1)h)^{\beta} \right)^{\alpha} - \left(t^{\beta} - (nh)^{\beta} \right)^{\alpha} \right]
\end{aligned}.
$$

Element $f_{nm}(\alpha,\beta),(n<m)$ of Matrix $F(\alpha,\beta)$ is determined by average value in $[(m-1)h,mh]$ like above.

$$f_{nm}(\alpha,\beta)=\frac{1}{h}\frac{1}{\Gamma(\alpha+1)\beta^{\alpha}}\int_{(m-1)h}^{mh}\left[\left(t^{\beta}-((n-1)h)^{\beta}\right)^{\alpha}-\left(t^{\beta}-(nh)^{\beta}\right)^{\alpha}\right]dt$$

$$\overset{\frac{t-(m-1)h}{(m-1)h}=x}{=}\frac{h^{\alpha\beta}}{\Gamma(\alpha+1)\beta^{\alpha}}(m-1)^{\alpha\beta+1}\int_0^{\frac{1}{m-1}}\left[\left((1+x)^{\beta}-\left(\frac{n-1}{m-1}\right)^{\beta}\right)^{\alpha}-\left((1+x)^{\beta}-\left(\frac{n}{m-1}\right)^{\beta}\right)^{\alpha}\right]dx$$

$$\overset{(x+1)^{\beta}\approx 1+\beta x}{\approx}\frac{h^{\alpha\beta}}{\Gamma(\alpha+2)}\frac{(m-1)^{1-\beta}}{\beta^{\alpha+1}}\left\{\left[(m-1+\beta)(m-1)^{\beta-1}-(n-1)^{\beta}\right]^{\alpha+1}-\left[(m-1)^{\beta}-(n-1)^{\beta}\right]^{\alpha+1}\right.$$
$$\left.-\left[(m-1+\beta)(m-1)^{\beta-1}-n^{\beta}\right]^{\alpha+1}+\left[(m-1)^{\beta}-n^{\beta}\right]^{\alpha+1}\right\}$$

To sum up, we get Eq. (3,4). (End of proof)

Conformable fractional calculus in sense of L-C of $f(t)$ using the definition above is approximated as follows;

$${}_0^{C\beta}I_t^{\alpha}f(t)\approx\mathbf{f}^T\ {}_0^{C\beta}I_t^{\alpha}(B_N(t))\approx\mathbf{f}^T F(\alpha,\beta)B_N(t)$$

3.2 Delayed conformable fractional Integral operating matrix in sense of L-C

From Eq. (3,2), absolutely integrable function with delay $f(t-\tau)$ is developed as follows;

$$f(t-\tau)\approx\sum_{n=1}^{N}f_n b_n(t-\tau)=\mathrm{f}^T B_N(t-\tau) \tag{3.5}$$

If time delay T is given as follows;

$$\tau=\frac{k}{N}T,\ k=0,1,\cdots,N-1$$

Then,

$$b_n(t-\tau)=\begin{cases}b_n(t) & (n-1)h\le t-kh<nh\\ 0 & otherwise\end{cases}=b_{(n+k)}(t)$$

Thus, delay matrix E_k is expressed as follows;

$$E_k=\begin{pmatrix}0&\cdots&0&\overset{k\text{-}th}{1}&0&\cdots&0\\ \cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots\\ 0&\cdots&0&0&0&\cdots&1\\ 0&\cdots&0&0&0&\cdots&0\\ \cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots\\ 0&\cdots&0&0&0&\cdots&0\end{pmatrix}_{N\times N}$$

Thus, conformable fractional calculus in sense of L-C of $f(t-\tau)$ is approximated as follows;

$$ {}_0^{C\beta}I_t^{\alpha} f(t-\tau) \approx \boldsymbol{f}^T \, {}_0^{C\beta}I_t^{\alpha}\left(B_N(t-\tau)\right) \approx \boldsymbol{f}^T F(\alpha,\beta) E_k B_N(t) \tag{3.6} $$

3.3 Identification for system with conformable fractional derivative

Study on Fractional Order System (FOS) expressed by conformable fractional derivative.

$$ \begin{aligned} &\sum_{i=0}^{n} a_i \cdot {}_0^{C\gamma} D_t^{\alpha_i} x(t) = \sum_{j=0}^{m} b_j \cdot {}_0^{C\gamma} D_t^{\beta_j} u_j\left(t-\tau_j\right) \\ &y(t) = x(t) + v(t) \end{aligned} \tag{3.7} $$

where $u(t), v(t), y(t), x(t)$ is input, noise, output signal and state parameter. It can be $a_n = 1$ objectively.

