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Mathematica Analysis of an RCHO Standard Model

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(Dated: September 6, 2024)

This "paper with code" is a *Mathematica*™ analysis of recent work by Furey on identifying connections between the Standard Model (SM), group theory, particle physics, and the hypercomplex number systems of $\mathbb{R}\otimes\mathbb{C}\otimes\mathbb{H}\otimes\mathbb{O}$. It is a brief overview with results of the analysis followed up with an ancillary notebook showing the full detail of the computations that validate the operation of the logical constructs behind the ideas. It uses a comprehensive symbolic computational environment purpose-built over several decades to study and visualize hypercomplex number systems, representational group theory, and the associated experimental and theoretical physics concepts that use them.

PACS numbers: 112.10.Dm, 2.60.Rc, 12.38.-t, 02.10.Hh, 12.90.+b
Keywords: octonions, quaternions, Standard Model, group theory, E8

I. INTRODUCTION

In a series of papers [1][2][3][4] and a talk presented in the *Octonions, Standard Model, and Unification* (OSMU 2023) conference series[5], Furey gives an in-depth overview of a novel approach to Standard Model (SM) Grand Unification Theories (GUTs) through the association of the hypercomplex algebra of $\mathbb{R}\otimes\mathbb{C}\otimes\mathbb{H}\otimes\mathbb{O}$ with various group theoretic structures from the most popular GUTs.

In an attempt to provide for computational integrity, transparency, and most importantly, the readability of the code, it is written in a way that maximizes the use of the same symbols and operators as used on paper and in PowerPoint slides. The code attempts to maximize the expressiveness of *Mathematica*™ (MTM) while avoiding the tedious use of cryptic algorithms or overly verbose logic. Some of the underlying code is presented where it helps the reader and omitted if it gets in the way of the analysis. Using the MTM symbolic computing platform allows the reader with basic familiarity of its capability to understand how this code represents the content of the papers.

A. Quaternions

While MTM seamlessly handles the $\mathbb{R}$eal and $\mathbb{C}$omplex algebraic number systems, the native implementation of $\mathbb{H}$amilton's quaternions is augmented to provide the flexibility needed to address the independent tensoring with the $\mathbb{O}$ctonions and the $\mathbb{C}$omplex algebras with the option to assume (or not) whether the $\mathbb{I}$maginary i in $\mathbb{C}$ is (or isn't) the same as the $I\in\mathbb{H}$ and/or the $e_1\in\mathbb{O}$.

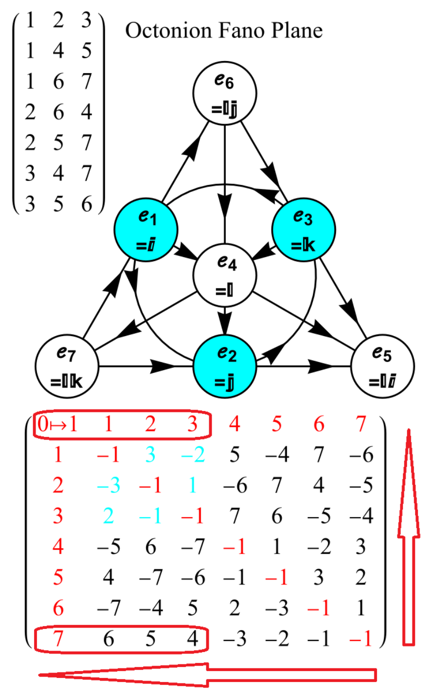

FIG. 1. An alternative set of structure constant triads, octonion Fano plane mnemonic, and multiplication table, with decorations showing the palindromic multiplication.

The tensoring also creates a split in the way conjugation must be handled. The MTM native conjugation (`\[Conjugate]` as a function or as an operator exponent) handles the imaginary components of $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$ (after loading the `Quaternions` package). This implementation operates on octonion symbols (i.e. $\texttt{oct}=\{e_0,e_1,e_2,e_3,e_4,e_5,e_6,e_7\}$) as real, but the octonion conjugation as a function (`octConjugate[p_, doComplexified_ : True]`) or asterisk exponent operates on the imaginary octonion symbols and can switch to MTM conjugation as needed. The dagger (†) exponent operator has been used for this pur-

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pose as the Hermitian conjugate exponent operator, but this implementation has reserved the dagger symbol for use as the `CongugateTranspose` exponent operator.

We have maximized the use of MTM's quaternion implementation, which uses the standard choice of $\texttt{I} ** \texttt{J} = \texttt{K}$ with canonical ordering and left to right non-commutative multiplication operator order (i.e. the MTM $**$ operator symbol for `NonCommutativeMultiply`) as opposed to the other multiplication order option of $\texttt{J} ** \texttt{I} = \texttt{K}$ or by using a right-to-left multiplication operator.

B. 𝕆ctonions

The quaternions within the octonions are defined by the ℝ*eal* part as $e_0 = 1$ in the first triad, where a variable $\texttt{triad1Rev} = \texttt{True}||\texttt{False}$ is used to provide a switch between the two quaternion sets. This choice applied to quaternions is the same idea as switching between right (R) vs. left (L) octonion non-commutative multiplication ($\circ$) applied in these papers to octonions. Visually, changing `triad1Rev` is represented as clockwise vs. counterclockwise arrows in the generation of the Fano plane's center circle, as shown in Fig. 1. It creates a 2X doubling of the main 240 sets of triads making up all octonion choices. These 240 are created from a canonically-ordered set of 30 base triads applied to 8 sets of 8 hexadecimal sign masks. The sign masks determine the ordering of each triad and are related to the Hadamard matrix and Hamming codes.

"Each representation of the octonions arranges the seven imaginary unit elements into seven triads, with no two distinct triads having a pair of elements in common. There is a natural mapping between the 30 distinct sets of triads and the 30 distinct versions of B7 which contain the vertex (0,0,0,0,0,0,0). Assigning a chirality to each triad of a given octonion representation then reveals that not all 128 possible chirality assignments result in a normed division algebra. In fact, for each of the 30 possible ways of grouping the elements, each of 16 chirality assignments (again corresponding to a B7) gives a distinct representation of the octonions for a total of 480 representations, and each of the other 112 (corresponding to D7) gives a distinct representation of the twisted octonion algebra."[6][7] This algorithm has been implemented here. With this description of how the octonions can be constructed, it is easier to see the connection between G_2 as the automorphism group of the octonions and E_8, as shown in the Freudenthal ℝℂℍ𝕆 magic square[8] shown in Fig. 2.

Most papers that work with octonions use one of a handful of popular octonion sets, which typically have easily-remembered numeric patterns in the triads for resolving the particular multiplication table. Here, we provide a mechanism for searching and selecting any triad set that provides a meaningful representation to the particular discussion. For the purposes of this analysis, we have generalized all references to particular octonion elements specific to Furey's choice of octonion triads (i.e. $\{124, 137, 156, 235, 267, 346, 457\}$ aka. $\texttt{triad1Rev} = \texttt{True}$ #150), such that the analysis holds for any set of triads. The output analysis provided uses the palindromic octonion introduced previously[9] as $\texttt{triad1Rev} = \texttt{True}$ #204 with triads $\{123, 145, 167, 264, 257, 347, 356\}$ shown in Fig. 1.

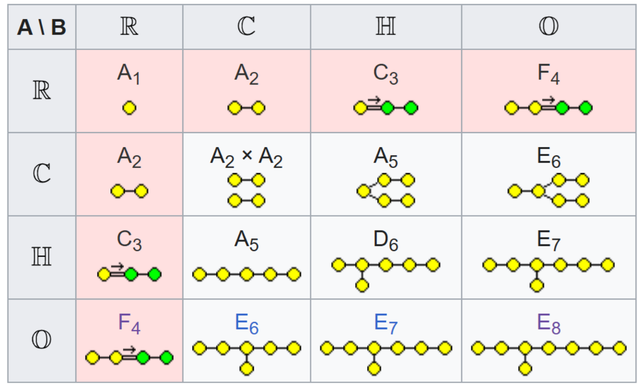

FIG. 2. *"The Freudenthal magic square includes all of the exceptional Lie groups apart from G2, and it provides one possible approach to justify the assertion that "the exceptional Lie groups all exist because of the octonions: G2 itself is the automorphism group of the octonions (also, it is in many ways like a classical Lie group because it is the stabilizer of a generic 3-form on a 7-dimensional vector space – see prehomogeneous vector space)."*

In addition to its palindromic nature, this alternative choice is desirable given that it naturally provides for the standard representation of quaternions $\{I, J, K\}$ within the octonions, where e_0 is the ℝ part of the first triad as 0123 (v.s. 0124, which allocates $K = e_4$). This comes in handy when dealing with biquaternions, which has the ℝ part of $\mathbb{C} \otimes \mathbb{H}$ in the *left* quaternion (i.e. `lQuat`e) and the imaginary part as the other 4 octonion-only elements (4567), which can be optionally referenced as the *right* quaternions (i.e. `rQuat`e). This facilitates using switches such as `splitOctonion` to utilize the octonion structure for ℝℂℍ as a biquaternion. As one can see, having a 0124 as `lQuat`e and 3567 as the `rQuat`e could cause confusion when interpreting ℝℂℍ𝕆 elements as biquaternions. The biquaternion multiplication construct used here uses the Sangwine implementation[10]. Interestingly, this left/right quaternion pattern emerged in this analysis when generalizing the Cl(0,8) Clifford algebra assignments of 3 generations of $SU(3)_C \times U(1)_{em})/\mathbb{Z}_3$ particles in [1] eq. (21). This is discussed in more detail in Section II.

C. $\mathbb{R} \otimes \mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$

Output containing ℝℂℍ𝕆 elements can be simplified and/or provide more insight when shown reduced to sub-

♯⇒	1	2	3	4	5	6	7	8	9	10	11	12	13	14	20	21	28	29	34	35
$\mathbb{RO}$⇒	$\frac{1}{2}(1-e_6)$	$\frac{e_0}{2}+\frac{e_6}{2}$	$\frac{1}{2}(e_6-1)$	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_0	$-e_1-1$	e_3-e_2	$-e_2-e_3$	e_2-e_3	$-e_1-1$	e_2+e_3	$1-e_1$	$-e_4-e_5$	e_3-e_2
$\mathbb{RCO}$⇒	$\frac{1}{2}(1+i\,e_7)$	$\frac{e_0}{2}+\frac{i\,e_7}{2}$	$\frac{1}{2}(-1+i\,e_7)$	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_0	$-1-i$	$(1+i)\,e_3$	$(-1-i)\,e_2$	$(-1-i)\,e_3$	$-1-i$	$(1+i)\,e_2$	$(-1-i)\,e_1$	$(-1-i)\,e_4$	$(1+i)\,e_3$
$\mathbb{RHO}$⇒	$\frac{1}{2}(1+i\,e_7)$	$\frac{e_0}{2}-\frac{i\,e_7}{2}$	$\frac{1}{2}(-1+i\,e_7)$	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_0	$(-1+i)\,e_1$	$(-1+i)\,e_2$	$(-1+i)\,e_3$	$(1+i)\,J\,e_1$	$(1+i)\,J\,e_2$	$(1+i)\,K\,e_1$	$(1+i)\,K\,e_2$	$(1+i)\,K\,e_7$	$(1+i)\,K$
$\mathbb{RCHO}$⇒	$\frac{1}{2}(1+i\,e_7)$	$\frac{e_0}{2}-\frac{i\,e_7}{2}$	$\frac{1}{2}(-1+i\,e_7)$	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_0	$(1+i)\,\mathbb{I}\,e_1$	$(1+i)\,\mathbb{I}\,e_2$	$(1+i)\,\mathbb{I}\,e_3$	$(1+i)\,\mathbb{J}\,e_1$	$(1+i)\,\mathbb{J}\,e_2$	$(1+i)\,\mathbb{K}\,e_1$	$(1+i)\,\mathbb{K}\,e_2$	$(1+i)\,\mathbb{K}\,e_7$	$(1+i)\,\mathbb{K}\,e_0$

