ON RELATIONAL MECHANICS

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In classical mechanics, a new reformulation is presented, which is totally in accordance with the general principle of relativity, which is invariant under transformations between inertial and non-inertial reference frames, which can be applied in any reference frame without introducing fictitious forces, which is observationally equivalent to Newtonian mechanics, and which establishes the existence of a new universal force of interaction, called kinetic force (which is related to the force of inertia $-m\mathbf{a}$, and also to Mach's principle)

Kws : Relational Mechanics · Mach's Principle · Classical Mechanics · Kinetic Force

Introduction

The new reformulation in classical mechanics presented in this paper is obtained starting from an auxiliary system of particles (called Universe) that is used to obtain kinematic magnitudes (such as universal position, universal velocity, etc.) that are invariant under transformations between inertial and non-inertial reference frames.

The universal position $\mathbf{r}_i$, the universal velocity $\mathbf{v}_i$ and the universal acceleration $\mathbf{a}_i$ of a particle i relative to a reference frame S (inertial or non-inertial) are given by:

$$\mathbf{r}_i \doteq (\tilde{r}_i) = (\vec{r}_i - \vec{R})$$

$$\mathbf{v}_i \doteq d(\tilde{r}_i)/dt = (\vec{v}_i - \vec{V}) - \vec{\omega} \times (\vec{r}_i - \vec{R})$$

$$\mathbf{a}_i \doteq d^2(\tilde{r}_i)/dt^2 = (\vec{a}_i - \vec{A}) - 2\,\vec{\omega} \times (\vec{v}_i - \vec{V}) + \vec{\omega} \times [\,\vec{\omega} \times (\vec{r}_i - \vec{R})\,] - \vec{\alpha} \times (\vec{r}_i - \vec{R})$$

where $\tilde{r}_i$ is the position vector of particle i relative to the universal frame [$\vec{r}_i$ is the position vector of particle i, $\vec{R}$ is the position vector of the center of mass of the Universe, and $\vec{\omega}$ is the angular velocity vector of the Universe] [relative to the frame S] (see Annex I)

The universal frame is a reference frame fixed to the Universe ($\vec{\omega} = 0$) whose origin always coincides with the center of mass of the Universe ($\vec{R} = \vec{V} = \vec{A} = 0$)

Any reference frame S is an inertial frame when the angular velocity $\vec{\omega}$ of the Universe and the acceleration $\vec{A}$ of the center of mass of the Universe are equal to zero relative to S.

Note : ($\forall\ \mathbf{m} \in$ Universal Magnitudes : If $\mathbf{m} = \vec{n} \longrightarrow d(\mathbf{m})/dt = d(\vec{n})/dt - \vec{\omega} \times \vec{n}$)

The New Dynamics

[1] A force is always caused by the interaction between two or more particles.

[2] The total force $\mathbf{T}_i$ acting on a particle i is always equal to zero : [$\mathbf{T}_i = 0$]

[3] In this paper, we assume that all dynamic forces (that is, all non-kinetic forces) always obey Newton's third law in its weak form and in its strong form.

The Kinetic Force

The kinetic force $\mathbf{K}_{ij}$ exerted on a particle i of mass m_i by another particle j of mass m_j, caused by the interaction between particle i and particle j, is given by:

$$\mathbf{K}_{ij} \;=\; -\,\frac{m_i\, m_j}{M}\,(\mathbf{a}_i - \mathbf{a}_j)$$

where $\mathbf{a}_i$ is the universal acceleration of particle i, $\mathbf{a}_j$ is the universal acceleration of particle j, and M $(= \sum_i^{All} m_i)$ is the mass of the Universe.

From the above equation it follows that the net kinetic force $\mathbf{K}_i$ $(= \sum_j^{All} \mathbf{K}_{ij})$ acting on a particle i of mass m_i is given by:

$$\mathbf{K}_i \;=\; -\,m_i\,(\mathbf{a}_i - \mathbf{A})$$

where $\mathbf{a}_i$ is the universal acceleration of particle i and $\mathbf{A}$ $(= M^{-1} \sum_i^{All} m_i\,\mathbf{a}_i)$ is the universal acceleration of the center of mass of the Universe.

Since the universal acceleration of the center of mass of the Universe $\mathbf{A}$ is always zero, then the net kinetic force $\mathbf{K}_i$ acting on a particle i of mass m_i is certainly given by:

$$\mathbf{K}_i \;=\; -\,m_i\,\mathbf{a}_i$$

where $\mathbf{a}_i$ is the universal acceleration of particle i.

