Proof of the Riemann Hypothesis using the decomposition ζ(z)= X(z) – Y(z)
And analysis of the distribution of the zeros of ζ(z) based on X(z) and Y(z)

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Abstract: Prime numbers are the atoms of mathematics and mathematics is needed to make sense of the real world. Finding the Prime number structure and eventually being able to crack their code is the ultimate goal in what is called Number Theory. From the evolution of species to cryptography, Nature finds help in Prime numbers.

One of the most important advances in the study of Prime numbers was the paper by Bernhard Riemann in November 1859 called "Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse" (On the number of primes less than a given quantity).

In that paper, Riemann gave a formula for the number of primes less than x in terms the integral of 1/log(x) and the roots (zeros) of the zeta function defined by:

$$[\text{RZF}] \quad \zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$$

Where ζ(z) is a function of a complex variable z that analytically continues the Dirichlet series. Riemann also formulated a conjecture about the location of the zeros of RZF, which fall into two classes: the "trivial zeros" -2, -4, -6, etc., and those whose real part lies between 0 and 1. Riemann's conjecture Riemann hypothesis [RH] was formulated as this:

[RH] The real part of every nontrivial zero z* of the RZF is 1/2.

Proving the RH is, as of today, one of the most important problems in mathematics. In this paper we will provide proof of the RH. The proof of the RH will be built following these five parts:

- PART 1: Description of the Riemann Zeta Function RZF $\zeta(z)$
  - Introducing s limit and an approximation

- PART 2: The C-transformation. An artifact to decompose $\zeta(z)$

- PART 3: Application of the C-transformation to $f(z) = \frac{1}{x^z}$ in Re(z)≥0 to obtain $\zeta(z) = X(z) - Y(z)$
  - Decomposition of $\zeta(z) = X(z) - Y(z)$
  - Analysis of $X(z), |X(z)|, |X(z)|^2$
  - Analysis of $Y(z), |Y(z)|, |Y(z)|^2$

- PART 4: Proof of the Riemann Hypothesis
  - Analysis of the values of z such that $X(z) = Y(z)$, that equates to $\zeta(z) = 0$
  - Proof that $|X(z)| = |Y(z)|$ only if $Re(z) = 1/2$
  - Conclude that $\zeta(z) = 0$ only if $Re(z) = 1/2 \; for \; Re(z) \geq 0$

- PART 5: On the distribution of the non-trivial zeros of Zeta in the critical line $\alpha = 1/2$.
  - Algorithm N1
  - Algorithm H1
  - Algorithm H2

Nomenclature and conventions

- ζ(z)=$\sum_{k=1}^{\infty} k^{-z}$ is the Riemann Zeta Function (RZF).
- z* is any nontrivial zero (NTZ) of the RZF verifying that ζ(z*) = 0.
- $ß^{*}(n)$ is the n^{th} zero of the Riemann function in the critical line Re(z)=1/2 in C
- α=Re(z) is the real part of z
- ß=Im(z) is the imaginary part of z
- If z=α+iß, Modulus(z)= $|z| = \sqrt{(\alpha^2 + ß^2)}$ and SquareAbsolute(z)= $|z|^2$

Comments

- Only relevant formulas will be numbered

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PART 1:

The Riemann Zeta function $\zeta(z)$ [RZF]

1. $\zeta(s)$ *in R*

As defined in literature (Sondow et al, Weisstein, Edwards, Jekel)

1.1. $\zeta(s) = \sum_{j=1}^{\infty} j^{-s}$ converges for $s \neq 1$

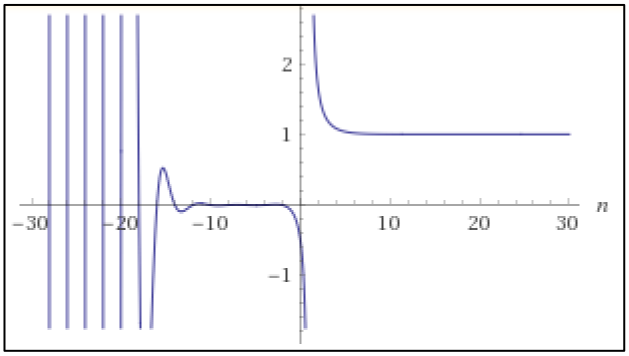


Figure. 1. Riemann Zeta function in R

1.2. Euler Product Formula that links $\zeta(s)$ with the distribution of prime numbers

$$\zeta(s) = \sum_{j=1}^{\infty} j^{-s} = \prod_{p=prime} \frac{1}{1-p^{-s}} \quad [1]$$

Example for k=2

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots = \frac{1}{1-2^{-2}} x \frac{1}{1-3^{-2}} x \frac{1}{1-5^{-2}} x \frac{1}{1-7^{-2}} x \ldots$$

1.3. Integral definition of $\zeta(s)$:

$$\zeta(s) = \sum_{j=1}^{\infty} j^{-s} = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{1}{e^x-1} x^s \frac{dx}{x}$$

Where $\Gamma(s)$, is the Gamma function

1.4. Analytical continuation of $\zeta(s)$ for :

Re(s)>0: [Dirichlet]

$$\zeta(s) = \frac{1}{s-1} \sum_{k=1}^{\infty} (\frac{n}{(n+1)^s} - \frac{n-s}{n^s}$$

0<Re(s)<1:

$$\zeta(s) = \frac{1}{1-2^{1-s}} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^s}$$

-k<Re(s) [Kopp, Konrad. 1945]:

$$\zeta(s) = \frac{1}{s-1}\sum_{k=1}^{\infty}\frac{k(k+1)}{2}\left(\frac{2k+3+s}{(k+1)^{s+2}} - \frac{2k-1-s}{k^{s+2}}\right)$$

1.5. Laurent series of $\zeta(s)$:

$$\zeta(s) = \frac{1}{s-1} + \sum_{k=}^{\infty}\frac{(-1)^k\gamma_n}{k!}(s-1)^k)$$

where γ_n are the Stieltjes constants.

1.6. Hurwitz function $\zeta(k,z)$:

$$\zeta(k,z) = \sum_{j=0}^{\infty}(j+z)^{-k} = \sum_{j=z}^{\infty} j^{-k} \qquad \text{converges for } k > 1$$

1.7. Generalized Harmonic Function $H_n^{(k)}$:

$$H_n^{(k)} = \sum_{j=1}^{n} j^{-k} = \left(\frac{1}{1^k} + \frac{1}{2^k} + \cdots + \frac{1}{n^k}\right) \qquad \text{converges for } k > 1$$

1.8. $\zeta(s)$ converges for s>1 to the following values (Sloane):

s	$\zeta(s)$	Known $\zeta(s)$ representations over π
2	1.6449179	$\pi^2/6$
4	1.0823232	$\pi^4/90$
6	1.0173431	$\pi^6/945$
8	1.0040774	$\pi^8/9450$

Table 1. Values of $\zeta(s)$

1.9. An approximation for the values of $\zeta(s)$ for s>1 in *R*

One can calculate that:

$$\lim_{s\to\infty}\left(\frac{\zeta(s)}{\zeta(s)+1}\right)^{\frac{1}{s}} = 1$$

And:

$$\lim_{s\to\infty}\left(\frac{\zeta(s)}{\zeta(s)-1}\right)^{\frac{1}{s}} = 2$$

Based on this expression, for s sufficiently large, one can represent $\zeta(s)$ as a multiple of π^s:

$$\zeta(s) = \frac{\pi^s}{K_s} \qquad \text{with } K_s = (2^s - 1) * \frac{\pi^s}{2^s}$$

with a very good approximation given by:

$$K_s^* = int\left((2^s - 1) * \frac{\pi^s}{2^s}\right) - 1 \qquad \text{where int(k) is the integer part of k.} \qquad [2]$$

The error between the K_s^* calculated and K_s actual is very small for s>4.