FIG. 3. Test inputs with assorted values of $\mathbb{RCHO}$ elements

```
In[∘]:=
(* This forms the L-iR BiQuaternions:
     {e0,e1,e2,e3}-i{e7,e6,e5,e4}/2 and
     i{e0,e1,e2,e3}+ {e7,e6,e5,e4}/2 *)
triads
Select[triads, MemberQ[#, 7] &]
%[[All, ;; 2]]
```

$$Out[\circ]= \begin{pmatrix} 1 & 2 & 3 \\ 1 & 4 & 5 \\ 1 & 6 & 7 \\ 2 & 6 & 4 \\ 2 & 5 & 7 \\ 3 & 4 & 7 \\ 3 & 5 & 6 \end{pmatrix}$$

$$Out[\circ]= \begin{pmatrix} 1 & 6 & 7 \\ 2 & 5 & 7 \\ 3 & 4 & 7 \end{pmatrix}$$

$$Out[\circ]= \begin{pmatrix} 1 & 6 \\ 2 & 5 \\ 3 & 4 \end{pmatrix}$$

```
In[∘]:=
lrQuat = {
  lQuat = Join[{0}, triads[[1]]],
  rQuat = Join[{7}, %[[All, 2]]]}
```

$$Out[\circ]= \begin{pmatrix} 0 & 1 & 2 & 3 \\ 7 & 6 & 5 & 4 \end{pmatrix}$$

```
In[∘]:=
(* Show the l/r quaternions are palindromic *)
rQuat == Reverse@Complement[Range@8 - 1, lQuat]
```

Out[∘]= True

```
In[∘]:=
octN
```

Out[∘]= $\{e_0, e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$

```
In[∘]:=
lrQuate = octN[[1 + #]] & /@ # & /@ lrQuat /. slRep
lQuate = lrQuate[[1]];
rQuate = lrQuate[[2]];
```

$$Out[\circ]= \begin{pmatrix} e_0 & e_1 & e_2 & e_3 \\ e_7 & e_6 & e_5 & e_4 \end{pmatrix}$$

FIG. 4. Generalized left/right quaternion assignments

sets such as $\mathbb{RO}, \mathbb{RCO}, \mathbb{RHO}$, or $\mathbb{RCHO}$. We provide for this using two logic switches `doℝℂℍ𝕆` and `doSymOct`. For example, we generate a group of operator inputs (e.g. for function `f[in_, doLeft : False]` that processes left or right octonion multiplication) with $\mathbb{RCHO}$ elements to test the logic and code as shown in Fig. 3. This list of test elements is aptly named for this function as `finRCHO`. It does this using two switches to show four different forms of separating (or mixing) $\mathbb{R}$, $\mathbb{C}$,$\mathbb{H}$, and $\mathbb{O}$.

II. DEFINE AND SHOW GROUP THEORY ELEMENTS

The definitions of $\mathbb{CO}$ operators s and S (and their conjugates s^*, S^*), $\alpha_{0,1,2,3}$, and the $\mathbb{CH}$ operator $\epsilon[in_-]_{ud_-}$, are defined in [3] eqs. (5-6) are shown in Appendix A Figs. 5-7. The operators for $SU(3)_C = \Lambda_{n=1-8}$ from [4] eq. (3) and [1] eq. (2), which includes the full $G_2 \simeq Aut(\mathbb{O})$ of $g_{n=9-14}$ are shown in Appendix A Figs. 8-9. The $Cl(10) = \Gamma_{n=1-11}$ elements from [2] eq. (25-27) and minimal left ideal (MLI) as Ψ_l (and minimal right ideal as Ψ_r) identified by eq. (31) are shown in Appendix A Fig. 10.

We have added an $\alpha_0 = \nu$ which references the octonionic elements e_0 and e_7. This addition completes a pattern for $\alpha_{0,1,2,3}$ where each references the corresponding position in both `lQuat`e and `rQuat`e as shown in eq. (1). This suggests a trivial reordering in the definition of $\alpha_{0,1,2,3}$ would be appropriate to seamlessly assign them to the `lrQuat`e definition which is predicated on the particular selection of octonion triads.

$$\alpha_{0,1,2,3} = L_{i\texttt{lQuat}e[[\{1,4,2,3\}]]} - L_{\texttt{rQuat}e[[\{1,4,2,3\}]]} \qquad (1)$$

III. DEFINE AND SHOW SM ELEMENTS

The Clifford algebra Cl(0,8) group structure is referenced in the assignment of 3 generations of $SU(3)_C \times U(1)_{em})/\mathbb{Z}_3$ particles[1] eq. (21) and is used extensively in the OSMU 2023 talk[5]. It is shown in Appendix B Fig. 11.

The operators for defining and showing $SU(3)_C \oplus SU(2)_L \oplus U(1)_Y$ are implemented as $l_{SM}[\Psi_l]$ using the boson-lepton (B-L) gauge operator and rY from [4] eq. (2-4). They are shown in Fig. 12.

The Ψ operators for individual $\mathcal{V}_L^{\downarrow} = \mathcal{V}_L, \overline{\mathcal{V}_R^{\downarrow}} = \overline{\mathcal{V}_R}$, $\mathcal{V}_R^{\uparrow} = \mathcal{V}_R, \overline{\mathcal{V}_L^{\downarrow}} = \overline{\mathcal{V}_L}$, $\mathcal{E}_L^{\downarrow} = \mathcal{E}_L$, $\overline{\mathcal{E}_R^{\downarrow}} = \overline{\mathcal{E}_R}$, $\mathcal{E}_R^{\uparrow} = \mathcal{E}_R, \overline{\mathcal{E}_L^{\downarrow}} = \overline{\mathcal{E}_L}$ are shown in Fig. 13.

The Ψ operators for individual $\mathcal{U}_L^{\downarrow} = \mathcal{U}_L, \overline{\mathcal{U}_R^{\downarrow}} = \overline{\mathcal{U}_R}$, $\mathcal{U}_R^{\uparrow} = \mathcal{U}_R, \overline{\mathcal{U}_L^{\downarrow}} = \overline{\mathcal{U}_L}$, $\mathcal{D}_L^{\downarrow} = \mathcal{D}_L$, $\overline{\mathcal{D}_R^{\downarrow}} = \overline{\mathcal{D}_R}$, $\mathcal{D}_R^{\uparrow} = \mathcal{D}_R, \overline{\mathcal{D}_L^{\downarrow}} = \overline{\mathcal{D}_L}$ are shown in Fig. 14.

The operators for combining $\{\mathcal{V}, \mathcal{E}, \mathcal{D}, \mathcal{U}\}_{R/L}$ components into Ψ_L and Ψ_R from [3] eq. (4 and 9, respectively) are shown in Fig. 15.

Finally, we come to the definition and analysis of the combined $\Psi_{LR} = \Psi_L^{\uparrow} \oplus \Psi_L^{\downarrow}$ from [3] eq. (15) shown in Fig. 16. This part of the analysis evaluates these operators with $l_{SM}[\Psi_l]$ as input.

Interestingly, when the Ψ_{LR} operator is evaluated with $l_{SM}[\Psi_l]$ as the input, it shows the palindromic nature of its complexified octonion ($\mathbb{R}\mathbb{C}\mathbb{O}$) when evaluated as the absolute values of their real and imaginary parts. With the rationals scaled to integers, they relate to the number of roots of $E8_{142}$, $E8_{241}$, and $E8_{241}$ polytopes. The details of this are shown in Fig. 17.

IV. CONCLUSION

This paper (with code) has presented a MTM analysis identifying connections between the Standard Model (SM), group theory, particle physics, and the hypercomplex number systems of $\mathbb{R} \otimes \mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$. The ancillary notebook file shows the detailed computations of the logical constructs behind the ideas. It uses a comprehensive symbolic computational environment to study and visualize hypercomplex number systems, representational group theory, and the associated experimental and theoretical physics concepts that use them.

ACKNOWLEDGMENTS

I would like to thank my wife for her love and patience and those in academia who have taken the time to review this work.

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Appendix A: *Code snippets defining and showing $\mathbb{R}\mathbb{C}\mathbb{H}\mathbb{O}$ operators used to generate the group theoretic SM representations*

Figs. 5-10

Appendix B: *The operators for group theoretic SM representations*

Figs. 11-17

```
In[•]:=
s@in_List := 1/2 (1 + opCheck[ⅈ, L[e7]octonion@in]) /. slRep;
s@in_ := s@oct2List@in;

In[•]:=
(* Only accepts a single symbolic octonion or an oct2List of one *)
S@in_List := 1/2 (1 + opCheck[ⅈ, R[e7]octonion@in]) /. slRep;
S@in_ := S@oct2List@in;

In[•]:=
Simplify[v == s@1 == S@1 == 1/2 (1 + ⅈ e7) /. slRep]

Out[•]= True

In[•]:=
(* String forms for symbolic labels *)
sSsym = {
   {"s", "S"},
   {"s", "S*"},
   {"s*", "S"},
   {"s*", "S*"}};

In[•]:=
(* Complex Conjugates *)
sS[in_, i_ : 1] := {(* Complex Conjugates *)
    {s[in /. slRep], S[in /. slRep]},
    {s[in /. slRep], octonion[oct2List[S[in /. slRep]]* /. octConjRep]},
    {octonion[oct2List[s[in /. slRep]]* /. octConjRep], S[in /. slRep]},
    {octonion[oct2List[s[in /. slRep]]* /. octConjRep], octonion[oct2List[S[in /. slRep]]* /. octConjRep]}}[[i]] /; doConjugate;

In[•]:=
(* This flag activates Complex Conjugates *)
doConjugate = True;
{#, sSsym[[#]], sS[1, #]} & /@ Range@4
```

Out[•]=

1	{s, S}	$\left\{\frac{1}{2}(1+i\,e_7), \frac{1}{2}(1+i\,e_7)\right\}$
2	{s, S^*}	$\left\{\frac{1}{2}(1+i\,e_7), \frac{1}{2}-\frac{i\,e_7}{2}\right\}$
3	{s^*, S}	$\left\{\frac{1}{2}-\frac{i\,e_7}{2}, \frac{1}{2}(1+i\,e_7)\right\}$
4	{s^*, S^*}	$\left\{\frac{1}{2}-\frac{i\,e_7}{2}, \frac{1}{2}-\frac{i\,e_7}{2}\right\}$

```
In[•]:=
(* octConjugate *)
sS[in_, i_ : 1] := {(* octConjugates *)
    {s[in /. slRep] , S[in /. slRep]},
    {s[in /. slRep] , S[in /. slRep]*},
    {s[in /. slRep]*, S[in /. slRep]},
    {s[in /. slRep]*, S[in /. slRep]*}}[[i]] /; ! doConjugate;
doConjugate = False;
{#, sSsym[[#]], sS[1, #]} & /@ Range@4
```

Out[•]=

1	{s, S}	$\left\{\frac{1}{2}(1+i\,e_7), \frac{1}{2}(1+i\,e_7)\right\}$
2	{s, S^*}	$\left\{\frac{1}{2}(1+i\,e_7), \frac{e_0}{2}+\frac{i\,e_7}{2}\right\}$
3	{s^*, S}	$\left\{\frac{e_0}{2}+\frac{i\,e_7}{2}, \frac{1}{2}(1+i\,e_7)\right\}$
4	{s^*, S^*}	$\left\{\frac{e_0}{2}+\frac{i\,e_7}{2}, \frac{e_0}{2}+\frac{i\,e_7}{2}\right\}$

FIG. 5. Code snippets defining and showing $\mathbb{C}\mathbb{O}$ operators s and S (and its conjugates s^*, S^*)