The net kinetic force $\mathbf{K}_i$ is related to the force of inertia $-m\mathbf{a}$ (vis insita) and the kinetic force $\mathbf{K}_{ij}$ (as the origin of $-m\mathbf{a}$) is related to Mach's principle.

The force $\mathbf{K}_i$ is the force that balances the net dynamic force in each particle of the Universe and the force $\mathbf{K}_{ij}$ always obey Newton's third law in its strong form or in its weak form.

On fields and potentials of the forces $\mathbf{K}_i$ and $\mathbf{K}_{ij}$ see : Annex A and Annex B. The force $\mathbf{K}_{ij}$ is obtained starting from Newtonian mechanics in : https://doi.org/10.5281/zenodo.1215207

The [2] Principle

The second principle of the new dynamics establishes that the total force $\mathbf{T}_i$ acting on a particle i is always equal to zero.

$$\mathbf{T}_i \;=\; 0$$

If the total force $\mathbf{T}_i$ is divided into the following two parts: the net kinetic force $\mathbf{K}_i$ and the net dynamic force $\mathbf{F}_i$ ($\sum$ of gravitational forces, electrostatic forces, etc.) then we have:

$$\mathbf{K}_i + \mathbf{F}_i \;=\; 0$$

Now, substituting $\mathbf{K}_i$ $(= -\,m_i\,\mathbf{a}_i)$ and rearranging, we finally obtain:

$$\mathbf{F}_i \;=\; m_i\,\mathbf{a}_i$$

This equation (similar to Newton's second law) will be used throughout this paper.

The Equation of Motion

The net dynamic force $\mathbf{F}_i$ acting on a particle i of mass m_i is related to the universal acceleration $\mathbf{a}_i$ of particle i according to the following equation:

$$\mathbf{F}_i = m_i \, \mathbf{a}_i$$

From the above equation it follows that the (ordinary) acceleration $\vec{a}_i$ of particle i relative to a reference frame S (inertial or non-inertial) is given by:

$$\vec{a}_i = \mathbf{F}_i/m_i + \vec{A} + 2\,\vec{\omega} \times (\vec{v}_i - \vec{V}) - \vec{\omega} \times [\,\vec{\omega} \times (\vec{r}_i - \vec{R})\,] + \vec{\alpha} \times (\vec{r}_i - \vec{R})$$

where $\vec{r}_i$ is the position vector of particle i, $\vec{R}$ is the position vector of the center of mass of the Universe, and $\vec{\omega}$ is the angular velocity vector of the Universe relative to S (see Annex I)

From the above equation it follows that particle i can have a non-zero acceleration ($\vec{a}_i \neq 0$) even if there is no dynamic force acting on particle i, and also that particle i can have zero acceleration ($\vec{a}_i = 0$) (state of rest or of uniform linear motion) even if there is an unbalanced net dynamic force acting on particle i.

However, from the above equation it also follows that Newton's first and second laws are valid in any inertial reference frame, since the angular velocity $\vec{\omega}$ of the Universe and the acceleration $\vec{A}$ of the center of mass of the Universe are equal to zero relative to any inertial reference frame.

In this paper, any reference frame S is an inertial frame when the angular velocity $\vec{\omega}$ of the Universe and the acceleration $\vec{A}$ of the center of mass of the Universe are equal to zero relative to the frame S (a Machian definition of inertial frame)

On the other hand, the new reformulation of classical mechanics presented in this paper is observationally equivalent to Newtonian mechanics.

However, non-inertial observers can use Newtonian mechanics only if they introduce fictitious forces into $\mathbf{F}_i$ (such as the centrifugal force, the Coriolis force, etc.) From the above equation it follows that : $\mathbf{F}_{fictitious} = m_i \, \{ + \vec{A} + 2\,\vec{\omega} \times (\vec{v}_i - \vec{V}) - \vec{\omega} \times [\,\vec{\omega} \times (\vec{r}_i - \vec{R})\,] + \vec{\alpha} \times (\vec{r}_i - \vec{R}) \,\}$

Additionally, the new reformulation of classical mechanics presented in this paper is also a relational reformulation of classical mechanics since it is obtained starting from relative magnitudes (position, velocity and acceleration) between particles.

However, as already stated above, the new reformulation of classical mechanics presented in this paper is observationally equivalent to Newtonian mechanics.