Some calculated values of K_s^* calculated and K_s actual:

s	Calculated	Actual
2	6.0	6.0
3	26.0	25.8
4	90.0	90.0
5	295.0	295.1
6	945.0	945.0
7	2,995.0	2,995.3
8	9,450.0	9,450.0
9	29,749.0	29,749.4
10	93,555.0	93,555.0
11	294,059.0	294,058.7

Table 2. Values of K_s^* calculated and K_s actual

One can use [2] to propose the following approximation for ζ(s):

$$\mathrm{CZ(s)} = \frac{1}{1-\pi^{-s}-2^{-s}} \qquad [3]$$

	s=3	s=4	s=10	s=14
ζ(s)	1.20206	1.0823	1.000994	1.0000612
CZ(s)	1.18659	1.0784	1.000988	1.0000611

Table 3. Comparing ζ(s) and CZ(s)

Graphically:

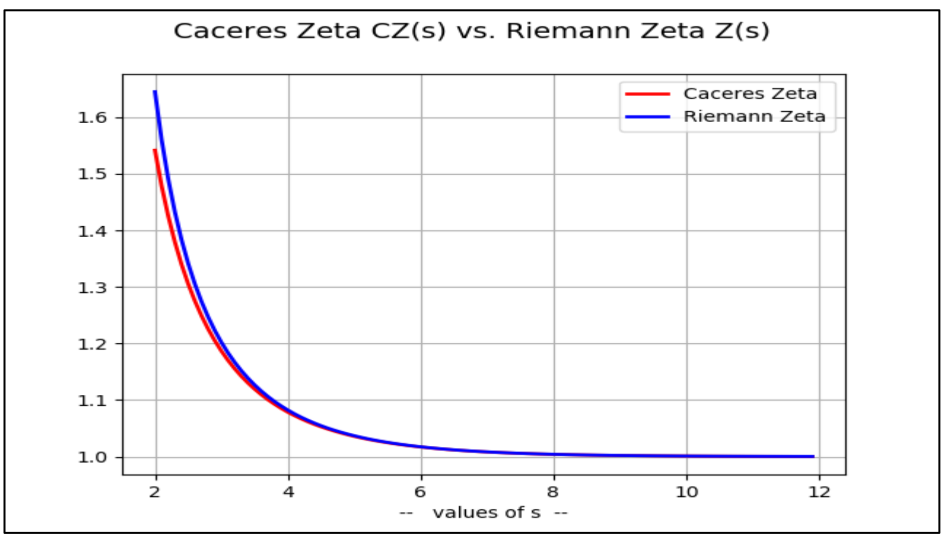


Figure. 2. Caceres' approximation for the Riemann Zeta function in R

1.10. Zeros of $\zeta(z)$ (Weisstein)

1.10.1. $\boldsymbol{Re(z) > 1}$

There are no zeros of ζ(s). ζ(s) converges for s>1.

1.10.2. $\boldsymbol{z = 1}$

ζ(1) is the Harmonic function and diverges, and:

$\lim_{s\to 1}(\zeta(s) - \frac{1}{1-s}) = \gamma$ (the Euler-Mascheroni constant)

1.10.3. $\boldsymbol{Re(z) < 1}$

ζ(z) has trivial zeros at $z = -2k$, with k natural numbers

ζ(z) has nontrivial zeros at $z = \frac{1}{2} \pm ßi$ at the critical line $Re(z) = \frac{1}{2}$ (Brentt)

The Riemann Hypothesis [RH] establishes that all nontrivial zeros of zeta have $Re(z) = \frac{1}{2}$

List of imaginary part of the first nontrivial zeros of zeta (Odlyzko):

ß(1) = 14.134725142
ß(3) = 25.010857580
ß(4) = 30.424876126
ß(6) = 37.586178159
ß(7) = 40.918719012
ß(9) = 48.005150881
ß(10) = 49.773832478

Representation of |ζ(z)|:

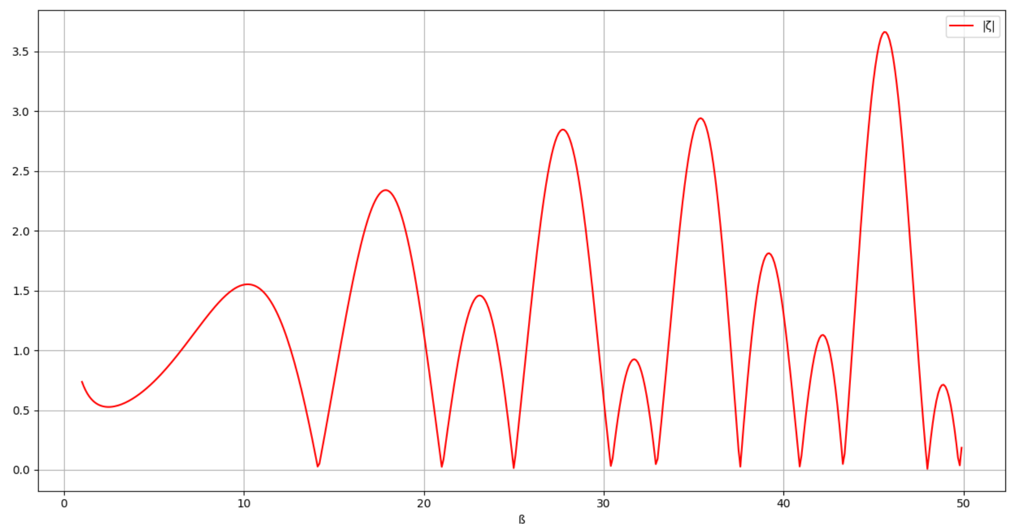


Figure. 3. Modulus of the Riemann Zeta function |ζ(z)|

PART 2:

The C-Transformation

1. C-Transformation of f(x)

Let's define the C-transformation of an integrable function f(x) by:

$$C_n\{f(x)\} = \sum_{k=1}^{n} f(k) - \int f(n)\ dn \qquad [4]$$

And the C-values is the limit, if it exists, of the C-transformation when $n \to \infty$:

$$C\{f(x)\} = \lim_{n\to\infty} C_n\{f(x)\} \qquad [5]$$

1.1. C-Transformation of $f(x) = \frac{1}{x}$ for $x \in R$:

$$C_n\left\{\frac{1}{x}\right\} = \sum_{\mathrm{k}=1}^{\mathrm{n}} \frac{1}{\mathrm{k}} - \int \frac{dn}{n}$$

And the C-Value of $f(x) = \frac{1}{x}$ is $\gamma = 0.5772$ (Euler-Mascheroni constant)

$$\mathrm{C}\left\{\frac{1}{x}\right\} = \lim_{n\to\infty} (\sum_{k=1}^{n} \frac{1}{k} - \ln(n)) = \gamma$$

1.2. C-Transformation of $f(x) = \frac{\ln(x)^m}{x}$ for $x \in R, m \in Z$:

$$C_n\left\{\frac{\ln(\mathrm{x})^{\mathrm{m}}}{x}\right\} = \sum_{\mathrm{k}=1}^{\mathrm{n}} \frac{\ln(k)^m}{\mathrm{k}} - \int \frac{\ln(n)^m\, dn}{n}$$

And the C-Value of $f(x) = \frac{\ln(x)^m}{x}$ are the Stieltjes constants that occur in the Laurent series expansion of the Riemann zeta function:

$$\mathrm{C}\left\{\frac{\ln(\mathrm{x})^{\mathrm{m}}}{x}\right\} = \lim_{\mathrm{m}\to\infty} (\sum_{\mathrm{k}=1}^{\mathrm{m}} \frac{(\ln k)^n}{\mathrm{k}} - \frac{(\ln n)^{m+1}}{m+1}) = \gamma_m$$

m	approximate value of γ_m
0	+0.5772156649015328606065120 9
1	−0.0728158454836767248605863 7
2	−0.0096903631928723184845303 8
3	+0.0020538344203033458661600 4

1.3. C-Transformation of $f(x) = m$, for $m \in R$ constant:

$$C_n\{m\} = \sum_{\mathrm{k}=1}^{\mathrm{n}} m - \int m\ dn$$

$$C_n\{\mathrm{m}\} = m * n - m * n = 0$$

and the C-values of f(x) =m constant is:

$$C\{m\} = 0$$

1.4. C-Transformation of $f(x) = \sin(x)$ for $x \in R$:

$$C_n\{\sin(x)\} = \sum_{k=1}^{n} \sin(k) - \int \sin(n)\, dn$$

$$C_n\{\sin(x)\} = \frac{1}{2\left(\sin(n) - \cot\left(\frac{1}{2}\right)\cos(n) + \cot\left(\frac{1}{2}\right) + \cos(n)\right)}$$