In[◦]:=

```
α0@in_ := 1/2 (L[in]{lQuate[[1]]} + opCheck[i, L[in]{rQuate[[1]]}]) //. octConjRep;
```

In[◦]:=

```
α0@1;
v;
s@1;
(1 + opCheck[i, e7]) / 2;
%%%% == %%% == %% == % == 1/2 (1 - e6) /. slRep
```

Out[◦]=

$$\frac{1}{2}(1+i\,e_7)=\frac{1}{2}(1+i\,e_7)=\frac{1}{2}(1+i\,e_7)=\frac{1}{2}(1+i\,e_7)=\frac{1}{2}(1-e_6)$$

In[◦]:=

```
α1@in_ := 1/2 (opCheck[i, L[in]{lQuate[[4]]}] - L[in]{rQuate[[4]]}) //. octConjRep;
```

In[◦]:=

```
α2@in_ := 1/2 (opCheck[i, L[in]{lQuate[[2]]}] - L[in]{rQuate[[2]]}) //. octConjRep;
```

In[◦]:=

```
α3@in_ := 1/2 (opCheck[i, L[in]{lQuate[[3]]}] - L[in]{rQuate[[3]]}) //. octConjRep;
```

In[◦]:=

```
doSymOct = True;
Table[αList[#][[i]], {i, 3}] & /@ lQuate
```

Out[◦]=

$$\begin{pmatrix} \frac{1}{2}(-e_4+i\,e_3) & \frac{1}{2}(-e_6+i\,e_1) & \frac{1}{2}(-e_5+i\,e_2) \\ \frac{1}{2}(-e_5-i\,e_2) & \frac{1}{2}(-e_7-i) & \frac{1}{2}(e_4+i\,e_3) \\ \frac{1}{2}(e_6+i\,e_1) & \frac{1}{2}(-e_4-i\,e_3) & \frac{1}{2}(-e_7-i) \\ \frac{1}{2}(-e_7-i) & \frac{1}{2}(e_5+i\,e_2) & \frac{1}{2}(-e_6-i\,e_1) \end{pmatrix}$$

In[◦]:=

```
(* octConjugates *)
Table[(αList[#][[i]])* /. octConjRep, {i, 3}] & /@ lQuate
```

Out[◦]=

$$\begin{pmatrix} \frac{1}{2}(-e_4-i\,e_3) & \frac{1}{2}(-e_6-i\,e_1) & \frac{1}{2}(-e_5-i\,e_2) \\ \frac{1}{2}(-e_5+i\,e_2) & \frac{1}{2}(-e_7+i) & \frac{1}{2}(e_4-i\,e_3) \\ \frac{1}{2}(e_6-i\,e_1) & \frac{1}{2}(-e_4+i\,e_3) & \frac{1}{2}(-e_7+i) \\ \frac{1}{2}(-e_7+i) & \frac{1}{2}(e_5-i\,e_2) & \frac{1}{2}(-e_6+i\,e_1) \end{pmatrix}$$

In[◦]:=

```
(* Normal Complex Conjugates *)
Table[(αList[#][[i]])* /. octConjRep, {i, 3}] & /@ lQuate
```

Out[◦]=

$$\begin{pmatrix} \frac{1}{2}(-e_4-i\,e_3) & \frac{1}{2}(-e_6-i\,e_1) & \frac{1}{2}(-e_5-i\,e_2) \\ \frac{1}{2}(-e_5+i\,e_2) & \frac{1}{2}(-e_7+i) & \frac{1}{2}(e_4-i\,e_3) \\ \frac{1}{2}(e_6-i\,e_1) & \frac{1}{2}(-e_4+i\,e_3) & \frac{1}{2}(-e_7+i) \\ \frac{1}{2}(-e_7+i) & \frac{1}{2}(e_5-i\,e_2) & \frac{1}{2}(-e_6+i\,e_1) \end{pmatrix}$$

In[◦]:=

```
doℝℂℍ𝕆 = False;
doSymOct = False;
```

FIG. 6. Code snippets defining and showing $\mathbb{C}\mathbb{O}$ operators $\alpha_{1,2,3}$

```
(* Create a symbolic Quaternion (∊) *)
∊List = {"↑↑", "↑↓", "↓↑", "↓↓"};
```

In[∘]:=

```
∊[in_]ud_ := Module[{part1, part2, out},
    part1 = 1/2 If[ud == "↑↑" || ud == "↓↓", 1, If[ud == "↑↓", -1, 1] getTriad1[[2]]] /.
      slRep;
    print["part1=", part1];
    part2 = If[doℝℂℍ𝕆 === True,
       opCheck[𝕚, If[ud == "↑↑" || ud == "↓↓",
         getTriad1[[3]],
         getTriad1[[1]]]],
       opCheck[𝕚, L[in]{If[ud=="↑↑"||ud=="↓↓",
                        getTriad1[[3]],
                        getTriad1[[1]]]}]] /. slRep;
    print["part2=", part2];
    out = If[ud == "↓↓", Subtract, Plus][part1, part2] /. slRep;
    print["out=", out];
    out];
```

In[∘]:=

```
Table[doℝℂℍ𝕆 = i[[1]]; doSymOct = i[[2]];
  Column[{
    Row@{"in=", j, ", Ψ=", fInℝℂℍ𝕆[[j]], getℝℂℍ𝕆@i},
    MatrixForm@Table[
      Row@{"∊[", fInℝℂℍ𝕆[[j]], "]", ""k, "=", ∊[fInℝℂℍ𝕆[[j]]]k},
      {k, ∊List}]}, Center],
  {i, {{False, False}, {False, True}, {True, False}, {Both, False}}},
  {j, 10, 11}] // MatrixForm
```

Out[∘]//MatrixForm=

$$
\begin{pmatrix}
\text{in=10, } \Psi{=}e_7\text{, } \mathbb{RO}\Rightarrow & \text{in=11, } \Psi{=}e_0\text{, } \mathbb{RO}\Rightarrow \\
\begin{pmatrix} \epsilon[e_7]_{\uparrow\uparrow}{=}e_5+\frac{1}{2} \\ \epsilon[e_7]_{\uparrow\downarrow}{=}e_7-\frac{e_2}{2} \\ \epsilon[e_7]_{\downarrow\uparrow}{=}\frac{e_2}{2}+e_7 \\ \epsilon[e_7]_{\downarrow\downarrow}{=}\frac{1}{2}-e_5 \end{pmatrix} &
\begin{pmatrix} \epsilon[e_0]_{\uparrow\uparrow}{=}\frac{1}{2}-e_2 \\ \epsilon[e_0]_{\uparrow\downarrow}{=}-\frac{e_2}{2}-1 \\ \epsilon[e_0]_{\downarrow\uparrow}{=}\frac{e_2}{2}-1 \\ \epsilon[e_0]_{\downarrow\downarrow}{=}e_2+\frac{1}{2} \end{pmatrix} \\
\text{in=10, } \Psi{=}e_7\text{, } \mathbb{RCO}\Rightarrow & \text{in=11, } \Psi{=}e_0\text{, } \mathbb{RCO}\Rightarrow \\
\begin{pmatrix} \epsilon[e_7]_{\uparrow\uparrow}{=}\frac{1}{2}+i\,e_4 \\ \epsilon[e_7]_{\uparrow\downarrow}{=}-\frac{e_2}{2}+i\,e_6 \\ \epsilon[e_7]_{\downarrow\uparrow}{=}\frac{e_2}{2}+i\,e_6 \\ \epsilon[e_7]_{\downarrow\downarrow}{=}\frac{1}{2}-i\,e_4 \end{pmatrix} &
\begin{pmatrix} \epsilon[e_0]_{\uparrow\uparrow}{=}\frac{1}{2}+i\,e_3 \\ \epsilon[e_0]_{\uparrow\downarrow}{=}-\frac{e_2}{2}+i\,e_1 \\ \epsilon[e_0]_{\downarrow\uparrow}{=}\frac{e_2}{2}+i\,e_1 \\ \epsilon[e_0]_{\downarrow\downarrow}{=}\frac{1}{2}-i\,e_3 \end{pmatrix} \\
\text{in=10, } \Psi{=}e_7\text{, } \mathbb{RHO}\Rightarrow & \text{in=11, } \Psi{=}e_0\text{, } \mathbb{RHO}\Rightarrow \\
\begin{pmatrix} \epsilon[e_7]_{\uparrow\uparrow}{=}\frac{1}{2}+i\,K \\ \epsilon[e_7]_{\uparrow\downarrow}{=}-\frac{J}{2}-1 \\ \epsilon[e_7]_{\downarrow\uparrow}{=}\frac{J}{2}-1 \\ \epsilon[e_7]_{\downarrow\downarrow}{=}\frac{1}{2}-i\,K \end{pmatrix} &
\begin{pmatrix} \epsilon[e_0]_{\uparrow\uparrow}{=}\frac{1}{2}+i\,K \\ \epsilon[e_0]_{\uparrow\downarrow}{=}-\frac{J}{2}-1 \\ \epsilon[e_0]_{\downarrow\uparrow}{=}\frac{J}{2}-1 \\ \epsilon[e_0]_{\downarrow\downarrow}{=}\frac{1}{2}-i\,K \end{pmatrix} \\
\text{in=10, } \Psi{=}e_7\text{, } \mathbb{RCHO}\Rightarrow & \text{in=11, } \Psi{=}e_0\text{, } \mathbb{RCHO}\Rightarrow \\
\begin{pmatrix} \epsilon[e_7]_{\uparrow\uparrow}{=}\frac{1}{2}+i\,\mathbb{K}\,e_7 \\ \epsilon[e_7]_{\uparrow\downarrow}{=}-\frac{\mathbb{J}}{2}+i\,\mathbb{I}\,e_7 \\ \epsilon[e_7]_{\downarrow\uparrow}{=}\frac{\mathbb{J}}{2}+i\,\mathbb{I}\,e_7 \\ \epsilon[e_7]_{\downarrow\downarrow}{=}\frac{1}{2}-i\,\mathbb{K}\,e_7 \end{pmatrix} &
\begin{pmatrix} \epsilon[e_0]_{\uparrow\uparrow}{=}\frac{1}{2}+i\,\mathbb{K} \\ \epsilon[e_0]_{\uparrow\downarrow}{=}-\frac{\mathbb{J}}{2}+i\,\mathbb{I} \\ \epsilon[e_0]_{\downarrow\uparrow}{=}\frac{\mathbb{J}}{2}+i\,\mathbb{I} \\ \epsilon[e_0]_{\downarrow\downarrow}{=}\frac{1}{2}-i\,\mathbb{K} \end{pmatrix}
\end{pmatrix}
$$

