Proof of the Collatz conjecture based on a directed graph

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Abstract

The Collatz conjecture considers recursively sequences of positive integers where n is succeeded by $\frac{n}{2}$, if n is even or $\frac{3n+1}{2}$, if n is odd. The conjecture states that for all starting values n the Collatz sequence eventually reaches a trivial cycle 1, 2, 1, 2……. If the Collatz conjecture is false, then either there is a nontrivial cycle, or one sequence goes to infinity. In this paper, we construct a directed graph based on the union of infinite number of basic Collatz directed graphs. Each basic Collatz directed graph relates to each positive integer. We show that the directed graph is connected and covers all positive integers. There is only a trivial cycle and no sequence goes to infinity.

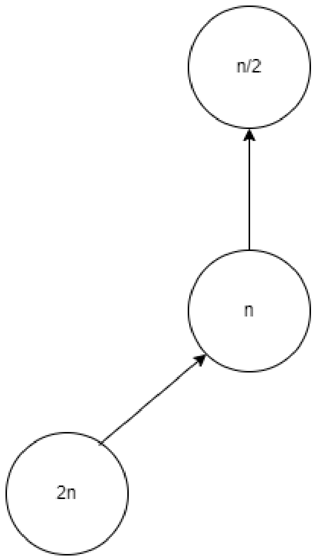
1. Introduction

The Collatz conjecture considers recursively sequences of positive integers where n is succeeded by $\frac{n}{2}$, if n is even, or $\frac{3n+1}{2}$, if n is odd. The conjecture states that for all starting values n the sequence eventually reaches the trivial cycle 1, 2, 1, 2……. If the Collatz conjecture is false, then either there is a nontrivial cycle, or one sequence goes to infinity [1-2].

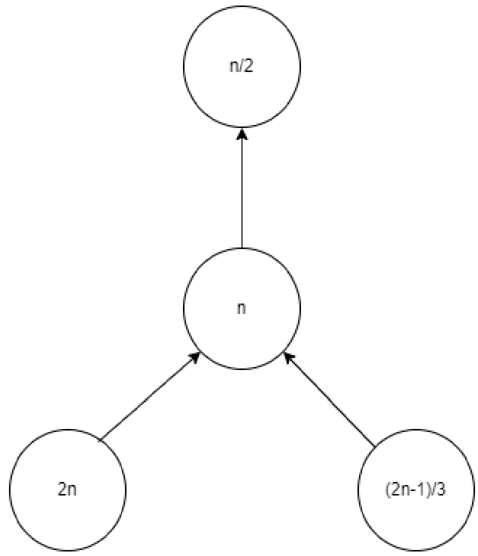
2. A basic Collatz directed graph

The basic directed graph is constructed for each natural number as follows:

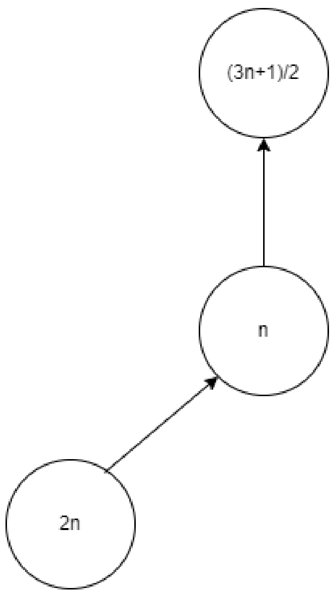
Let n be a positive integer node. Its parent node is $\frac{n}{2}$, if n is even or $\frac{3n+1}{2}$, if n is odd. Its left child is 2n. Its right child is $\frac{2n-1}{3}$, if $n \equiv 2 \bmod 3$, or no right child , if $n \neq 2 \bmod 3$. Thus there are four types of basis Collatz directed graph as shown in Figure 1.



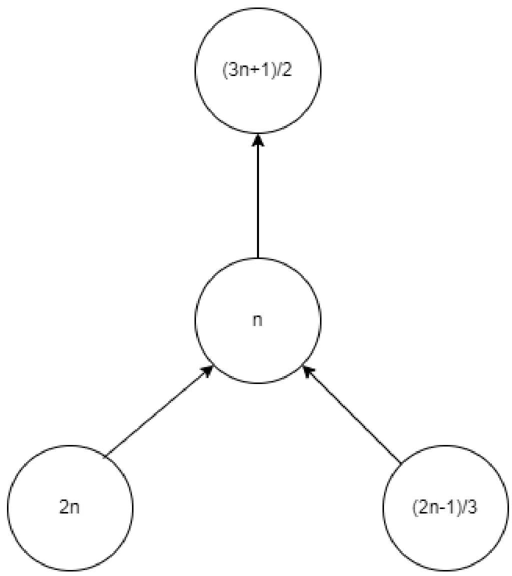
(a) n is even and not equal to 2 mod 3



(b) n is even and equals to 2 mod 3



(c) n is odd and not equal to 2 mod 3



(d) n is odd and equals to 2 mod 3

Figure 1, Four types of basic Collatz directed graphs

Examples of basic Collatz directed graphs shown in Figure 2.