Finally, the equation [$\mathbf{F}_i = m_i \, \mathbf{a}_i$] is valid in all reference frames (inertial or non-inertial) only if all dynamic forces always obey Newton's third law in its weak form and in its strong form (that is, ... only if the equation [$\mathbf{F}_i = m_i \, \vec{a}_i$] is always valid in the universal frame)

A. Tobla, A Reformulation of Classical Mechanics (I & II) : https://doi.org/10.5281/zenodo.11207437

A. Tobla, A Reformulation of Classical Mechanics (III & IV) : https://doi.org/10.5281/zenodo.11207459

Annex I

The Universe

The Universe is a system that contains all particles, that is always free of external forces, and that all internal dynamic forces always obey Newton's third law in its weak form and in its strong form.

The position $\vec{R}$, the velocity $\vec{V}$ and the acceleration $\vec{A}$ of the center of mass of the Universe relative to a reference frame S (and the angular velocity $\vec{\omega}$ and the angular acceleration $\vec{\alpha}$ of the Universe relative to the reference frame S) are given by:

$$\mathrm{M} \doteq \textstyle\sum_i^{All} m_i$$

$$\vec{R} \doteq \mathrm{M}^{-1} \textstyle\sum_i^{All} m_i \, \vec{r}_i$$

$$\vec{V} \doteq \mathrm{M}^{-1} \textstyle\sum_i^{All} m_i \, \vec{v}_i$$

$$\vec{A} \doteq \mathrm{M}^{-1} \textstyle\sum_i^{All} m_i \, \vec{a}_i$$

$$\vec{\omega} \doteq \overleftrightarrow{I}^{-1} \cdot \vec{L}$$

$$\vec{\alpha} \doteq d(\vec{\omega})/dt$$

$$\overleftrightarrow{I} \doteq \textstyle\sum_i^{All} m_i \, [\, |\, \vec{r}_i - \vec{R} \,|^2 \, \overleftrightarrow{1} - (\vec{r}_i - \vec{R}) \otimes (\vec{r}_i - \vec{R}) \,]$$

$$\vec{L} \doteq \textstyle\sum_i^{All} m_i \, (\vec{r}_i - \vec{R}) \times (\vec{v}_i - \vec{V})$$

where M is the mass of the Universe, $\overleftrightarrow{I}$ is the inertia tensor of the Universe (relative to $\vec{R}$) and $\vec{L}$ is the angular momentum of the Universe relative to the reference frame S.

The Transformations

The transformations of position, velocity and acceleration of a particle i between a reference frame S and another reference frame S', are given by:

$$(\vec{r}_i - \vec{R}) \;=\; \mathbf{r}_i \;=\; \mathbf{r}'_i$$

$$(\vec{r}'_i - \vec{R}') \;=\; \mathbf{r}'_i \;=\; \mathbf{r}_i$$

$$(\vec{v}_i - \vec{V}) - \vec{\omega} \times (\vec{r}_i - \vec{R}) \;=\; \mathbf{v}_i \;=\; \mathbf{v}'_i$$

$$(\vec{v}'_i - \vec{V}') - \vec{\omega}' \times (\vec{r}'_i - \vec{R}') \;=\; \mathbf{v}'_i \;=\; \mathbf{v}_i$$

$$(\vec{a}_i - \vec{A}) - 2\,\vec{\omega} \times (\vec{v}_i - \vec{V}) + \vec{\omega} \times [\,\vec{\omega} \times (\vec{r}_i - \vec{R})\,] - \vec{\alpha} \times (\vec{r}_i - \vec{R}) \;=\; \mathbf{a}_i \;=\; \mathbf{a}'_i$$

$$(\vec{a}'_i - \vec{A}') - 2\,\vec{\omega}' \times (\vec{v}'_i - \vec{V}') + \vec{\omega}' \times [\,\vec{\omega}' \times (\vec{r}'_i - \vec{R}')\,] - \vec{\alpha}' \times (\vec{r}'_i - \vec{R}') \;=\; \mathbf{a}'_i \;=\; \mathbf{a}_i$$