FIG. 7. Code snippets defining and showing $\mathbb{CH}$ operator $\epsilon[in\_]_{ud\_}$

```
Define and show Λ[[1 - 8]][in_]                    from [5] eq .2 and from [10] eq. 3

In[•]:=
(* from [10] eq. 3 *)
doℝℂℍ𝕆 = False;
doSymOct = True;
Clear[Λ1, Λ2, Λ3, Λ4, Λ5, Λ6, Λ7, Λ8, g9, g10, g11, g12, g13, g14];
ΛListStr = {"Λ1", "Λ2", "Λ3", "Λ4", "Λ5", "Λ6", "Λ7", "Λ8", "g9", "g10", "g11", "g12", "g13", "g14"};
ΛListList = ToExpression[# <> "List" & /@ ΛListStr];
ΛList = ToExpression@ΛListStr;

In[•]:=
Λf[Λ_, f_] := Λ@f;

In[•]:=
(*)          lr24,ll42,rl44,lr34,ll43,lr32,rr23     reverse l↔r *)
Λ1@in_ := 1/2 opCheck[i, L[in]{lQuate[[2]],rQuate[[4]]}] - 1/2 opCheck[i, L[in]{rQuate[[2]],lQuate[[4]]}] /. slRep;
Λ2@in_ := 1/2 opCheck[i, L[in]{lQuate[[4]],lQuate[[2]]}] + 1/2 opCheck[i, L[in]{rQuate[[4]],rQuate[[2]]}] /. slRep;
Λ3@in_ := - 1/2 opCheck[i, L[in]{rQuate[[4]],lQuate[[4]]}] - 1/2 opCheck[i, L[in]{lQuate[[2]],rQuate[[2]]}] /. slRep;
Λ4@in_ := 1/2 opCheck[i, L[in]{lQuate[[3]],rQuate[[4]]}] - 1/2 opCheck[i, L[in]{rQuate[[3]],lQuate[[4]]}] /. slRep;
Λ5@in_ := 1/2 opCheck[i, L[in]{lQuate[[4]],lQuate[[3]]}] + 1/2 opCheck[i, L[in]{rQuate[[4]],rQuate[[3]]}] /. slRep;
Λ6@in_ := 1/2 opCheck[i, L[in]{lQuate[[3]],rQuate[[2]]}] - 1/2 opCheck[i, L[in]{rQuate[[3]],lQuate[[2]]}] /. slRep;
Λ7@in_ := 1/2 opCheck[i, L[in]{rQuate[[2]],rQuate[[3]]}] + 1/2 opCheck[i, L[in]{lQuate[[2]],lQuate[[3]]}] /. slRep;

In[•]:=
Λ8@in_ := - 1/(2 √3) opCheck[i, L[in]{rQuate[[2]],lQuate[[2]]}] - 1/(2 √3) opCheck[i, L[in]{rQuate[[4]],lQuate[[4]]}] + 1/(1 √3) opCheck[i, L[in]{rQuate[[3]],lQuate[[3]]}] /. slRep;

In[•]:=
(*)                    lr24,ll42,lr43,ll34,lr32,ll23 - same but reverse l↔r (& even #)   - All e7=rQuate[[1]] w/alt pos,l3,r3,l2,r2,l4,r4 *)
g9@in_ := - 1/(2 √3) opCheck[i, L[in]{lQuate[[2]],rQuate[[4]]}] - 1/(2 √3) opCheck[i, L[in]{rQuate[[2]],lQuate[[4]]}] + 1/√3 opCheck[i, L[in]{rQuate[[1]],lQuate[[3]]}] /. slRep;
g10@in_ := - 1/(2 √3) opCheck[i, L[in]{lQuate[[4]],lQuate[[2]]}] - 1/(2 √3) opCheck[i, L[in]{rQuate[[2]],rQuate[[4]]}] + 1/√3 opCheck[i, L[in]{rQuate[[3]],rQuate[[1]]}] /. slRep;
g11@in_ := - 1/(2 √3) opCheck[i, L[in]{lQuate[[4]],rQuate[[3]]}] - 1/(2 √3) opCheck[i, L[in]{rQuate[[4]],lQuate[[3]]}] + 1/√3 opCheck[i, L[in]{rQuate[[1]],lQuate[[2]]}] /. slRep;
g12@in_ := - 1/(2 √3) opCheck[i, L[in]{lQuate[[3]],lQuate[[4]]}] - 1/(2 √3) opCheck[i, L[in]{rQuate[[4]],rQuate[[3]]}] + 1/√3 opCheck[i, L[in]{rQuate[[2]],rQuate[[1]]}] /. slRep;
g13@in_ := - 1/(2 √3) opCheck[i, L[in]{rQuate[[3]],lQuate[[2]]}] - 1/(2 √3) opCheck[i, L[in]{lQuate[[3]],rQuate[[2]]}] + 1/√3 opCheck[i, L[in]{rQuate[[1]],lQuate[[4]]}] /. slRep;
g14@in_ := - 1/(2 √3) opCheck[i, L[in]{lQuate[[2]],lQuate[[3]]}] - 1/(2 √3) opCheck[i, L[in]{rQuate[[3]],rQuate[[2]]}] + 1/√3 opCheck[i, L[in]{rQuate[[4]],rQuate[[1]]}] /. slRep;

(* Furey's octonion G2 octonion automorphism mapping calculated w/output saved,
   without evaluating the octonion multiplication in order to show consistency with the paper's non-generalized assignments.
   Otherwise, the evaluation of the octonion elements reduce to 0 due to octonion non-associativity. *)

(* Furey's octonion lrQuat mapping *)
lrQuat;
(0 1 2 4
 7 3 6 5);
```

$$
\begin{pmatrix}
1 & \frac{1}{2}\, i((f\circ e_5)\circ e_1) - \frac{1}{2}\, i((f\circ e_4)\circ e_3) \\
2 & \frac{1}{2}\, i((f\circ e_1)\circ e_4) + \frac{1}{2}\, i((f\circ e_3)\circ e_5) \\
3 & -\frac{1}{2}\, i((f\circ e_3)\circ e_1) - \frac{1}{2}\, i((f\circ e_4)\circ e_5) \\
4 & \frac{1}{2}\, i((f\circ e_5)\circ e_2) - \frac{1}{2}\, i((f\circ e_4)\circ e_6) \\
5 & \frac{1}{2}\, i((f\circ e_2)\circ e_4) + \frac{1}{2}\, i((f\circ e_6)\circ e_5) \\
6 & \frac{1}{2}\, i((f\circ e_3)\circ e_2) - \frac{1}{2}\, i((f\circ e_1)\circ e_6) \\
7 & \frac{1}{2}\, i((f\circ e_2)\circ e_1) + \frac{1}{2}\, i((f\circ e_6)\circ e_3) \\
8 & -\frac{i((f\circ e_1)\circ e_3)}{2\sqrt{3}} + \frac{i((f\circ e_2)\circ e_6)}{\sqrt{3}} - \frac{i((f\circ e_4)\circ e_5)}{2\sqrt{3}} \\
9 & \frac{i((f\circ e_2)\circ e_7)}{\sqrt{3}} - \frac{i((f\circ e_4)\circ e_3)}{2\sqrt{3}} - \frac{i((f\circ e_5)\circ e_1)}{2\sqrt{3}} \\
10 & -\frac{i((f\circ e_1)\circ e_4)}{2\sqrt{3}} - \frac{i((f\circ e_5)\circ e_3)}{2\sqrt{3}} + \frac{i((f\circ e_7)\circ e_6)}{\sqrt{3}} \\
11 & \frac{i((f\circ e_1)\circ e_7)}{\sqrt{3}} - \frac{i((f\circ e_2)\circ e_5)}{2\sqrt{3}} - \frac{i((f\circ e_6)\circ e_4)}{2\sqrt{3}} \\
12 & -\frac{i((f\circ e_4)\circ e_2)}{2\sqrt{3}} - \frac{i((f\circ e_6)\circ e_5)}{2\sqrt{3}} + \frac{i((f\circ e_7)\circ e_3)}{\sqrt{3}} \\
13 & -\frac{i((f\circ e_1)\circ e_6)}{2\sqrt{3}} - \frac{i((f\circ e_3)\circ e_2)}{2\sqrt{3}} + \frac{i((f\circ e_4)\circ e_7)}{\sqrt{3}} \\
14 & -\frac{i((f\circ e_2)\circ e_1)}{2\sqrt{3}} - \frac{i((f\circ e_3)\circ e_6)}{2\sqrt{3}} + \frac{i((f\circ e_7)\circ e_5)}{\sqrt{3}}
\end{pmatrix};
$$

```
{#, ΛList[[#]][f]} & /@ Range@14 // MatrixForm

Out[•]//MatrixForm=
```

$$
\begin{pmatrix}
1 & 0 \\ 2 & 0 \\ 3 & 0 \\ 4 & 0 \\ 5 & 0 \\ 6 & 0 \\ 7 & 0 \\ 8 & 0 \\ 9 & 0 \\ 10 & 0 \\ 11 & 0 \\ 12 & 0 \\ 13 & 0 \\ 14 & 0
\end{pmatrix}
$$

FIG. 8. The operators for $SU(3)_C = \Lambda_{n=1-8}$ from [4] eq. (3) and [1] eq. (2), which includes the full $G_2 \simeq Aut(\mathbb{O})$ with $g_{n=9-14}$

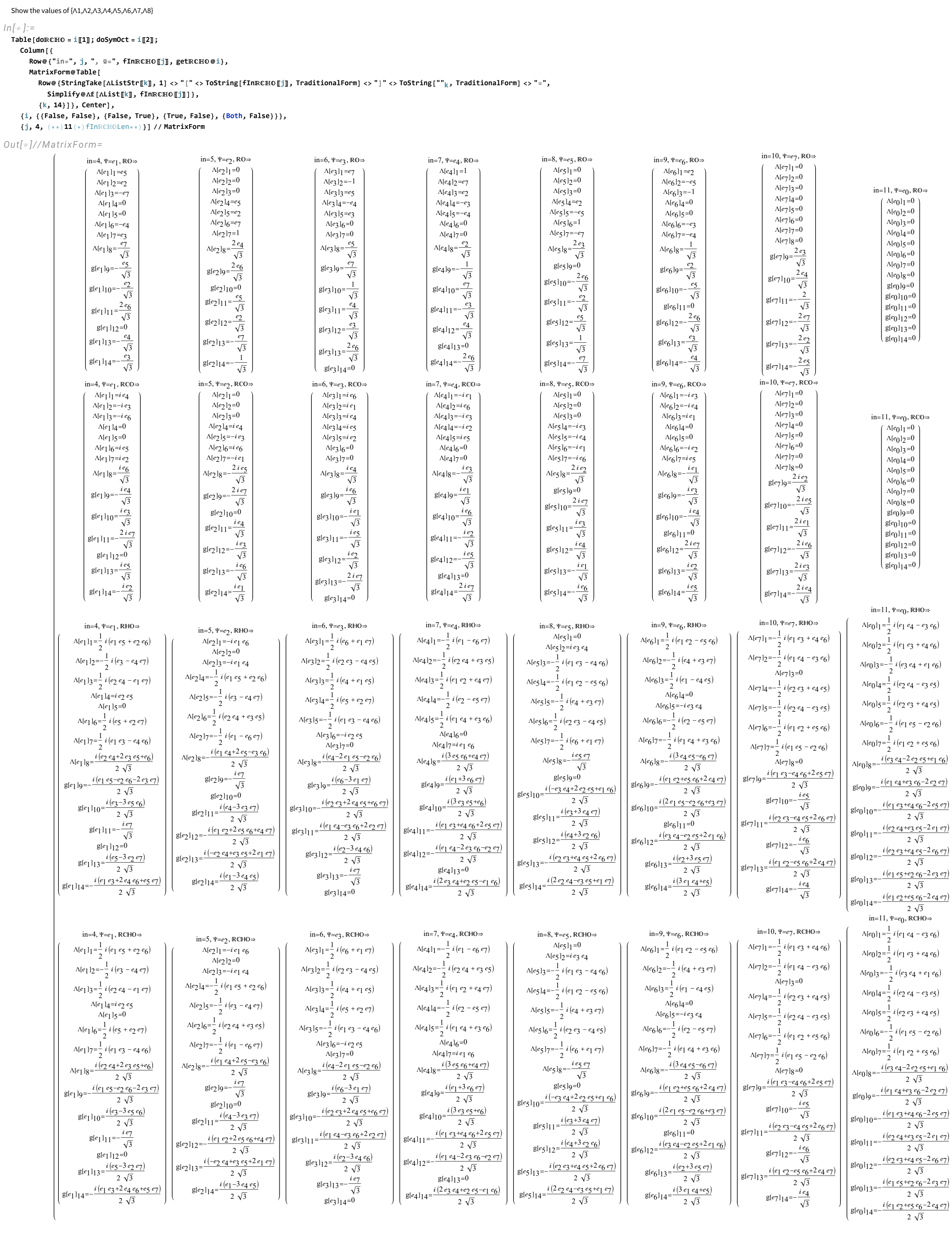

FIG. 9. The analysis data for the operators of $SU(3)_C = \Lambda_{n=1-8}$ from [4] eq. (3) and [1] eq. (2), which includes the full $G_2 \simeq Aut(\mathbb{O})$ with $g_{n=9-14}$