And the C-values of $f(x) = \sin(x)$ are in the interval:

$$C\{\sin(x)\} \in \left[\frac{1}{2}\left(2\cot\left(\frac{1}{2}\right) - 3\right), \frac{3}{2}\right]$$

One can also calculate that:

$$C\{\cos(x)\} \in \left[\frac{1}{2}\left(\cot\left(\frac{1}{2}\right) - 4\right), \frac{1}{2}\left(2 - \cot\left(\frac{1}{2}\right)\right)\right]$$

1.5. C-Transformation of $f(x) = e^{-x}$ for $x \in R$:

$$C_n\{e^{-x}\} = \sum_{k=1}^{n} e^{-k} - \int e^{-n} dn$$

$$C_n\{\sin(x)\} = \sum_{k=1}^{n} e^{-k} + \frac{e^{-n}}{n}$$

And the C-values of $f(x) = e^{-x}$ are:

$$C\{e^{-x}\} = \frac{1}{e - 1}$$

1.6. C-Transformation of $f(x) = x^{-s}$ for $x, s \in R$, s>1:

$$C_n\left\{\frac{1}{x^s}\right\} = \sum_{k=1}^{n} \frac{1}{k^s} - \int \frac{dn}{n^s}$$

$$C_n\left\{\frac{1}{x^s}\right\} = \sum_{k=1}^{n} \frac{1}{k^s} - \frac{n^{1-s}}{1 - s}$$

and the C-value of $f(x) = \frac{1}{x^s}$ is the Riemann Zeta function for s>1:

$$C\left\{\frac{1}{x^s}\right\} = \lim_{n\to\infty}\left(\sum_{k=1}^{n} \frac{1}{k^s} - \frac{n^{1-s}}{1 - s}\right) = \lim_{n\to\infty}\left(\sum_{k=1}^{n} \frac{1}{k^s}\right) - \lim_{n\to\infty}\left(\frac{n^{1-s}}{1 - s}\right) = \zeta(s)$$

1.7. C-Transformation of $f(z) = \frac{1}{x^z}$ for $z \in C, Re(z) \geq 0, z \neq 1$

$$C_n\left\{\frac{1}{x^z}\right\} = \sum_{k=1}^{n} \frac{1}{k^z} - \int \frac{dn}{n^z}$$

One can use Euler's identity:

$$e^x = \cos(x) + i * \sin(x)$$

To calculate for z=α+ßi:

$$k^{-z} = k^{-\alpha} \ [\cos(ß * \ln k) - i \ (\sin \ (ß * \ln \mathrm{k})]$$

And:

$$\int \frac{dn}{n^z} = n^{(1-\alpha)} [\cos(ß * ln(n) - i \, sin(ß * ln(n)] * \frac{[(1-\alpha) + iß]}{[(1-\alpha)^2 + ß^{2]}}$$

One can now express the real and imaginary components of $C_n\{f\}$ as:

$$Re(C_n\{f\}) = \sum_{k=1}^{n} k^{-\alpha} \ (\cos \ (ß * \ln(k)) + + \frac{1}{[(1-\alpha)^2+ß^{2]}} \ (n^{(1-\alpha)} \ [(1-\alpha)*\cos(ß*\ln(n))+ß* \sin(ß*\ln(n))]) \quad [6]$$

$$Im(C_n\{f\}) = -\sum_{k=1}^{n} k^{-\alpha} \ (\sin \ (ß * \ln(k))+ + \frac{1}{[(1-\alpha)^2+ß^{2]}} \ (n^{(1-\alpha)} \ [ß*\cos(ß*\ln(n)) - (1-\alpha)*\sin(ß*\ln(n))]) \quad [7]$$

One can calculate that, for *α=Re(z)>1*, and for any ϵ arbitrarily small, there is a value of n=N such that for n>N, $C_N\{f\}$ - $\zeta(z) < \epsilon$, as the following table shows:

α	***ß***	***$C_N\{f\}$ for N=500***	***ζ(z)***	***|C_N {f} −ζ(z)|***
2	0	1.644934068	1.654934067	$< 10^{-8}$
2	1	1.150355702 + 0.437530865 i	1.150355703+0.437530866 i	$< 10^{-8}$
3	0	1.202056903	1.202056903	$< 10^{-9}$

Table 4. Values of $C_n\{f(n) = k^{-z}\}$ for α=Re(z)>1 for N=500

That shows that the C-values of f(z) $= \frac{1}{x^z}$ for Re(z)>1 is ζ(z).

PART 3:

A decomposition of ζ(z) based on the C-transformation of $f(x) = \frac{1}{x^z}$ for $z \in C, 0 \leq Re(z) < 1$

1. C-Transformation of $f(z) = \frac{1}{x^z}$ for $z \in C, 0 \leq Re(z) < 1$

The C-values of $f(z) = \frac{1}{x^z}$ from [6] and [7] are equal to the $\zeta(z)$ when Re(z)>1, this error $|C_n\{f\}$ - $\zeta(z)|$ grows significantly in the critical strip $for\ 0 \leq Re(z) < 1$ as observed in the following table:

α	ß	$C_n\{f\}$	$\zeta(z)$	$\lvert C\_n\ \{f\} - \zeta(z)\rvert$
0.0	0	$C_N\{f\}$ *for N=500*	-0.5	0.5
0.2	2	0.399824505+0.322650799 i	0.360103 + 0.266246 i	> 0.05
0.7	0	-2.777900606	-2.7783884455	$> 10^{-4}$

Table 5. Values of $C_n\{f(n) = k^{-z}\}$ *for 0≤Re(z)<1 for N=500*

To understand better the value of the difference $C_n\left\{\frac{1}{k^z}\right\}$ - $\zeta(z)$, one can plot the difference for $\alpha \in [0,1)$ and ß $= 0$: (Similar exponential charts occur for all values of $\alpha \in [0,1)$ for any given value of ß)

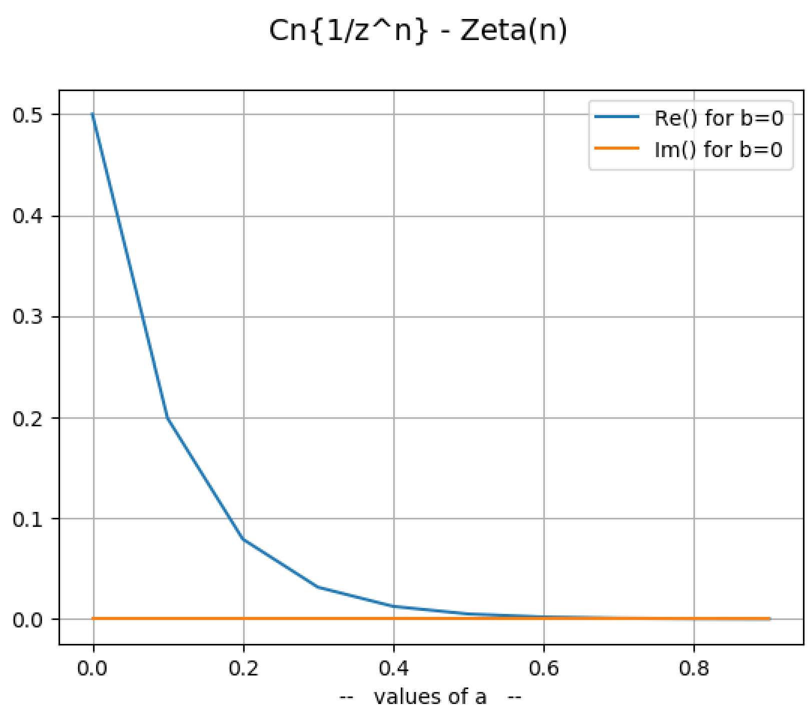


Figure. 4a. Where a=Re(z) and b=Im(z)

And plot the difference for variable values of ß $\in [0,1)$ and $\alpha = 0$: (Similar sine charts occur for all values of ß $\in [0,1)$ for any given value of α)