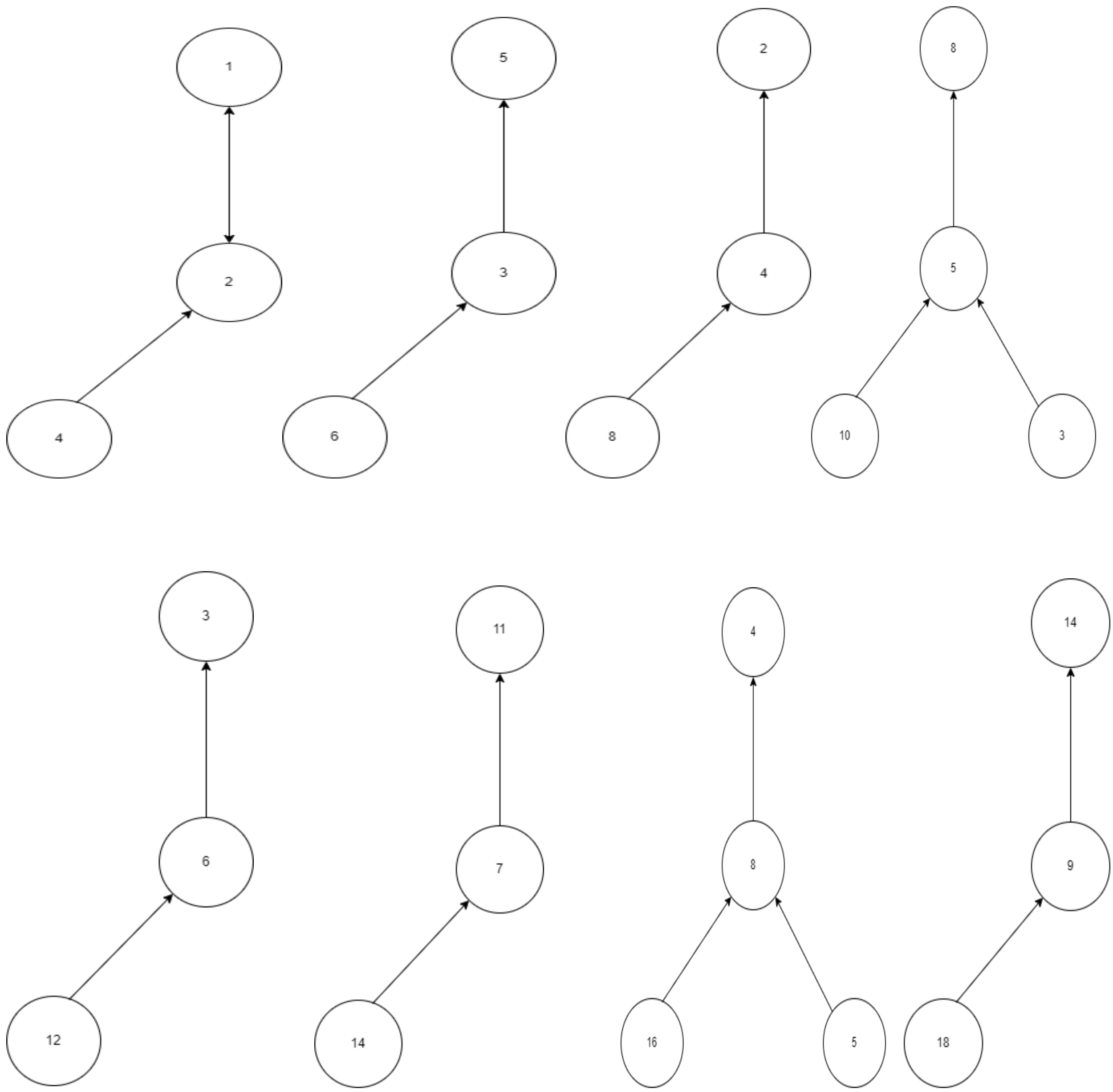


Figure 2. Basic Collatz directed graphs of 2, 3, 4, 5, 6, 7, 8, and 9

3. The union of two basic Collatz directed graphs

The union of two basic Collatz directed graphs for different cases shown in Figure 3.