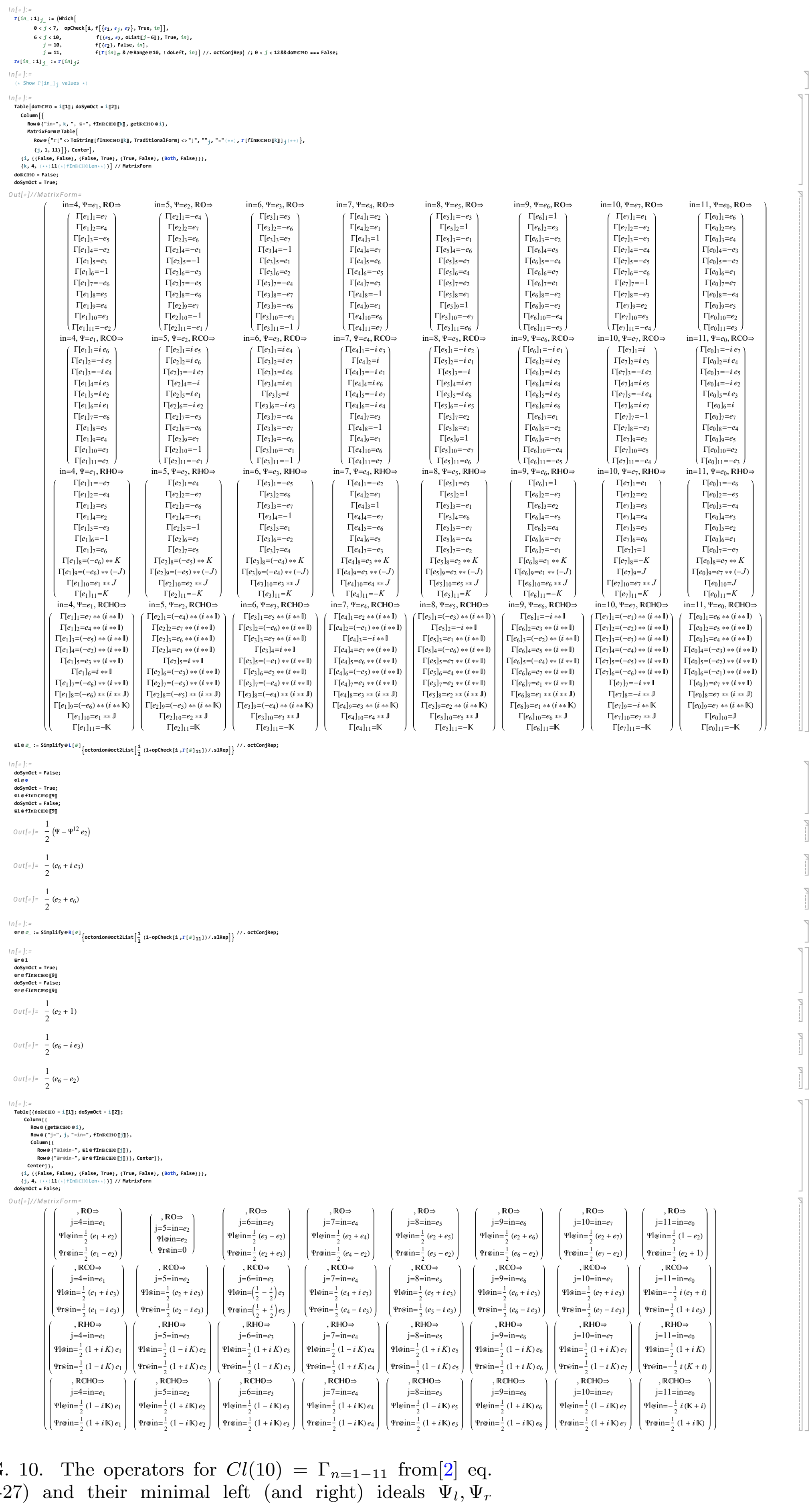

FIG. 10. The operators for $Cl(10) = \Gamma_{n=1-11}$ from[2] eq. (25-27) and their minimal left (and right) ideals Ψ_l, Ψ_r identified by eq. (31)

FIG. 11. The operators for the Clifford algebra Cl(0,8) based assignments of 3 generations of $SU(3)_C \times U(1)_{em})/\mathbb{Z}_3$ particles

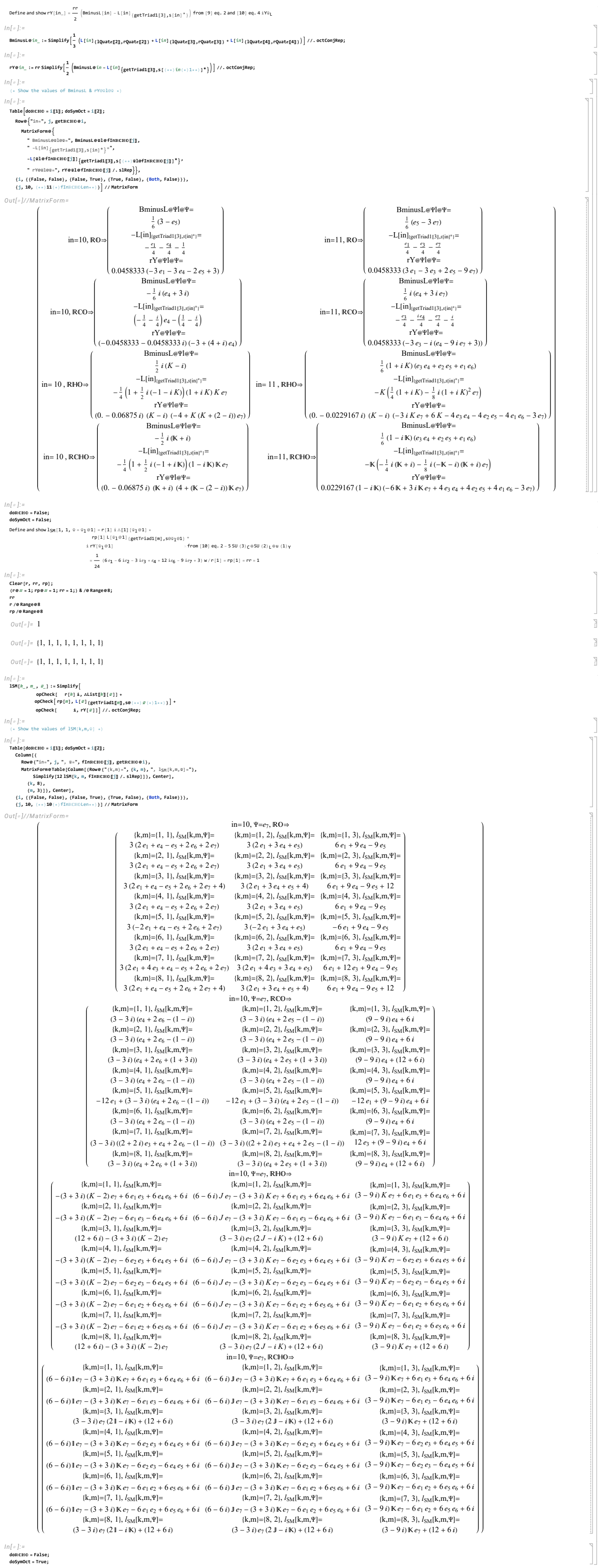

FIG. 12. The operators for $SU(3)_C \oplus SU(2)_L \oplus U(1)_Y$ implemented as $l_{SM}[\Psi_l]$ using boson-lepton (B-L) gauge operator and rY from [4] eq. (2-4)

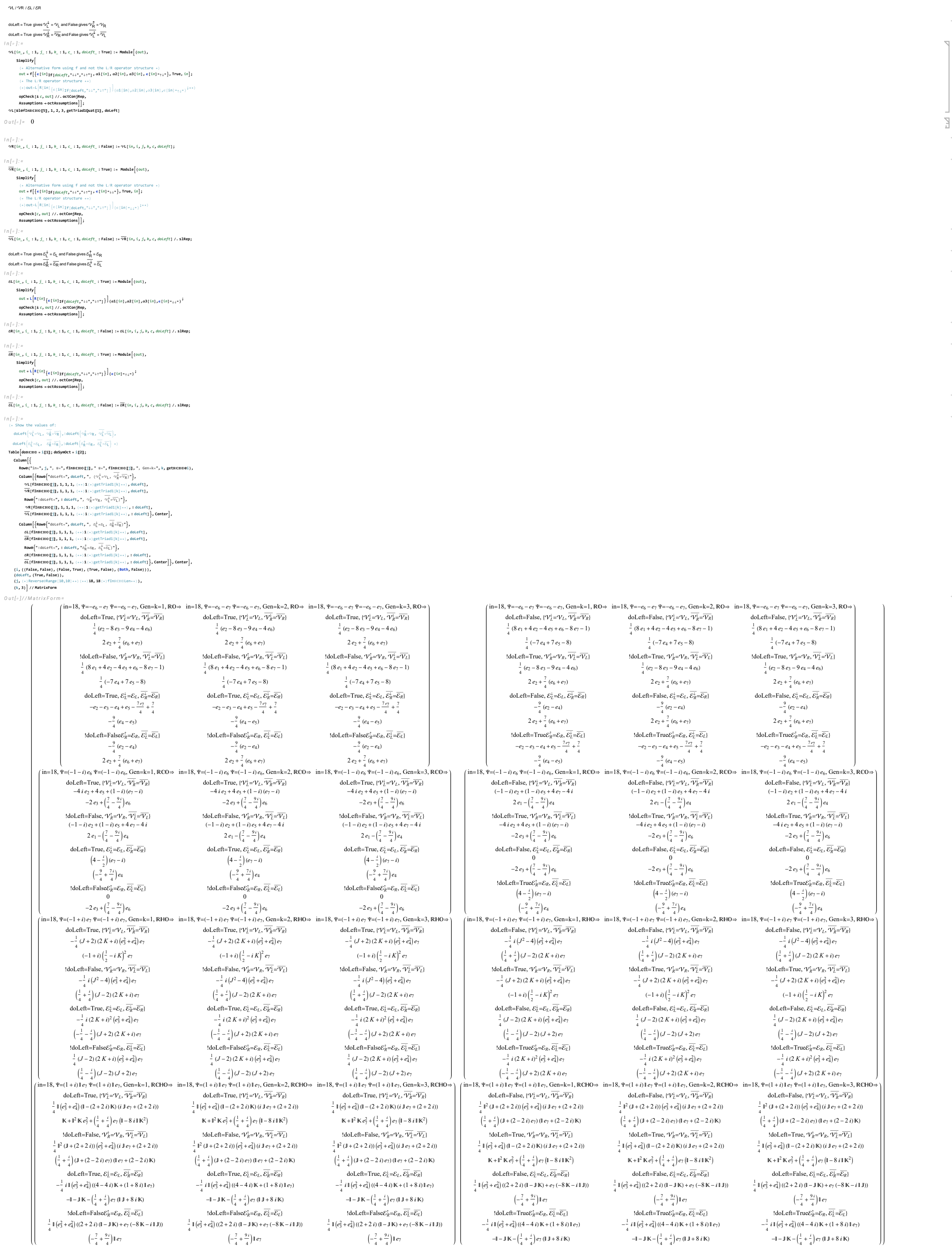

FIG. 13. The Ψ operators for $\mathcal{V}_L^{\downarrow} = \mathcal{V}_L, \overline{\mathcal{V}_R^{\downarrow}} = \overline{\mathcal{V}_R}$, $\mathcal{V}_R^{\uparrow} = \mathcal{V}_R, \overline{\mathcal{V}_L^{\downarrow}} = \overline{\mathcal{V}_L}$, $\mathcal{E}_L^{\downarrow} = \mathcal{E}_L$, $\overline{\mathcal{E}_R^{\downarrow}} = \overline{\mathcal{E}_R}$, $\mathcal{E}_R^{\uparrow} = \mathcal{E}_R, \overline{\mathcal{E}_L^{\downarrow}} = \overline{\mathcal{E}_L}$