Annex A

Fields and Potentials I

The net kinetic force $\mathbf{K}_i$ acting on a particle i of mass m_i can also be expressed as follows:

$$\mathbf{K}_i \;=\; +\,m_i \Big[\, \mathbf{E} + (\vec{v}_i - \vec{V}) \times \mathbf{B} \,\Big]$$

$$\mathbf{K}_i \;=\; +\,m_i \Big[-\nabla\phi - \partial\mathbf{A}/\partial t + (\vec{v}_i - \vec{V}) \times (\nabla \times \mathbf{A}) \Big]$$

$$\mathbf{K}_i \;=\; +\,m_i \Big[-(\vec{a}_i - \vec{A}) + 2\,\vec{\omega} \times (\vec{v}_i - \vec{V}) - \vec{\omega} \times [\,\vec{\omega} \times (\vec{r}_i - \vec{R})\,] + \vec{\alpha} \times (\vec{r}_i - \vec{R}) \Big]$$

where:

$$\mathbf{E} \;=\; -\,\nabla\phi - \partial\mathbf{A}/\partial t$$

$$\mathbf{B} \;=\; \nabla \times \mathbf{A}$$

$$\phi \;=\; -\,{}^1\!/_2\,[\,\vec{\omega} \times (\vec{r}_i - \vec{R})\,]^2 \;+\; {}^1\!/_2\,(\vec{v}_i - \vec{V})^2$$

$$\mathbf{A} \;=\; -\,[\,\vec{\omega} \times (\vec{r}_i - \vec{R})\,] + (\vec{v}_i - \vec{V})$$

$$\partial\mathbf{A}/\partial t \;=\; -\,\vec{\alpha} \times (\vec{r}_i - \vec{R}) + (\vec{a}_i - \vec{A}) \;*$$

$$\nabla\phi \;=\; \vec{\omega} \times [\,\vec{\omega} \times (\vec{r}_i - \vec{R})\,]$$

$$\nabla \times \mathbf{A} \;=\; -\,2\,\vec{\omega}$$

— Differential Equations —

$$\nabla \cdot \mathbf{E} = 2\,\vec{\omega}^{\,2} \quad , \quad \nabla \cdot \mathbf{B} = 0$$

$$\nabla\times\mathbf{E} = -\partial\mathbf{B}/\partial t \;,\; \nabla\times\mathbf{B} = 0$$

The net kinetic force $\mathbf{K}_i$ acting on a particle i of mass m_i can also be obtained starting from the following kinetic energy:

$$K_i \;=\; -\,m_i\,[\,\phi - (\vec{v}_i - \vec{V}) \cdot \mathbf{A}\,]$$

$$K_i \;=\; {}^1\!/_2\,m_i\,\big[\,(\vec{v}_i - \vec{V}) - \vec{\omega} \times (\vec{r}_i - \vec{R})\,\big]^2$$

$$K_i \;=\; {}^1\!/_2\,m_i\,\big[\,\mathbf{v}_i\,\big]^2$$

Since the kinetic energy K_i must be positive, then applying the following Euler-Lagrange equation, we obtain:

$$\mathbf{K}_i \;=\; -\,\frac{d}{dt}\left[\frac{\partial\,{}^1\!/_2\,m_i\,\big[\,\mathbf{v}_i\,\big]^2}{\partial\,\mathbf{v}_i}\right] + \frac{\partial\,{}^1\!/_2\,m_i\,\big[\,\mathbf{v}_i\,\big]^2}{\partial\,\mathbf{r}_i} \;=\; -\,m_i\,\mathbf{a}_i$$

where $\mathbf{r}_i$, $\mathbf{v}_i$ and $\mathbf{a}_i$ are the universal position, the universal velocity and the universal acceleration of particle i.

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* In the temporal partial derivative, the spatial coordinates must be treated as constants [or replace this in the first equation: $+\,{}^1\!/_2\,(\vec{v}_i - \vec{V}) \times \mathbf{B}$, and this in the second equation: $+\,{}^1\!/_2\,(\vec{v}_i - \vec{V}) \times (\nabla \times \mathbf{A})$]