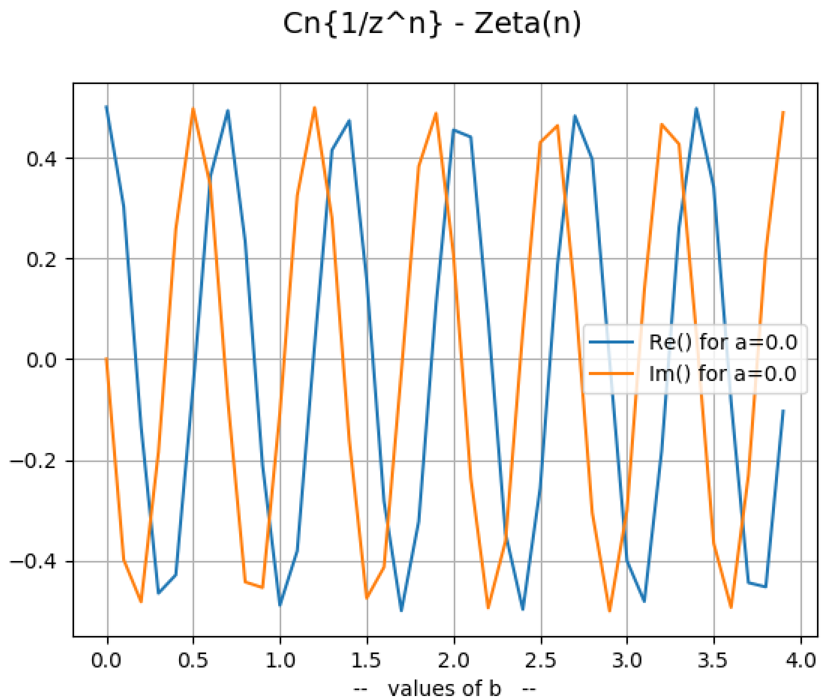


Figure. 4b. Where a=Re(z) and b=Im(z)

These charts lead to the following calculation of the difference $C_n\left\{\frac{1}{k^z}\right\} - \zeta(z)$:

$$\mathrm{Re}[C_n\left\{\frac{1}{k^z}\right\} - \zeta(z)] = \frac{1}{2}\, n^{-a} * \cos\,(ß * \ln\,(n)) + O(\frac{1}{n})$$

$$\mathrm{Im}[C_n\left\{\frac{1}{k^z}\right\} - \zeta(z)] = \frac{1}{2}\, n^{-a} * \sin\,(ß * \ln\,(n)) + O(\frac{1}{n})$$

With $O(1/n\,) \text{—>}\ 0$ when $n \text{—>}\ \infty$.

And one can finally write:

$$Re(C_n\{f\}) = \sum_{k=1}^{n} k^{-\alpha}\,(\cos\,(ß * \ln(k)) +$$

$$+ \frac{1}{[(1-\alpha)^2+ß^2]}\,(n^{(1-\alpha)}\ [(1-\alpha)*\cos(ß*\ln(n))+ß*\sin(ß*\ln(n))])$$

$$+\frac{1}{2}\, n^{-a} * \cos(ß * \ln(n)) \qquad [8]$$

$$Im(C_n\{f\}) = -\sum_{k=1}^{n} k^{-\alpha}\,(\sin\,(ß * \ln(k))+$$

$$+ \frac{1}{[(1-\alpha)^2+ß^2]}\,(n^{(1-\alpha)}\ [ß*\cos(ß*\ln(n)) - (1-\alpha)*\sin(ß*\ln(n))])$$

$$+\frac{1}{2}\, n^{-a} * \sin(ß * \ln(n)) \qquad [9]$$

and the C-value of $f(x) = \frac{1}{x^z}$ for $z \in C, Re(z) \geq 0, z \neq 1$ is the Riemann Zeta function ζ(z).

2. Decomposition of $\zeta(z) = X(z) - Y(z)$

One can rewrite [8] and [9] creating the $X(z,n)$ and $Y(z,n)$ functions:

$$\zeta(z) = \lim_{n\to\infty} [X(z,n) - Y(z,n)], \text{ where:}$$

$$X(z,n) = (\sum_{k=1}^{n} k^{-\alpha} * cos\ (ß * ln(k)) + \frac{1}{2}\ n^{-\alpha} cos\ (ß\ ln(n)) + \\ + i * (\sum_{k=1}^{n} k^{-\alpha} * sin\ (ß * ln(k)) + \frac{1}{2}\ n^{-\alpha}\ sin(ß\ ln(n)))) \quad [10]$$

$$Y(z,n) = n^{(1-\alpha)} \frac{1}{[(1-\alpha)^2 + ß^2]} [((1-\alpha) * \cos(ß \ln(n)) + ß * \sin(ß \ln(n))) + \\ + i\ (ß * \cos(ß \ln(n)) - (1-\alpha) * \sin(ß \ln(n)))] \quad [11]$$

and define:

$$X(z) = \lim_{n\to\infty} X(z,n) \text{ and}$$

$$Y(z) = \lim_{n\to\infty} Y(z,n)$$

to write:

$$\zeta(z) = X(z) - Y(z) \quad [12]$$

The following table compared the values of $\zeta(z)$ and $X(z) - Y(z)$:

z= 0 +j* 0 and n=500
Zeta(z) = -0.5 + i* 0.0 X(z)-Y(z) = -0.5 +i* 0.0 ---> Error = 0.0 +i* 0.0
z= 0.2 +j* 2 and n=500
Zeta(z) = 0.360102590022591 + i* -0.266246199765574 X(z)-Y(z) = 0.360102741838091 +i* -0.266246128959438 ---> Error= -1.5181550 e-7 +i* -7.080613 e-8
z= 0.4 +j* 0 and n=500
Zeta(z) = -1.13479778386698 + i* 0.0 X(z)-Y(z) = -1.1347977871726 +i* 0.0 ---> Error= 3.305619 e-9 +i* 0.0

Table 6. ζ(z) compared to X(z) - Y(z)

The highest error for α∈ [0,1), ß ∈ [0,100], n=1000 is $8x10^{-6}$.

3. Representation of the function $\zeta(z) = X(z) - Y(z)$ for Re(z)=1/2

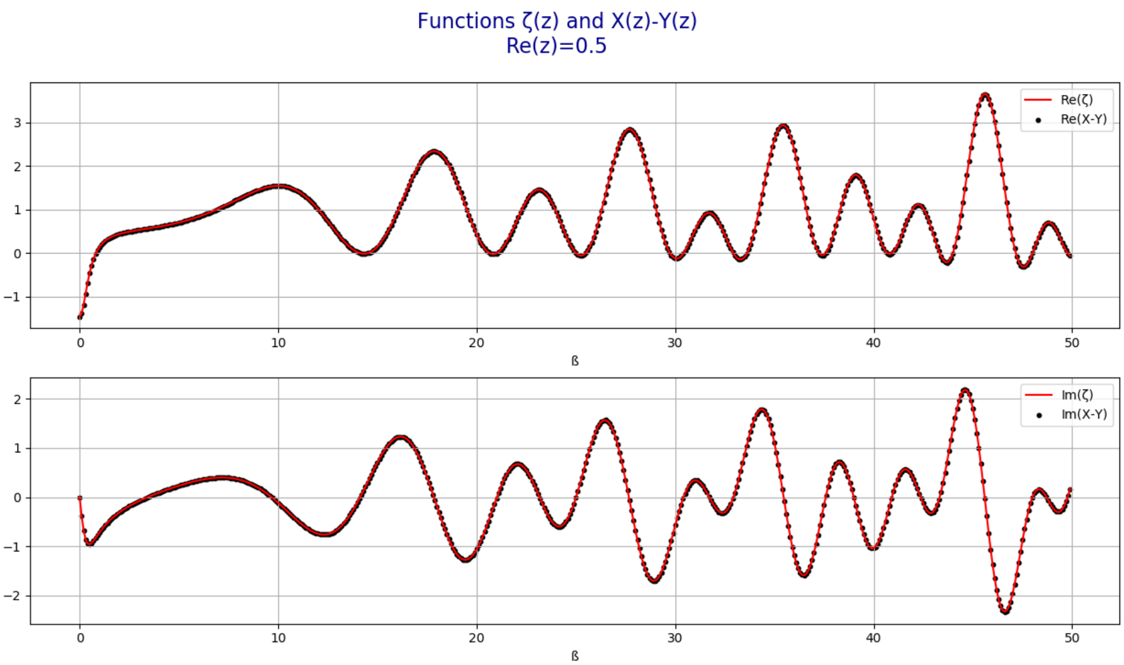


Figure. 5: ζ(z) = X(z) - Y(z)