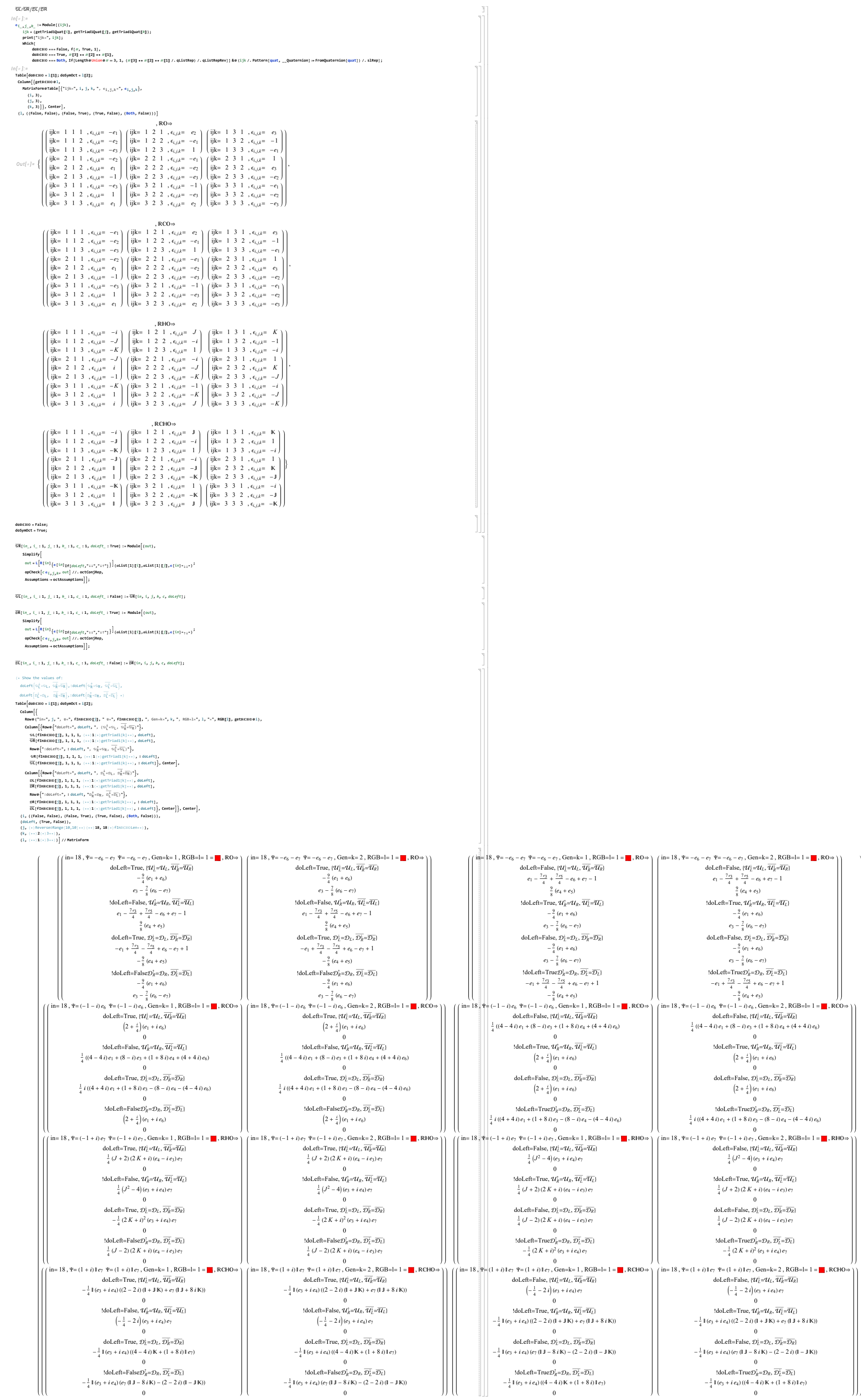

FIG. 14. The Ψ operators for
$\mathcal{U}_L^{\downarrow} = \mathcal{U}_L, \overline{\mathcal{U}_R^{\downarrow}} = \overline{\mathcal{U}_R},$
$\mathcal{U}_R^{\uparrow} = \mathcal{U}_R, \overline{\mathcal{U}_L^{\downarrow}} = \overline{\mathcal{U}_L},$
$\mathcal{D}_L^{\downarrow} = \mathcal{D}_L,\ \overline{\mathcal{D}_R^{\downarrow}} = \overline{\mathcal{D}_R},$
$\mathcal{D}_R^{\uparrow} = \mathcal{D}_R, \overline{\mathcal{D}_L^{\downarrow}} = \overline{\mathcal{D}_L}$

```
ΨL = ΨL↓
In[•]:=
ΨL[in_, i_ : 1, j_ : 1, k_ : 1, c_ : 1, doLeft_ : True] := Simplify@Module[{sm},
      sm =
        𝒱L[in, i, j, k, c, doLeft] +
        ℰL[in, i, j, k, c, doLeft] +
        𝒟̄R[in, i, j, k, c, doLeft] +
        𝒰̄R[in, i, j, k, c, doLeft] + (* negated in 𝒰L *)

        𝒰L[in, i, j, k, c, doLeft] +
        𝒟L[in, i, j, k, c, doLeft] +
        ℰ̄R[in, i, j, k, c, doLeft] + (* sterile ν *)
        If[! doLeft, -1, 1] 𝒱̄R[in, i, j, k, c, doLeft];
      Simplify[
       opCheck[c, L[L[]sS[1,4]]sm],
       Assumptions → octAssumptions]] //. octConjRep;
In[•]:=
 (* Show the values of: doLeft{ΨL},!doLeft{ΨL} *)
In[•]:=
Table[{doℝℂℍ𝕆 = i⟦1⟧; doSymOct = i⟦2⟧;
   Column[{
     Row@{"Ψ=", j, "=", fInℝℂℍ𝕆⟦j⟧, ", Gen=k=", k, ", RGB=l=", l, "=", RGB⟦l⟧, getℝℂℍ𝕆@i},
     Row@{"ΨL=", ΨL[fInℝℂℍ𝕆⟦j⟧, l, 2, 3, (**)2^2 ∘ (**)getTriad1⟦k⟧]}},
    Center]},
  {i, {{False, False}, {False, True}, {True, False}, {Both, False}}},
  {j, (*)Reverse@Range[10,10]**) (**)10, 10(*)fInℝℂℍ𝕆Len**)},
  {k, (**)1(*)3**)},
  {l, (**)1(*)3**)}] // MatrixForm
Out[•]//MatrixForm=
```

$$\Psi = 10 = e_7\,,\ \text{Gen=k=}\ 1\,,\ \text{RGB=l=}\ 1 = \blacksquare\,,\ \mathbb{R}\mathbb{O}\Rightarrow$$
$$\Psi\text{L}=\frac{1}{2}\,i\,((5+12\,i)\,e_1-10\,e_2-(5+20\,i)\,e_3-(10-16\,i)\,e_4+(3-6\,i)\,e_5-(10-8\,i)\,e_6-(5-10\,i)\,e_7+6)$$

$$\Psi = 10 = e_7\,,\ \text{Gen=k=}\ 1\,,\ \text{RGB=l=}\ 1 = \blacksquare\,,\ \mathbb{R}\mathbb{C}\mathbb{O}\Rightarrow$$
$$\Psi\text{L}=e_1\,((-3+9\,i)\,e_6+3\,i\,e_7+e_1\,(-6\,i\,e_2+(10+8\,i)\,e_3-(8-10\,i)\,e_4+6\,e_5+3\,i\,e_6+(13-9\,i)\,e_7-10\,i))$$

$$\Psi = 10 = e_7\,,\ \text{Gen=k=}\ 1\,,\ \text{RGB=l=}\ 1 = \blacksquare\,,\ \mathbb{R}\mathbb{H}\mathbb{O}\Rightarrow$$
$$\Psi\text{L}=(1-2\,i\,K)\,(e_7+i)\,(e_2\,(J+2\,i\,K+1)-i\,(e_3\,(J-2\,i\,K+3)+e_4\,(i\,J+2\,K+3\,i)+(e_5-(1-2\,i))\,(J+2\,i\,K+1)))$$

$$\Psi = 10 = e_7\,,\ \text{Gen=k=}\ 1\,,\ \text{RGB=l=}\ 1 = \blacksquare\,,\ \mathbb{R}\mathbb{C}\mathbb{H}\mathbb{O}\Rightarrow$$
$$\begin{aligned}\Psi\text{L}=&\\ &-\mathbb{I}\,(-(2-4\,i)\,\mathbb{I}\,\mathbb{J}\,\mathbb{K}+(-1+2\,i)\,\mathbb{I}\,\mathbb{J}+(4-8\,i)\,\mathbb{I}\,\mathbb{K}-(3-6\,i)\,\mathbb{I}+(4-8\,i)\,\mathbb{K}-i\,\mathbb{I}\,\mathbb{J}\,\mathbb{K}\,e_4+3\,\mathbb{I}\,\mathbb{J}\,\mathbb{K}\,e_5+\mathbb{I}\,e_3\,(1-i\,e_7)\,(4\,\mathbb{I}\,\mathbb{K}-\mathbb{I}-\mathbb{J}\,\mathbb{K}+\mathbb{J}-2\,\mathbb{K}-3)+(4+2\,i)\,\mathbb{I}\,\mathbb{J}\,\mathbb{K}\,e_7-\mathbb{I}\,\mathbb{J}\,\mathbb{K}\,e_4\,e_7-\\ &\quad i\,\mathbb{I}\,\mathbb{J}\,\mathbb{K}\,e_5\,e_7+\mathbb{I}\,e_2\,(e_7\,(4\,\mathbb{I}\,\mathbb{K}+\mathbb{I}+\mathbb{J}\,\mathbb{K}+\mathbb{J}-2\,\mathbb{K}+3)+i\,(4\,\mathbb{I}\,\mathbb{K}+3\,\mathbb{I}+3\,\mathbb{J}\,\mathbb{K}+\mathbb{J}-6\,\mathbb{K}+3))+i\,\mathbb{I}\,\mathbb{J}\,e_4+\mathbb{I}\,\mathbb{J}\,e_5+(2+i)\,\mathbb{I}\,\mathbb{J}\,e_7+\\ &\quad \mathbb{I}\,\mathbb{J}\,e_4\,e_7-i\,\mathbb{I}\,\mathbb{J}\,e_5\,e_7-2\,i\,\mathbb{I}\,\mathbb{K}\,e_4-6\,\mathbb{I}\,\mathbb{K}\,e_5-(8+4\,i)\,\mathbb{I}\,\mathbb{K}\,e_7-2\,\mathbb{I}\,\mathbb{K}\,e_4\,e_7+2\,i\,\mathbb{I}\,\mathbb{K}\,e_5\,e_7-3\,i\,\mathbb{I}\,e_4+3\,\mathbb{I}\,e_5-3\,\mathbb{I}\,e_4\,e_7-3\,i\,\mathbb{I}\,e_5\,e_7+\\ &\quad (6+3\,i)\,\mathbb{I}\,e_7-4\,i\,\mathbb{K}\,e_4-4\,\mathbb{K}\,e_5-(8+4\,i)\,\mathbb{K}\,e_7-4\,\mathbb{K}\,e_4\,e_7+4\,i\,\mathbb{K}\,e_5\,e_7+i\,e_4-3\,e_5+e_4\,e_7+i\,e_5\,e_7-(4+2\,i)\,e_7+(2-4\,i))\end{aligned}$$

```
ΨR = ΨL↑

ΨR[in_, i_ : 1, j_ : 1, k_ : 1, c_ : 1, doLeft_ : False] := Simplify@Module[{sm},
      sm =
        𝒱R[in, i, j, k, c, doLeft] +
        ℰR[in, i, j, k, c, doLeft] +
        𝒟̄L[in, i, j, k, c, doLeft] + (* negated in 𝒰L *)
        𝒰̄L[in, i, j, k, c, doLeft] +

        𝒰R[in, i, j, k, c, doLeft] +
        𝒟R[in, i, j, k, c, doLeft] +
        ℰ̄L[in, i, j, k, c, doLeft] + (* sterile ν *)
        If[! doLeft, -1, 1] 𝒱̄L[in, i, j, k, c, doLeft];
      Simplify[
       opCheck[c, L[L[]sS[1,4]]sm],
       Assumptions → octAssumptions]] //. octConjRep;

 (* Show the values of: doLeft{ΨR},!doLeft{ΨR} *)

Table[{doℝℂℍ𝕆 = i⟦1⟧; doSymOct = i⟦2⟧;
   Column[{
     Row@{"Ψ=", j, "=", fInℝℂℍ𝕆⟦j⟧, ", Gen=k=", k, ", RGB=l=", l, "=", RGB⟦l⟧, getℝℂℍ𝕆@i},
     Row@{"ΨR=", ΨR[fInℝℂℍ𝕆⟦j⟧, l, 2, 3, (**)2^2 ∘ (**)getTriad1⟦k⟧]}},
    Center]},
  {i, {{False, False}, {False, True}, {True, False}, {Both, False}}},
  {j, (*)Reverse@Range[10,10]**) (**)10, 10(*)fInℝℂℍ𝕆Len**)},
  {k, (**)1(*)3**)},
  {l, (**)1(*)3**)}] // MatrixForm
```