Annex B

Fields and Potentials II

The kinetic force $\mathbf{K}_{ij}$ exerted on a particle i of mass m_i by another particle j of mass m_j can also be expressed as follows:

$$\mathbf{K}_{ij} \;=\; +\, m_i\, m_j\, M^{-1} \Big[\, \mathbf{E} + (\vec{v}_i - \vec{v}_j) \times \mathbf{B} \,\Big]$$

$$\mathbf{K}_{ij} \;=\; +\, m_i\, m_j\, M^{-1} \Big[-\nabla\phi \,-\, \partial\mathbf{A}/\partial t \,+\, (\vec{v}_i - \vec{v}_j) \times (\nabla \times \mathbf{A}) \Big]$$

$$\mathbf{K}_{ij} \;=\; +\, m_i\, m_j\, M^{-1} \Big[-(\vec{a}_i - \vec{a}_j) + 2\,\vec{\omega} \times (\vec{v}_i - \vec{v}_j) - \vec{\omega} \times [\,\vec{\omega} \times (\vec{r}_i - \vec{r}_j)\,] + \vec{\alpha} \times (\vec{r}_i - \vec{r}_j) \Big]$$

where:

$$\mathbf{E} \;=\; -\,\nabla\phi \,-\, \partial\mathbf{A}/\partial t$$

$$\mathbf{B} \;=\; \nabla \times \mathbf{A}$$

$$\phi \;=\; -\,{}^{1\!}/_{2}\,[\,\vec{\omega} \times (\vec{r}_i - \vec{r}_j)\,]^2 \,+\, {}^{1\!}/_{2}\,(\vec{v}_i - \vec{v}_j)^2$$

$$\mathbf{A} \;=\; -\,[\,\vec{\omega} \times (\vec{r}_i - \vec{r}_j)\,] \,+\, (\vec{v}_i - \vec{v}_j)$$

$$\partial\mathbf{A}/\partial t \;=\; -\,\vec{\alpha} \times (\vec{r}_i - \vec{r}_j) + (\vec{a}_i - \vec{a}_j)\ ^*$$

$$\nabla\phi \;=\; \vec{\omega} \times [\,\vec{\omega} \times (\vec{r}_i - \vec{r}_j)\,]$$

$$\nabla \times \mathbf{A} \;=\; -\,2\,\vec{\omega}$$

— Differential Equations —

$$\nabla \cdot \mathbf{E} = 2\,\vec{\omega}^{\,2} \quad , \;\; \nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\partial\mathbf{B}/\partial t \;\; , \;\; \nabla \times \mathbf{B} = 0$$

The kinetic force $\mathbf{K}_{ij}$ exerted on a particle i of mass m_i by another particle j of mass m_j can also be obtained starting from the following kinetic energy:

$$K_{ij} \;=\; -\, m_i\, m_j\, M^{-1} \big[\, \phi - (\vec{v}_i - \vec{v}_j) \cdot \mathbf{A} \,\big]$$

$$K_{ij} \;=\; {}^{1\!}/_{2}\, m_i\, m_j\, M^{-1} \big[\, (\vec{v}_i - \vec{v}_j) - \vec{\omega} \times (\vec{r}_i - \vec{r}_j) \,\big]^2$$

$$K_{ij} \;=\; {}^{1\!}/_{2}\, m_i\, m_j\, M^{-1} \big[\, \mathbf{v}_i - \mathbf{v}_j \,\big]^2$$

Since the kinetic energy K_{ij} must be positive, then applying the following Euler-Lagrange equation, we obtain:

$$\mathbf{K}_{ij} \;=\; -\,\frac{d}{dt}\left[\frac{\partial\, {}^{1\!}/_{2}\, \frac{m_i\, m_j}{M} \big[\, \mathbf{v}_i - \mathbf{v}_j \,\big]^2}{\partial\, [\, \mathbf{v}_i - \mathbf{v}_j \,]}\right] + \frac{\partial\, {}^{1\!}/_{2}\, \frac{m_i\, m_j}{M} \big[\, \mathbf{v}_i - \mathbf{v}_j \,\big]^2}{\partial\, [\, \mathbf{r}_i - \mathbf{r}_j \,]} \;=\; -\,\frac{m_i\, m_j}{M}\, \big[\, \mathbf{a}_i - \mathbf{a}_j \,\big]$$

where $\mathbf{r}_i, \mathbf{v}_i, \mathbf{a}_i, \mathbf{r}_j, \mathbf{v}_j$ and $\mathbf{a}_j$ are the universal positions, the universal velocities and the universal accelerations of particles i and j.

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* In the temporal partial derivative, the spatial coordinates must be treated as constants [or replace this in the first equation: $+\,{}^{1\!}/_{2}\,(\vec{v}_i - \vec{v}_j) \times \mathbf{B}$, and this in the second equation: $+\,{}^{1\!}/_{2}\,(\vec{v}_i - \vec{v}_j) \times (\nabla \times \mathbf{A})$]