Assume that differential orders $\alpha_i, \beta_j, \gamma \in [0,1)$, $i = 0,1,\cdots,n,\ j = 0,1,\cdots m$ is given and

$0 = \alpha_0 < \alpha_1 < \cdots < \alpha_n, 0 = \beta_0 < \beta_1 < \beta_2 < \cdots < \beta_m, \alpha_n > \beta_m$.

Adopt α_n order Conformable Fractional Integral (CFI) to both sides and Eq. (3.4) and (3.6), then

$$ \sum_{i=0}^{n} a_i \mathrm{Y}^T F(\alpha_n - \alpha_i, \gamma) B_N(t) = \sum_{j=0}^{m} b_j \mathrm{U}^T F(\beta_j - \alpha_i, \gamma) E_{k_j} B_N(t). $$

where $\tau_j = k_j h = k_j \dfrac{T}{N},\ j = 0,1,\cdots,m$.

Form the structural characteristic of Conformable Fractional Integral Operational Matrix (CFIOM), the following recurrence equation is obtained;

$$ \sum_{i=0}^{n} A_i \mathrm{Y}^i(r) = \sum_{j=0}^{m} B_j Z^j\left(r - k_j\right) \tag{3.8} $$

Where

$$ \begin{aligned} &a_i \frac{h^{(\alpha_n-\alpha_i)\gamma}}{\Gamma(\alpha_n - \alpha_i + 2)} = A_i, \quad b_j \frac{h^{(\alpha_n-\beta_j)\gamma}}{\Gamma(\alpha_n - \beta_j + 2)} = B_j \\ &\mathrm{Y}^i(r) = \eta_{1r}^{(\alpha_n-\alpha_i)} \mathrm{Y}_1 + \eta_{2r}^{(\alpha_n-\alpha_i)} \mathrm{Y}_2 + \cdots + \eta_{rr}^{(\alpha_n-\alpha_i)} \mathrm{Y}_r, (i = 0,\cdots,n) \\ &\mathrm{Z}^j(r) = \eta_{1r}^{(\alpha_n-\beta_j)} \mathrm{U}_1 + \eta_{2r}^{(\alpha_n-\beta_j)} \mathrm{U}_2 + \cdots + \eta_{rr}^{(\alpha_n-\beta_j)} \mathrm{U}_r, (j = 0,\cdots,m) \end{aligned} . $$

Caution: $Z^j\left(r - k_j\right) = 0,\ k_j \ge r, 1 \le r \le N$, $A_n = 1$.

Recurrence equation (3.8) is as follows;

$$ \mathrm{Y}_n(r) = \varphi^T(r,k)\theta \tag{3.9} $$

Where

$$\varphi(r,k)=\left[-Y^0(r),-Y^1(r),\cdots,-Y^{n-1}(r),Z^0(r-k_0),\cdots,Z^m(r-k_m)\right]^T$$
$$\theta=\left[A_0,A_1,\cdots,A_{n-1},B_0,\cdots,B_m\right]^T$$

The estimated output from Eq. (3.9) is

$$\hat{Y}_n(r)=\varphi^T\left(r,\hat{k}\right)\hat{\theta}. \tag{3.10}$$

The optimum value can be obtained by minimizing target function.

$$\left(\hat{\theta},\hat{k}\right)=arg\min_{\theta\in\Omega}\frac{1}{N}\sum_{k=1}^{N}\left[y(kh)-\hat{y}(kh)\right]^2 \tag{3.11}$$

where Ω is allowance region.

4. Simulation result

Consider the following model Conformable Fractional Order System (CFOS).

$$\begin{aligned} &{}^{C0.75}_{\;0}D_t^{0.8}x(t)+a_1\,{}^{C0.75}_{\;0}D_t^{0.5}x(t)+a_2x(t)=bu(t-\tau)\\ &y(t)=x(t)+v(t)\end{aligned} \tag{4.1}$$

where $a_1=0.8, a_2=0.5, b=1.5, \tau=0.6$. Assume that $\theta=\left[a_1,a_2,b,\tau\right]$ is a parameter to be estimated. The number of BPFs is $N=2000$.

Fig.1 shows step response when CFOS (4.1) has several input delay in case there's no noise.