4. Representation of the function $|\zeta(z)| = |X(z) - Y(z)|$ for Re(z)=1/2

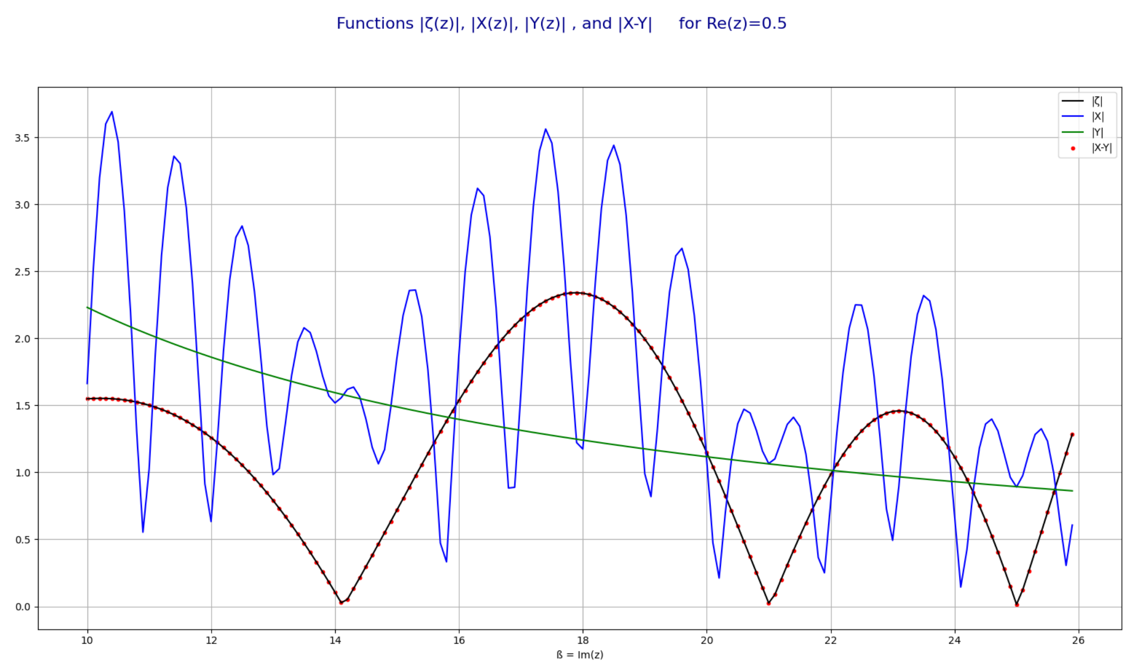


Figure. 6: |ζ(z)| = |X(z) - Y(z)|

5. Representation of the function $X(z, n)$

The following chart represents $X(z,n)$ for $a = 1/2$ and $b \in [1,6]$ and $n = 250$

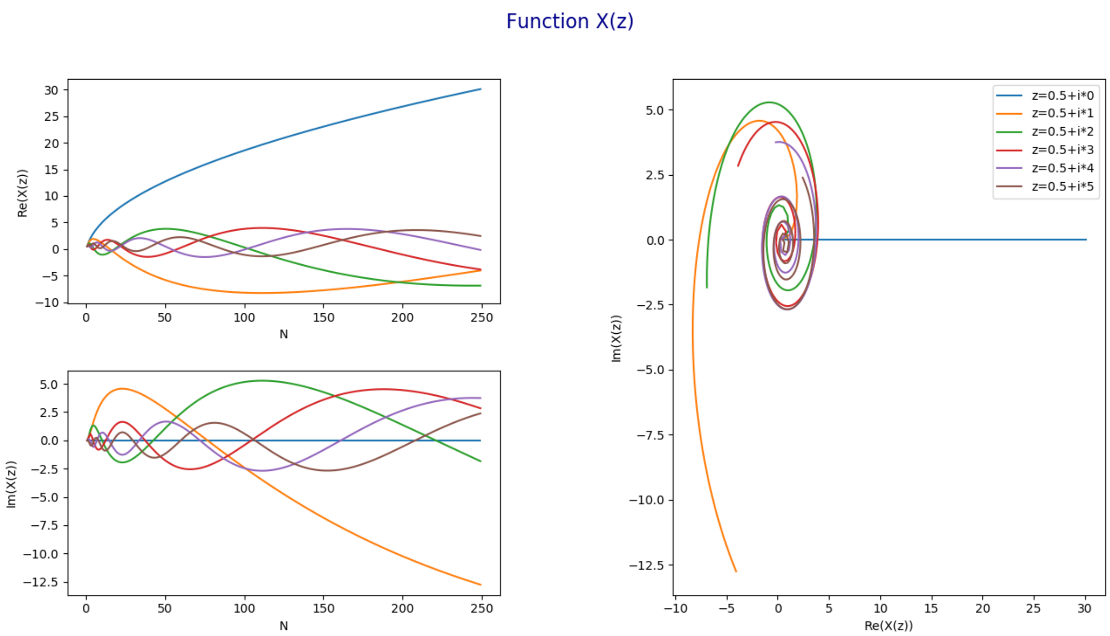


Figure. 7: $X(z,n)$

The following chart represents $X(z,n)$ for $a \in [1,6]$ and $b = 1$ and $n = 250$

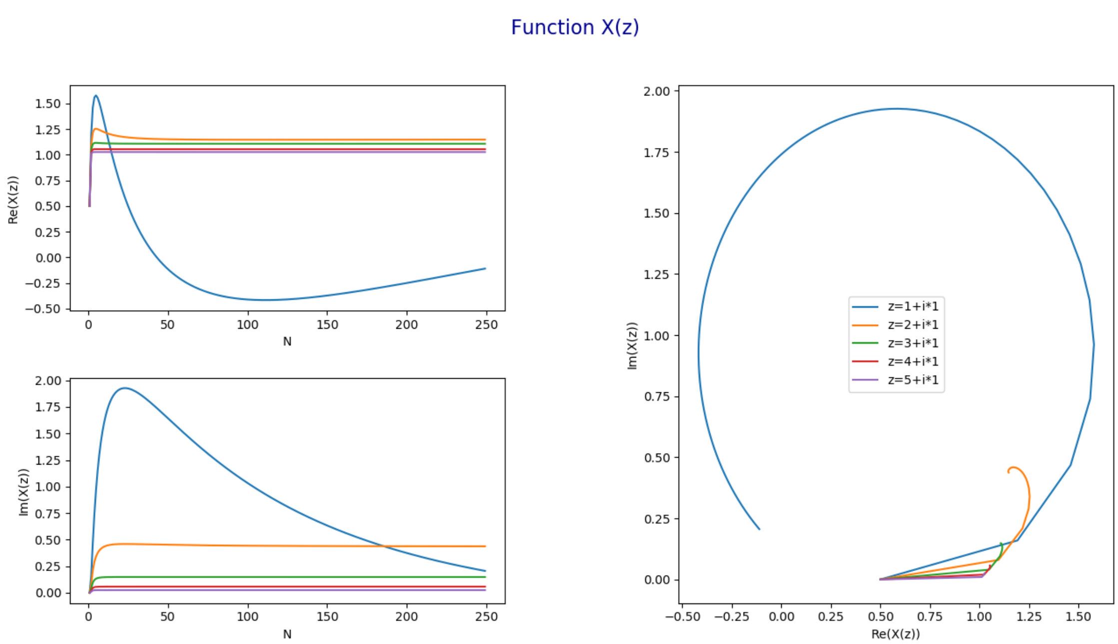


Figure. 8: $X(z,n)$

6. Representation of the function $\boldsymbol{Y(z, n)}$

The following chart represents $Y(z, n)$ for $a = 1/2$ and $b \in [1,6]$ and $n = 250$