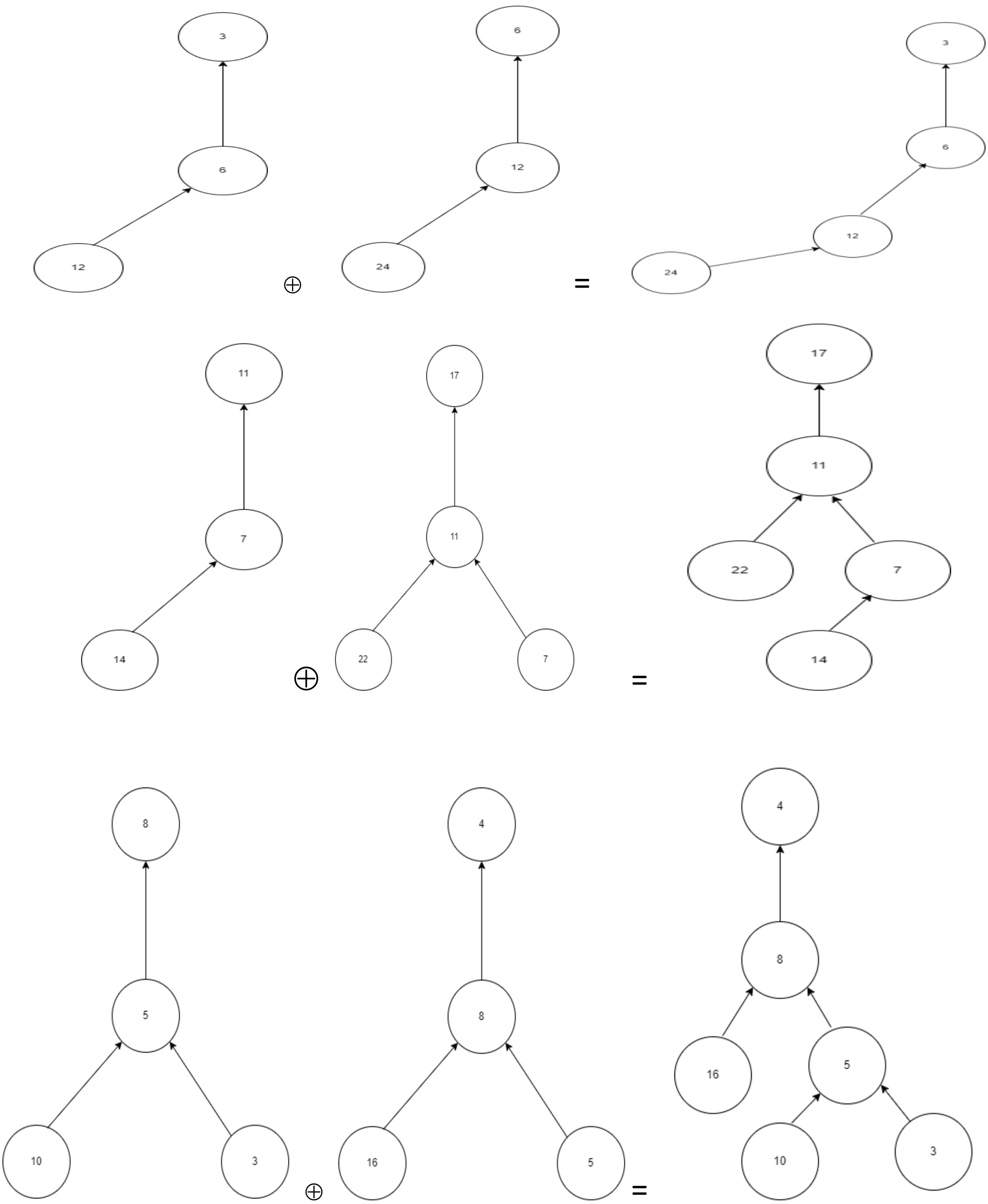


Figure 3. Various cases for the union of two basic Collatz directed graphs

4. A Complete Collatz directed graph

The complete Collatz directed graph arranged in levels 0 to ∞ with node 1 in the level 0 as shown in Figure 4.There is no nontrivial cycle or divergence sequence in this graph**.** Assume that G represents a directed graphs different from the complete Collatz directed graph. We have to prove that G is the same as the complete Collatz directed graph.