$$\Psi = 10 = e_7\,,\ \text{Gen=k=}\ 1\,,\ \text{RGB=l=}\ 1 = \blacksquare\,,\ \mathbb{R}\mathbb{O}\Rightarrow$$
$$\Psi\text{R}=\left(-6-\frac{13\,i}{2}\right)e_1+3\,i\,e_2-\left(6-\frac{5\,i}{2}\right)e_3-(8-3\,i)\,e_4+\left(3+\frac{5\,i}{2}\right)e_5+(4-5\,i)\,e_6+\left(5+\frac{3\,i}{2}\right)e_7+5\,i$$

$$\Psi = 10 = e_7\,,\ \text{Gen=k=}\ 1\,,\ \text{RGB=l=}\ 1 = \blacksquare\,,\ \mathbb{R}\mathbb{C}\mathbb{O}\Rightarrow$$
$$\Psi\text{R}=e_1\,((3-4\,i)\,e_6+(8+3\,i)\,e_7+i\,e_1\,(10\,e_2+(8+10\,i)\,e_3-(10-8\,i)\,e_4+10\,i\,e_5+3\,e_6+(4-7\,i)\,e_7-10)-8\,i)$$

$$\Psi = 10 = e_7\,,\ \text{Gen=k=}\ 1\,,\ \text{RGB=l=}\ 1 = \blacksquare\,,\ \mathbb{R}\mathbb{H}\mathbb{O}\Rightarrow$$
$$\begin{aligned}\Psi\text{R}=&(e_7+i)\,(-2\,i\,J\,K\,e_4+2\,J\,K\,e_5+e_2\,(J\,(-1+2\,i\,K)-4\,i\,K-3)+\\ &e_3\,(-2\,J\,K-i\,J+4\,K+7\,i)+(-2-4\,i)\,J\,K+J\,e_4+i\,J\,e_5+(2-i)\,J+4\,i\,K\,e_4-4\,K\,e_5+(4+8\,i)\,K-7\,e_4+3\,i\,e_5-(14+3\,i))\end{aligned}$$

$$\Psi = 10 = e_7\,,\ \text{Gen=k=}\ 1\,,\ \text{RGB=l=}\ 1 = \blacksquare\,,\ \mathbb{R}\mathbb{C}\mathbb{H}\mathbb{O}\Rightarrow$$
$$\begin{aligned}\Psi\text{R}=&\frac{1}{2}\,\mathbb{I}\,(-(4+8\,i)\,\mathbb{I}\,\mathbb{J}\,\mathbb{K}+(-2-4\,i)\,\mathbb{I}\,\mathbb{J}+(6-12\,i)\,\mathbb{I}-(8+16\,i)\,\mathbb{K}+2\,i\,\mathbb{I}\,\mathbb{J}\,\mathbb{K}\,e_4+6\,\mathbb{I}\,\mathbb{J}\,\mathbb{K}\,e_5-(8-4\,i)\,\mathbb{I}\,\mathbb{J}\,\mathbb{K}\,e_7+2\,\mathbb{I}\,\mathbb{J}\,\mathbb{K}\,e_4\,e_7-2\,i\,\mathbb{I}\,\mathbb{J}\,\mathbb{K}\,e_5\,e_7+\\ &\quad \mathbb{I}\,e_2\,(e_7+3\,i)\,(-2\,\mathbb{I}+2\,\mathbb{J}\,\mathbb{K}-i\,e_7\,(-4\,\mathbb{I}\,\mathbb{K}+\mathbb{J}-3))+\mathbb{I}\,e_3\,(e_7+i)\,(-e_7\,(-4\,\mathbb{I}\,\mathbb{K}+\mathbb{J}+3)+2\,i\,(\mathbb{I}-\mathbb{J}\,\mathbb{K}))+i\,\mathbb{I}\,\mathbb{J}\,e_4+\mathbb{I}\,\mathbb{J}\,e_5-(4-2\,i)\,\mathbb{I}\,\mathbb{J}\,e_7+\mathbb{I}\,\mathbb{J}\,e_4\,e_7-3\,i\,\mathbb{I}\,\mathbb{J}\,e_5\,e_7+3\,i\,\mathbb{I}\,e_4-\\ &\quad 3\,\mathbb{I}\,e_5+3\,\mathbb{I}\,e_4\,e_7+9\,i\,\mathbb{I}\,e_5\,e_7-(12+6\,i)\,\mathbb{I}\,e_7+4\,i\,\mathbb{K}\,e_4+4\,\mathbb{K}\,e_5-(16-8\,i)\,\mathbb{K}\,e_7+4\,\mathbb{K}\,e_4\,e_7-12\,i\,\mathbb{K}\,e_5\,e_7+2\,i\,e_4+6\,e_5+2\,e_4\,e_7-2\,i\,e_5\,e_7-(8-4\,i)\,e_7-(4+8\,i))\end{aligned}$$

FIG. 15. The operator defintions for Ψ_L and Ψ_R from [3] eq. (4 and 9) combined into (15).

Evaluate $\Psi_L[1_{SM}[k,m,\Psi_l@1],i,j,k,c]$

In[∘]:=
```
doℝℂℍ𝕆 = False;
doSymOct = True;
i = (**)2(*)2(**)3**);
j = (*)1**)2(*)3**);
k = (*)1(**)2**)3(**);
m = n = c = 1;
in = lSM[m, n, Ψl@1]
```

Out[∘]= $\frac{1}{24}(6e_1 - 6ie_2 - 3ie_3 + e_4 + 12ie_6 - 9ie_7 + 3)$

In[∘]:=
```
doLeft = True;
sm = {
    𝒱L[in, i, j, k, c, doLeft],
    ℰL[in, i, j, k, c, doLeft],
    𝒟R̅[in, i, j, k, c, doLeft],
    𝒰R̅[in, i, j, k, c, doLeft],

    𝒰L[in, i, j, k, c, doLeft],
    𝒟L[in, i, j, k, c, doLeft],
    ℰR̅[in, i, j, k, c, doLeft],
    If[! doLeft, -1, 1] 𝒱R̅[in, i, j, k, c, doLeft]} /. slRep;
{#, oct2List@#} & /@ % // MatrixForm
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} -\frac{i(1568e_1-12132ie_2-12340ie_3-6950e_4+10770e_5+9140ie_6+16119ie_7+21819)}{663552} & \left\{-\frac{7273i}{221184}, -\frac{49i}{20736}, -\frac{337}{18432}, -\frac{3085}{165888}, \frac{3475i}{331776}, -\frac{1795i}{110592}, \frac{2285}{165888}, \frac{199}{8192}\right\} \\ \frac{5(2391e_1-941ie_2-285ie_3+285e_4+617e_5+4143ie_6+1719ie_7+3861)}{1990656} & \left\{\frac{715}{73728}, \frac{3985}{663552}, -\frac{4705i}{1990656}, -\frac{475i}{663552}, \frac{475}{663552}, \frac{3085}{1990656}, \frac{6905i}{663552}, \frac{955i}{221184}\right\} \\ 0 & \{0,0,0,0,0,0,0,0\} \\ 0 & \{0,0,0,0,0,0,0,0\} \\ -\frac{i(1500e_1+966ie_2+3748ie_3+446e_4-3852e_5+1500ie_6+357ie_7-267)}{41472} & \left\{\frac{89i}{13824}, -\frac{125i}{3456}, \frac{161}{6912}, \frac{937}{10368}, -\frac{223i}{20736}, \frac{107i}{1152}, \frac{125}{3456}, \frac{119}{13824}\right\} \\ \frac{5(122e_1-429ie_2+459ie_3-279e_4-429e_5+122ie_6+279ie_7+459)}{41472} & \left\{\frac{85}{1536}, \frac{305}{20736}, -\frac{715i}{13824}, \frac{85i}{1536}, -\frac{155}{4608}, -\frac{715}{13824}, \frac{305i}{20736}, \frac{155i}{4608}\right\} \\ \frac{i(87e_1-423ie_2+606ie_3+1356e_4+657e_5-439ie_6-288ie_7-606)}{3456} & \left\{-\frac{101i}{576}, \frac{29i}{1152}, \frac{47}{384}, -\frac{101}{576}, \frac{113i}{288}, \frac{73i}{384}, \frac{439}{3456}, \frac{1}{12}\right\} \\ -\frac{5(78e_1-78ie_2+351ie_3+85e_4-108e_5+48ie_6-45ie_7-15)}{3456} & \left\{\frac{25}{1152}, -\frac{65}{576}, \frac{65i}{576}, -\frac{65i}{128}, -\frac{425}{3456}, \frac{5}{32}, -\frac{5i}{72}, \frac{25i}{384}\right\} \end{pmatrix}$$

In[∘]:=
```
Simplify@Total@sm
```

Out[∘]= $\frac{1}{1990656}((-183405 - 26592i)e_1 + (253620 + 116975i)e_2 - (206172 + 902145i)e_3 - (310335 - 780498i)e_4 + (211165 + 531018i)e_5 + (352284 - 88245i)e_6 + (231381 + 205155i)e_7 + (172665 - 401697i))$

In[∘]:=
```
octAssumptions
Simplify[c L[L[] sS[1,4]] Total@sm, Assumptions → octAssumptions]
```

Out[∘]= $\{e_1 \in \mathbb{R}, e_2 \in \mathbb{R}, e_3 \in \mathbb{R}, e_4 \in \mathbb{R}, e_5 \in \mathbb{R}, e_6 \in \mathbb{R}, e_7 \in \mathbb{R}, e_0 \in \mathbb{R}\}$

Out[∘]= $\frac{1}{1990656}((-135825 - 189438i)e_1 + (392319 - 47095i)e_2 + (287163 - 295905i)e_3 + (295905 + 287163i)e_4 + (47095 + 392319i)e_5 + (189438 - 135825i)e_6 + (316539 + 188910i)e_7 + (188910 - 316539i))$

Evaluate $\Psi_R[1_{SM}[k,m,\Psi_l@1],i,j,k,c]$

In[∘]:=
```
doLeft = False;
sm = {
    𝒱R[in, i, j, k, c, doLeft],
    ℰR[in, i, j, k, c, doLeft],
    𝒟L̅[in, i, j, k, c, doLeft],
    𝒰L̅[in, i, j, k, c, doLeft],