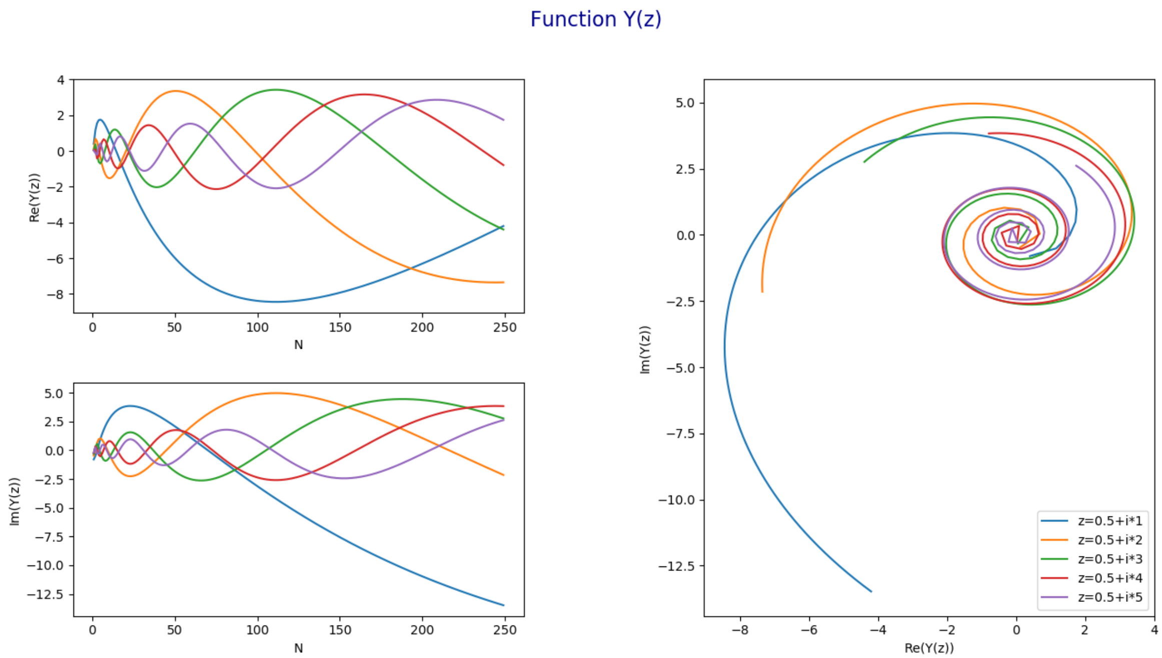


Figure. 9: $Y(z, n)$

The following chart represents $Y(z, n)$ for $a \in [1,6]$ and $b = 1$ and $n = 250$

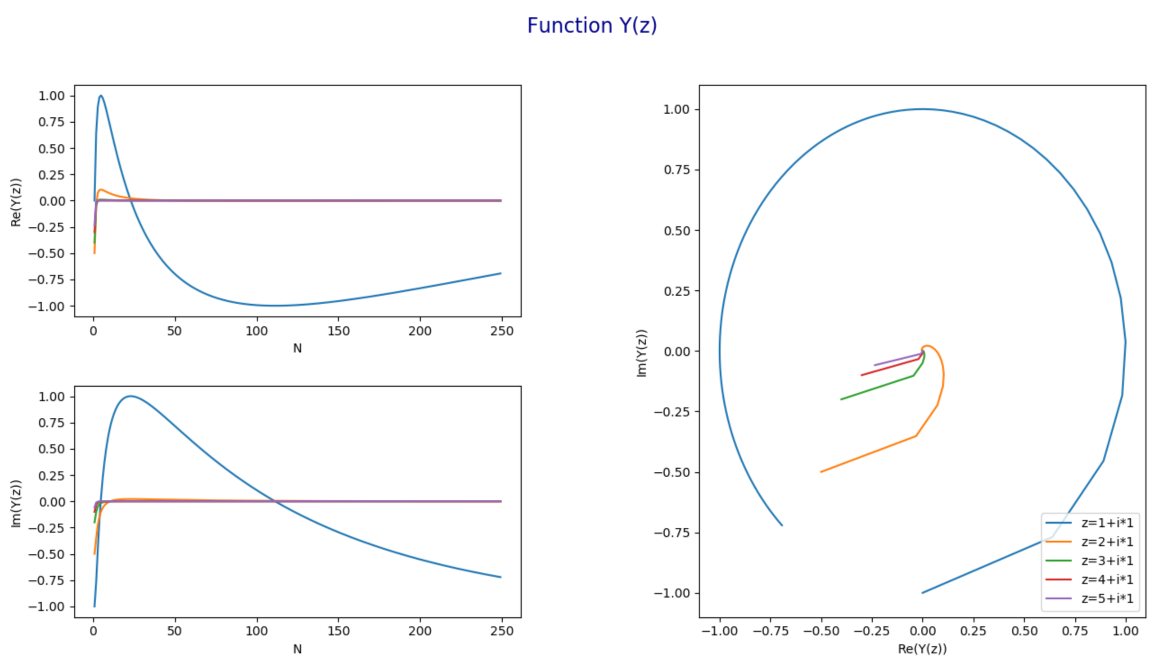
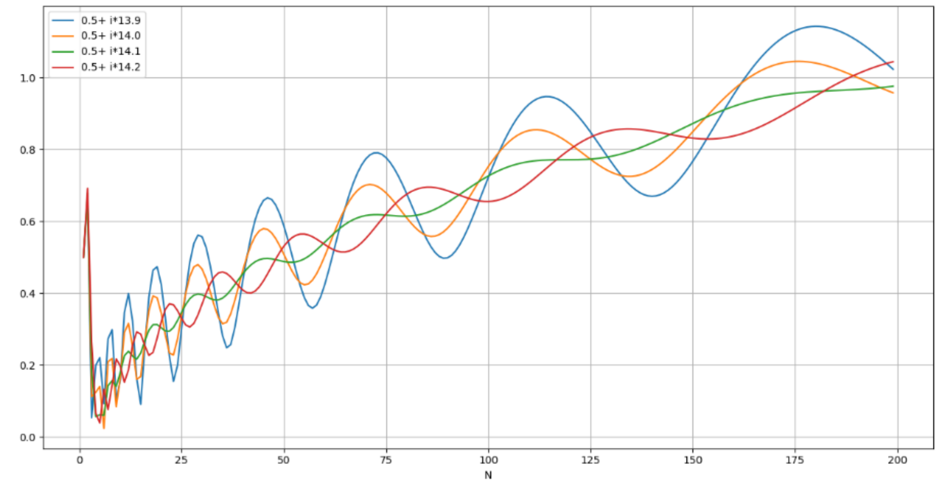