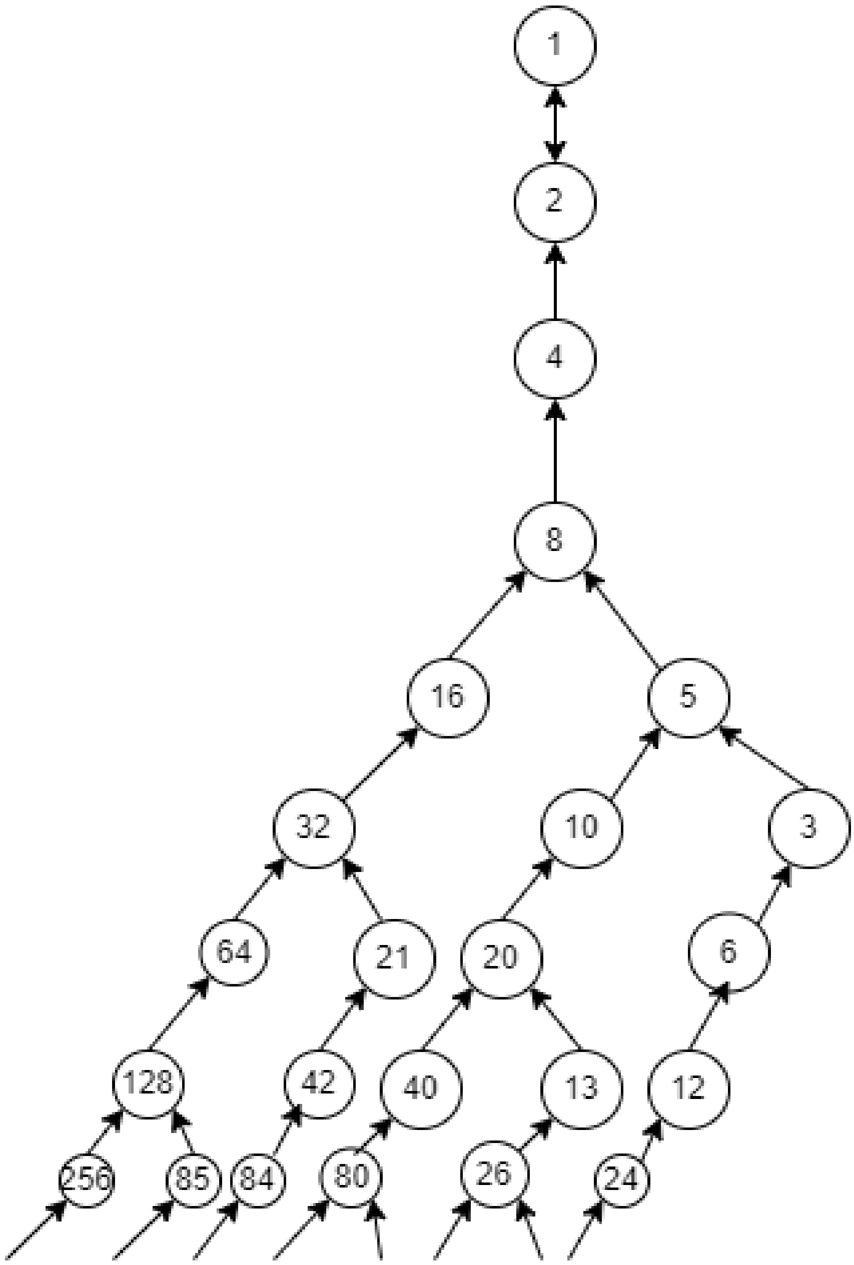


Figure 4. A complete Collatz directed graph

Because G is a directed graph and covers a countable set positive integers arranged in levels. At level i, there could be only six types of positive integers 6n+2, 6n+5, 6n+1. 6n+4, 6n, and 6n+3; n= 0, 1, 2, …….. Let N_i be number of nodes in level i. There are two cases to consider:

Case 1. Node 6n+2 or 6n+5 is in level i. N_i is less than N_{i+1} because this node has two children.

Case 2. Node 6n+2 or 6n+5 is not in level i. N_i equals to N_{i+1} because each node has only one child.

Thus, N_i is less than or equal to N_{i+1}. There is only finite number of i such that N_i equals to N_{i+1} because by starting with 6n+1. 6n+4, 6n, and 6n+3 in level i and following the Collatz rules it will reach level j < i such that there is only node 8 in this level which is the same as node 8 in the complete Collatz directed graph. This conclusion proves that G is the same as the complete Collatz directed graph as shown in Figure 4.

Conclusion

A complete Collatz directed graph is the same as G which covers all positive integers**.** By starting at any node in this complete Collatz directed graph, there is a unique path from that node to a node 1. Thus, the Collatz conjecture is proved to be valid.

References

[1] R . Terras, (1976). " A stopping time problem on the positive integers". Acta Arithmetica, 30(3), 241-252.

[2] J. C. Lagarias. The 3x+1 Problem: An Overview. https://arxiv.org/abs/2111.02635, 2021.