    𝒰R[in, i, j, k, c, doLeft],
    𝒟R[in, i, j, k, c, doLeft],
    ℰL̅[in, i, j, k, c, doLeft],
    If[! doLeft, -1, 1] 𝒱L̅[in, i, j, k, c, doLeft]} /. slRep;
{#, oct2List@#} & /@ % // MatrixForm
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} \frac{6170e_1-i(9537e_2+784e_3-10798ie_4+2889ie_5+5912e_6+13341e_7-11895i)}{331776} & \left\{-\frac{3965}{110592}, \frac{3085}{165888}, -\frac{3179i}{110592}, -\frac{49i}{20736}, -\frac{5399}{165888}, \frac{107}{12288}, -\frac{739i}{41472}, -\frac{4447i}{110592}\right\} \\ -\frac{i(20775e_1-633ie_2-5455ie_3+5455e_4+14790e_5+38478ie_6+8814ie_7+30138)}{995328} & \left\{-\frac{5023i}{165888}, -\frac{6925i}{331776}, -\frac{211}{331776}, -\frac{5455}{995328}, -\frac{5455i}{995328}, -\frac{2465i}{165888}, \frac{6413}{165888}, \frac{1469}{165888}\right\} \\ 0 & \{0,0,0,0,0,0,0,0\} \\ 0 & \{0,0,0,0,0,0,0,0\} \\ \frac{823e_1+978ie_2-696ie_3+2298e_4+960e_5+823ie_6-2295ie_7-711}{20736} & \left\{-\frac{79}{2304}, \frac{823}{20736}, \frac{163i}{3456}, -\frac{29i}{864}, \frac{383}{3456}, \frac{5}{108}, \frac{823i}{20736}, -\frac{85i}{768}\right\} \\ -\frac{i(1314e_1-2400ie_2+5131ie_3+2195e_4-4212e_5+1314ie_6-1893ie_7+3621)}{41472} & \left\{-\frac{1207i}{13824}, -\frac{73i}{2304}, -\frac{25}{432}, \frac{5131}{41472}, -\frac{2195i}{41472}, \frac{13i}{128}, \frac{73}{2304}, -\frac{631}{13824}\right\} \\ \frac{195e_1+141ie_2-723ie_3-73e_4-387e_5-327ie_6+45ie_7+51}{1728} & \left\{\frac{17}{576}, \frac{65}{576}, \frac{47i}{576}, -\frac{241i}{576}, -\frac{73}{1728}, -\frac{43}{192}, -\frac{109i}{576}, \frac{5i}{192}\right\} \\ -\frac{i(1755e_1-747ie_2+390ie_3+492e_4+45e_5-187ie_6+144ie_7-390)}{3456} & \left\{\frac{65i}{576}, -\frac{65i}{128}, -\frac{83}{384}, \frac{65}{576}, -\frac{41i}{288}, -\frac{5i}{384}, -\frac{187}{3456}, \frac{1}{24}\right\} \end{pmatrix}$$

In[∘]:=
```
Simplify@Total@sm
```

Out[∘]= $-\frac{1}{995328}i((557751 + 170334i)e_1 - (99549 + 273369i)e_2 + (452208 + 230009i)e_3 + (199831 + 35862i)e_4 - (73338 + 168165i)e_5 + (166584 + 16158i)e_6 + (124263 + 4854i)e_7 + (4722 - 40437i))$

In[∘]:=
```
Simplify[c L[L[] sS[1,4]] Total@sm, Assumptions → octAssumptions]
```

Out[∘]= $-\frac{1}{1990656}i((573909 + 3750i)e_1 - (267714 + 200031i)e_2 + (488070 + 30178i)e_3 - (30178 - 488070i)e_4 + (200031 - 267714i)e_5 - (3750 - 573909i)e_6 + (164700 + 9576i)e_7 + (9576 - 164700i))$

FIG. 16. The operators for Ψ_L and Ψ_R with $l_{SM}[\Psi_l]$ as input and shown as lists of their $\{\mathcal{V},\mathcal{E},\mathcal{D},\mathcal{U},\overline{\mathcal{U},\mathcal{D},\mathcal{E},\mathcal{V}}\}_{R/L}$ components.

Evaluate $\Psi_{L+R}[1_{SM}[k,m,\Psi l@1],i,j,k,c]$

In[•]:=

```
Simplify[(**) 2^(13+7×1+If[doSymOct,0,1]) × 3^(5+1×1) (**) {%, %%%%%%}]
Total@% /. slRep
(* Note the use of octMix to create the palindromic sequence when the octonion isn't naturally palindromic w/octMix=Range@8
  Also, some octonions don't produce the same numbers. *)
testAll = oct2List[%]〚octMix〛 // MatrixForm
```

Out[•]= $\{-384\, i\, ((573\,909 + 3750\, i)\, e_1 - (267\,714 + 200\,031\, i)\, e_2 + (488\,070 + 30\,178\, i)\, e_3 - (30\,178 - 488\,070\, i)\, e_4 + (200\,031 - 267\,714\, i)\, e_5 - (3750 - 573\,909\, i)\, e_6 + (164\,700 + 9576\, i)\, e_7 + (9576 - 164\,700\, i)),$
$384\, ((-135\,825 - 189\,438\, i)\, e_1 + (392\,319 - 47\,095\, i)\, e_2 + (287\,163 - 295\,905\, i)\, e_3 + (295\,905 + 287\,163\, i)\, e_4 + (47\,095 + 392\,319\, i)\, e_5 + (189\,438 - 135\,825\, i)\, e_6 + (316\,539 + 188\,910\, i)\, e_7 + (188\,910 - 316\,539\, i))\}$

Out[•]= $384\, ((-135\,825 - 189\,438\, i)\, e_1 + (392\,319 - 47\,095\, i)\, e_2 + (287\,163 - 295\,905\, i)\, e_3 + (295\,905 + 287\,163\, i)\, e_4 + (47\,095 + 392\,319\, i)\, e_5 + (189\,438 - 135\,825\, i)\, e_6 + (316\,539 + 188\,910\, i)\, e_7 + (188\,910 - 316\,539\, i)) -$
$384\, i\, ((573\,909 + 3750\, i)\, e_1 - (267\,714 + 200\,031\, i)\, e_2 + (488\,070 + 30\,178\, i)\, e_3 - (30\,178 - 488\,070\, i)\, e_4 + (200\,031 - 267\,714\, i)\, e_5 - (3750 - 573\,909\, i)\, e_6 + (164\,700 + 9576\, i)\, e_7 + (9576 - 164\,700\, i))$

Out[•]//MatrixForm=

$$\begin{pmatrix} 9\,296\,640 - 125\,228\,160\, i \\ -50\,716\,800 - 293\,125\,248\, i \\ 73\,838\,592 + 84\,717\,696\, i \\ 121\,858\,944 - 301\,046\,400\, i \\ 301\,046\,400 + 121\,858\,944\, i \\ -84\,717\,696 + 73\,838\,592\, i \\ 293\,125\,248 - 50\,716\,800\, i \\ 125\,228\,160 + 9\,296\,640\, i \end{pmatrix}$$

Note : the reversal of ($\mathbb{R}$) eal and ($\mathbb{I}$) maginary of the (l) eft quaternion lQuat$e$ = (e_0 e_1 e_2 e_3)
vs. (r) ight quaternion rQuat$e$ = (e_7 e_6 e_5 e_4),

In[•]:=

```
Abs /@ % // MatrixForm
```

Out[•]//MatrixForm=

$$\begin{pmatrix} 17\,280\, \sqrt{52\,808\,453} \\ 1152\, \sqrt{66\,682\,494\,226} \\ 384\, \sqrt{85\,647\,418\,105} \\ 384\, \sqrt{715\,322\,110\,906} \\ 384\, \sqrt{715\,322\,110\,906} \\ 384\, \sqrt{85\,647\,418\,105} \\ 1152\, \sqrt{66\,682\,494\,226} \\ 17\,280\, \sqrt{52\,808\,453} \end{pmatrix}$$

$\{ (45 \times 384 = 17\,280)\ \sqrt{52\,808\,453},$
$(3 \times 384 = 1152)\ \sqrt{66\,682\,494\,226},$
$384\ \sqrt{85\,647\,418\,105},$
$384\ \sqrt{715\,322\,110\,906},$
$384\ \sqrt{715\,322\,110\,906},$
$384\ \sqrt{85\,647\,418\,105},$
$(3 \times 384 = 1152)\ \sqrt{66\,682\,494\,226},$
$(45 \times 384 = 17\,280)\ \sqrt{52\,808\,453} \}$

where the number of vertices in $E_8\ (1_{42}) = 17\,280 = 2^7 \times 3^3 \times 5$ is $3^2 \times [$
$1920 = 1152 + 384 + 384 = 5 \times 384 = 10 \times 192 = 30 \times 64 = 2^3 \times 240 \times$
is the nunber of vertices in $E_8\ (2_{41}) - E_8\ (4_{21}) = 2160 - 240 = 2^4 \times 3 \times 5\ (3^2 - 1)\ ]$,
and $384 = 2^7 \times 3$ and $1152 = 2^7 \times 3^2 = 3 \times 384$,

This is all due to using the minimum factor of $2^6 \times 3^5 = 15,552$ in order to remove rationals in the result,
and another factor of $(2^7 \times 3) = 384$ for a total of $2^{13} \times 3^6 = 2^6 \times 3^3\ (2^7 \times 3^3 \times 5) / 5$

In[•]:=

```
Norm@%
N[%, 10]
FactorInteger@%%^2
```

Out[•]= $5376\, \sqrt{15\,388\,256\,065}$

Out[•]= 6.668896269×10^8

Out[•]=

$$\begin{pmatrix} 2 & 16 \\ 3 & 2 \\ 5 & 1 \\ 7 & 3 \\ 13 & 1 \\ 461 & 1 \\ 73\,363 & 1 \end{pmatrix}$$

In[•]:=

```
Abs@ReIm@testAll〚1〛
PalindromeQ@Flatten@%
```

Out[•]=

$$\begin{pmatrix} 9\,296\,640 & 125\,228\,160 \\ 50\,716\,800 & 293\,125\,248 \\ 73\,838\,592 & 84\,717\,696 \\ 121\,858\,944 & 301\,046\,400 \\ 301\,046\,400 & 121\,858\,944 \\ 84\,717\,696 & 73\,838\,592 \\ 293\,125\,248 & 50\,716\,800 \\ 125\,228\,160 & 9\,296\,640 \end{pmatrix}$$

Out[•]= True

The palindromic octonion vector has a Norm $= 384 \times 14\ \sqrt{15\,388\,256\,065}$,
where the number under the radical has prime factors of $\{5, 7, 13, 461, 73\,363\}$.

FIG. 17. This shows the palindromic nature of the complexified octonion ($\mathbb{RCO}$) of $\Psi_{LR} = \Psi_L^{\uparrow} \oplus \Psi_L^{\downarrow}$ when evaluated as the absolute values of their real and imaginary parts. With the rationals are scaled to integers, they relate to the number of roots of $E8_{142}$, $E8_{241}$, and $E8_{241}$ polytopes.