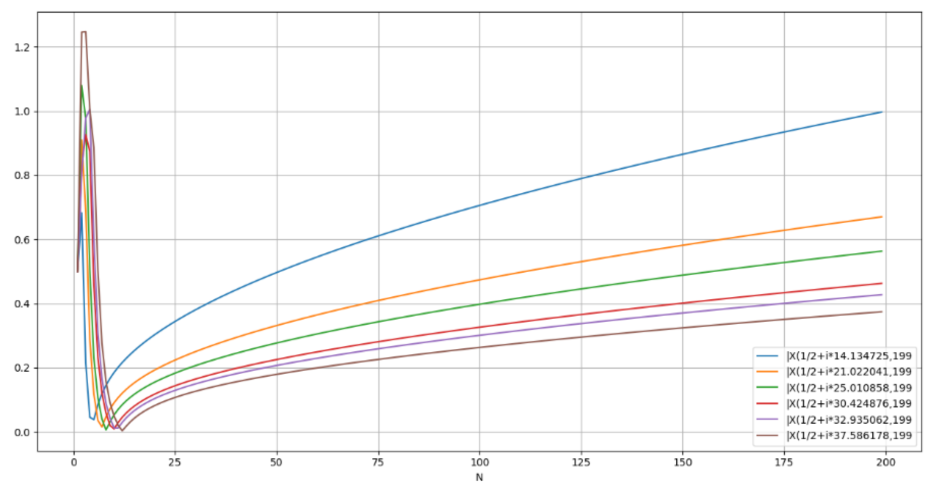
Figure. 10: $Y(z, n)$

7. Representation of $|X(z,n)|$

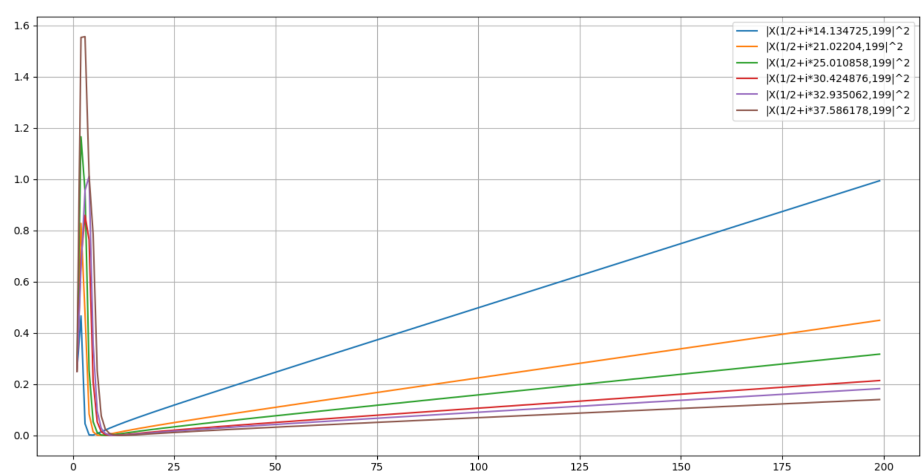
Wave representation for $|X(z,n)|$ $for\ Re(z) = 1/2\ and\ Im(z)\ variable$. [Figure. 11]



Parabolic representation for $|X(z,n)|$ $for\ (z)$ a nontrivial zero of Riemann Zeta. [Figure. 12]

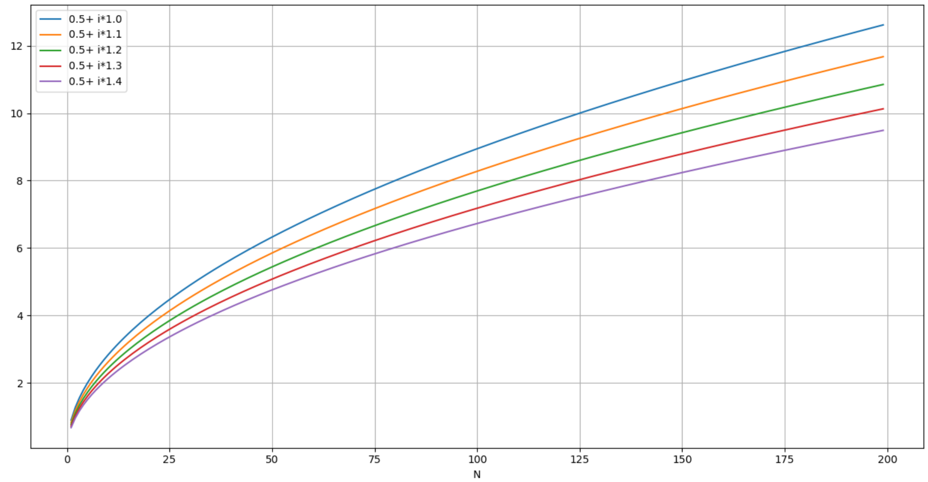


Linear representation for $|X(z,n)|^2$ $for\ (z)$ a nontrivial zero of Riemann Zeta. [Figure. 13]

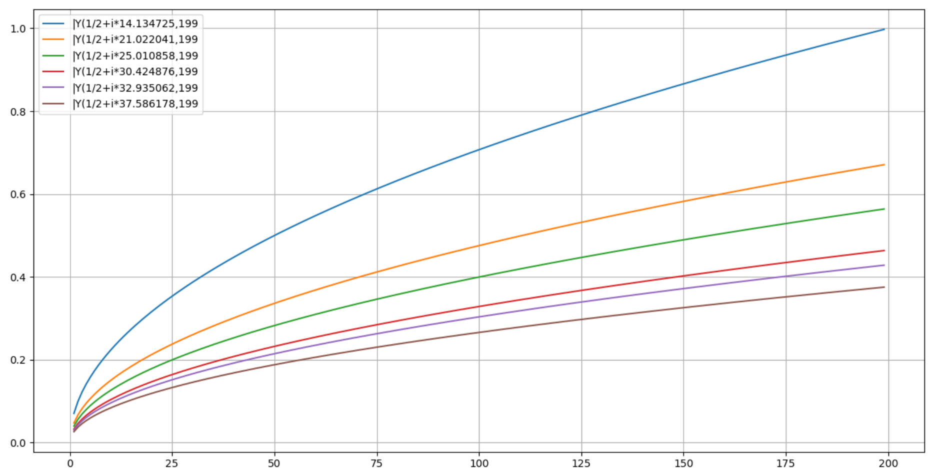


8. Representation of $|\boldsymbol{Y}(\boldsymbol{z}, \boldsymbol{n})|$

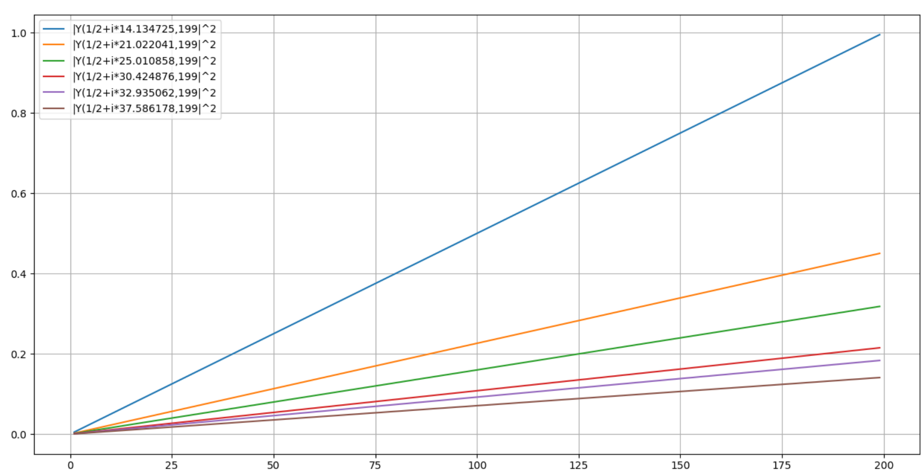
<u>Polynomial</u> representation for $|Y(z,n)|\ for\ Re(z) = 1/2\ and\ Im(z)\ variable$. [Figure. 14]



<u>Parabolic</u> representation for $|Y(z,n)|\ for\ (z)$a nontrivial zero of Riemann Zeta. [Figure. 15]



<u>Linear</u> representation for $|Y(z,n)|^2\ for\ (z)$ a nontrivial zero of Riemann Zeta. [Figure. 16]



9. Analysis of Absolute Square $|Y(z,n)|^2$

$$|Y(z,n)|^2 = [\left(n^{(1-\alpha)} \frac{1}{[(1-\alpha)^2 + ß^2]} \left[(1-\alpha) * cos(ßln(n)) + ß * sin(ßln(n))\right]\right)^2 + \left(n^{(1-\alpha)} \frac{1}{[(1-\alpha)^2 + ß^2]} \left[ß * cos(ßln(n)) - (1-\alpha) * sin(ßln(n))\right]\right)^2]$$

$$|Y(z,n)|^2 = n^{2(1-\alpha)} * \frac{1}{[ß^2+(1-\alpha)^2]} \quad \text{Polynomial representation} \qquad [13]$$

This could be observed in Figure. 14, 15, 16.