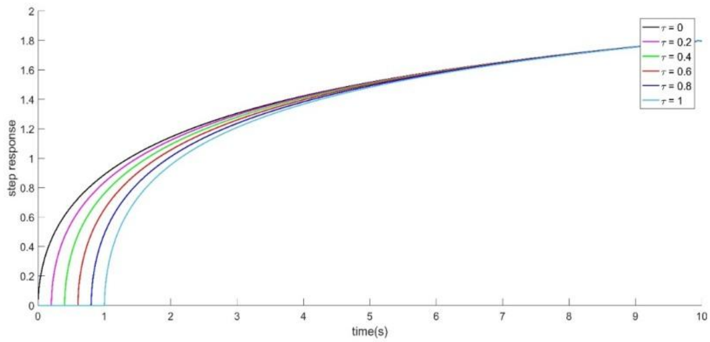

Fig.1 output characteristic with or without output delay

Fig.2 shows output characteristic in case of classical fractional derivative and conformable fractional derivative without noise when the following filtered output signal is given.

$$u(t)=\sum_{k=0}^{25}sin\left(0.1\pi kt-\frac{k(k-1)\pi}{25}\right) \tag{4.2}$$

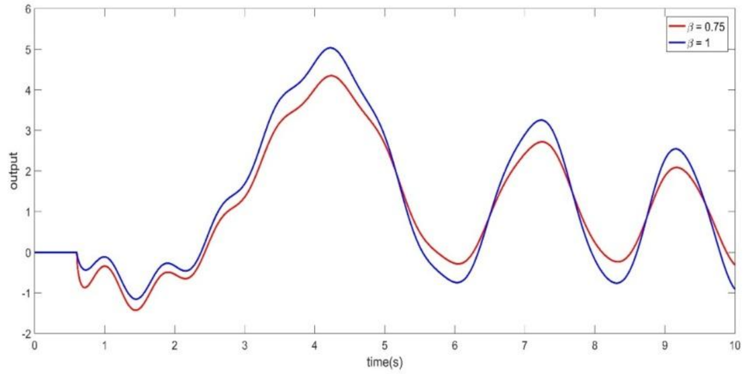

Fig.2 output characteristic of classical fractional calculus and conformable fractional calculus

Table 1 gives sample mean and dispersion on the estimated system parameters from 50 simulations according to identification algorithm proposed in different noise environment.

Table 1 sample mean and dispersion on the estimated system parameters

θ	**Real value**	**SNR=20dB**	**SNR=10dB**
		$\hat{\theta}\left(\hat{\sigma}^2\right)$	$\hat{\theta}\left(\hat{\sigma}^2\right)$
a_3	0.8	0.8010(0.0030)	0.8015(0.0105)
a_2	0.5	0.5005(0.0042)	0.5012(0.0207)
b	1.5	1.5007(0.0020)	1.5010(0.0012)
τ	0.6	0.6000(0.0003)	0.6000(0.0003)
MSE	-	0.0031	0.0102

Table 3 shows the relationship between estimated output curve and real one in different noise (SNR=20dB, 10dB) environment.

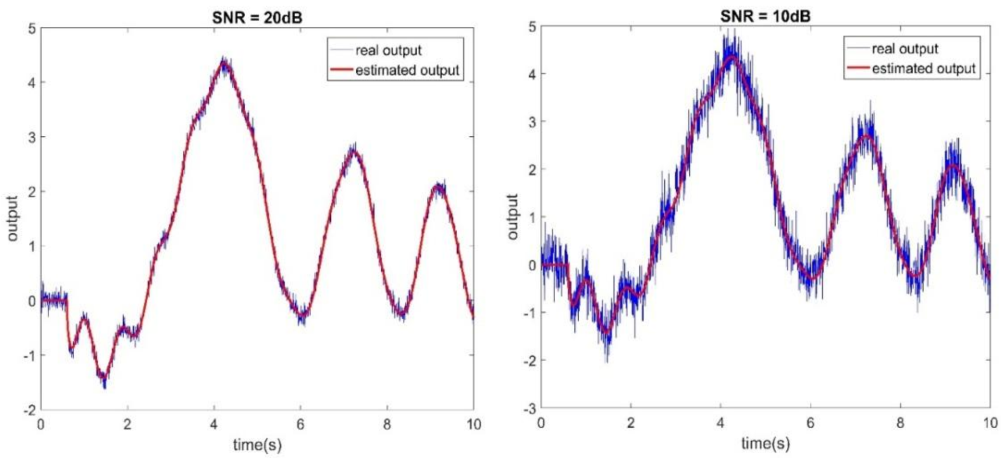

Fig. 3 output characteristic of identified conformable fractional system

5. Conclusion

This paper newly suggests conformable fractional operational matrix and, on this basis, proposes system identification approach. Having ensured that output characteristic of fraction expressed by conformable fractional derivative, this paper proposes requirements to identify output delay. It has been proved that conformable fractional identification approach estimate accurately parameters in different noise levels through simulation.

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