9.1. $|Y(z,n)|^2$ is a straight line if and only if $\alpha = \frac{1}{2}$

The slope of $|Y(z,n)|^2$ with respect to n is given by:

$$slope(|Y(z,n)|^2) = d(|Y(z,n)|^2)/dn$$

Which equals to:

$$d(|Y(z,n)|^2)/dn = 2(1-\alpha)\, n^{1-2\alpha} * \frac{1}{[ß^2 + (1-\alpha)^2]}$$

$|Y(z,n)|^2$ can only be a line when the slope is constant, which can only happen if and only if:

$$(1 - 2\alpha) = 0$$

therefore:

$|Y(z,n)|^2$ is a straight line if and only if $\alpha = \frac{1}{2}$ [14]

9.2. Summary for $|Y(z,n)|^2$ for $\alpha = \frac{1}{2}$:

- ⇨ the slope $|Y(z,n)|^2$ is constant if and only if $\alpha = \frac{1}{2}$
- ⇨ When α=1/2, $|Y(z,n)|^2 = \frac{n}{[ß^2+\frac{1}{4}]}$
- ⇨ The slope for $|Y(z,n)|^2$ is $\frac{1}{[ß^2+\frac{1}{4}]}$ for $\alpha = \frac{1}{2}$

10. Analysis of Absolute Square $|X(z,n)|^2$:

$$|X(z,n)|^2 = (\tfrac{1}{2}n^{-\alpha}\cos(ß\ln(n)) + \textstyle\sum k^{-\alpha} cos(ß\, ln(k)))^2 + (\tfrac{1}{2}n^{-\alpha}\sin(ß\ln(n)) + \textstyle\sum k^{-\alpha} sin(ß\, ln(k)))^2$$

[15]

Applying properties of infinite series (Kopp):

$$|X(z,n)|^2 = \frac{1}{4}n^{-2\alpha}(\cos^2(ß\ln(n)) + \sin^2(ß\ln(n))) + (\sum k^{-\alpha} cos(ß\, ln(k)))^2 + (\sum k^{-\alpha} sin(ß\, ln(k)))^2 + + n^{-\alpha}[\cos(ß\ln(n)) * \textstyle\sum k^{-\alpha} cos(ß\, ln(k))] + + n^{-\alpha}[\sin(ß\ln(n)) * \sum k^{-\alpha} sin(ß\, ln(k))]$$

$$|X(z,n)|^2 = \sum_{k=1}^{n}\sum_{j=1}^{n} k^{-\alpha} * j^{-\alpha} * cos\left(ß\left(ln\left(\frac{k}{j}\right)\right)\right) + \sum_{k=1}^{n} k^{-2\alpha} +$$

$$+\frac{1}{4}n^{-2a} + n^{-a}[\cos(ß\ln(n)) * \sum k^{-\alpha} cos(ß\, ln(k))] +$$
$$+ n^{-\alpha}[\sin(ß\ln(n)) * \sum k^{-\alpha} sin(ß\, ln(k))]$$

One can express the previous expression replacing:

$$R(n) = \frac{1}{4}n^{-2a} + n^{-a}\,[\cos(ß\ln(n)) * \sum k^{-\alpha} cos(ß\, ln(k)) + \sin(ß\ln(n)) * \sum k^{-\alpha} sin(ß\, ln(k))]$$

With:

$$\lim_{n\to\infty} R(n) = 0 \text{ if } \alpha>0\text{, therefore,} \qquad [16]$$

$$|X(z,n)|^2 = \sum_{k=1}^{n}\sum_{j=1}^{n} k^{-\alpha} * j^{-\alpha} * cos\left(ß * ln\left(\frac{k}{j}\right)\right) + \sum_{k=1}^{n} k^{-2\alpha} + R(n) \qquad [17]$$

10.1. When $|X(z,n)|^2$ is represented graphically, one can observe that:

- $|X(z,n)|^2$ is a wave that converges when n → ∞ and α>1 (Figure. 17)
- $|X(z,n)|^2$ is a wave that does not converge when n → ∞ and α<1 (Figure. 18)
- $|X(z,n)|^2$ is a wave that collapses to a line when n → ∞ and α=1/2 and ß=Im($\zeta(z^*)$) (Figure. 19) where z* is a nontrivial zero of RZF.

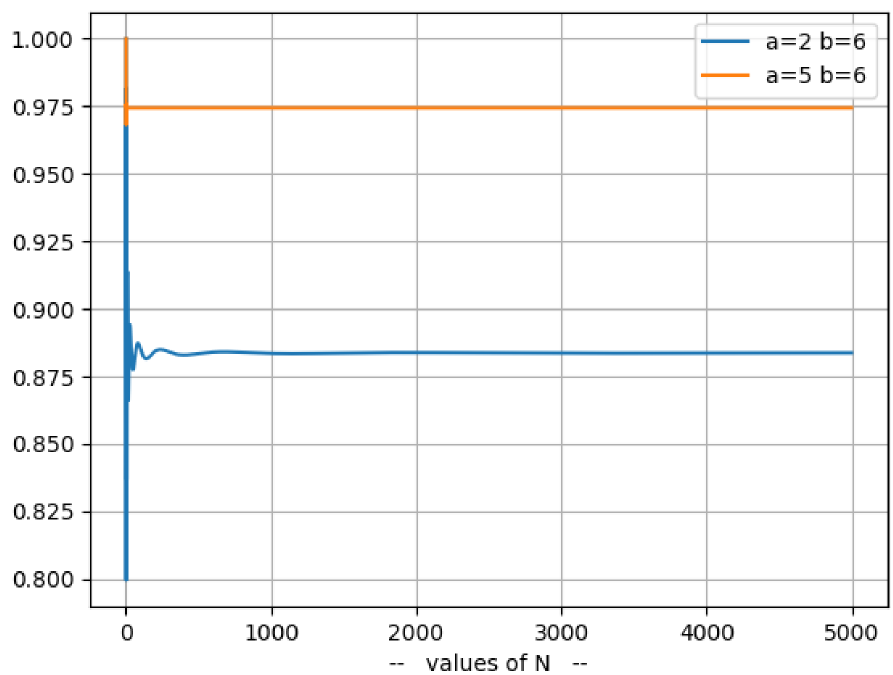


Figure. 17. $|X(z,n)|^2$ *for α>1*

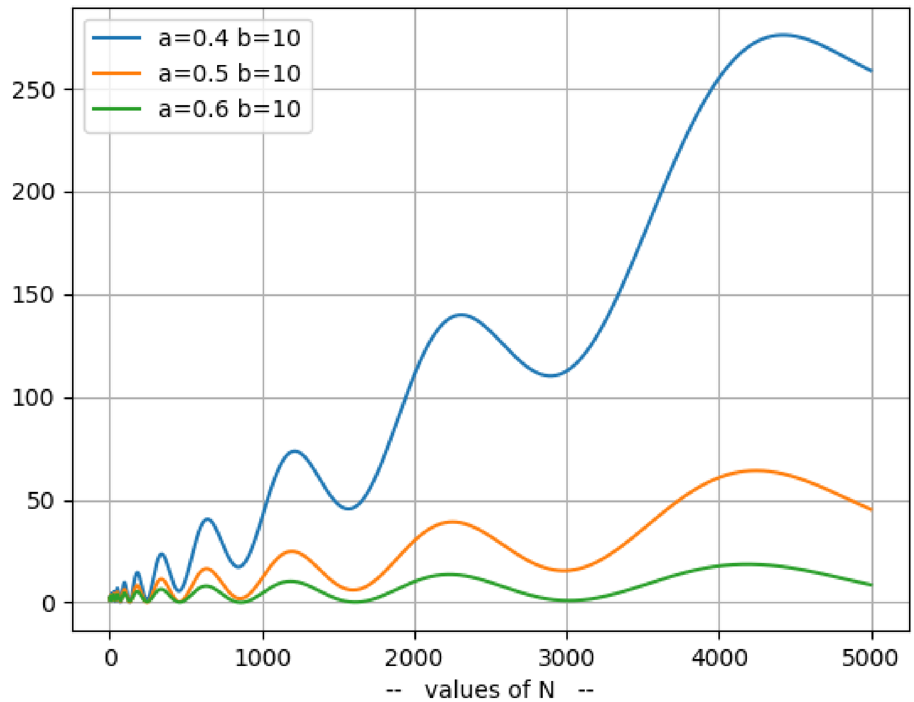


Figure. 18. $|X(z,n)|^2$ *for α<1*

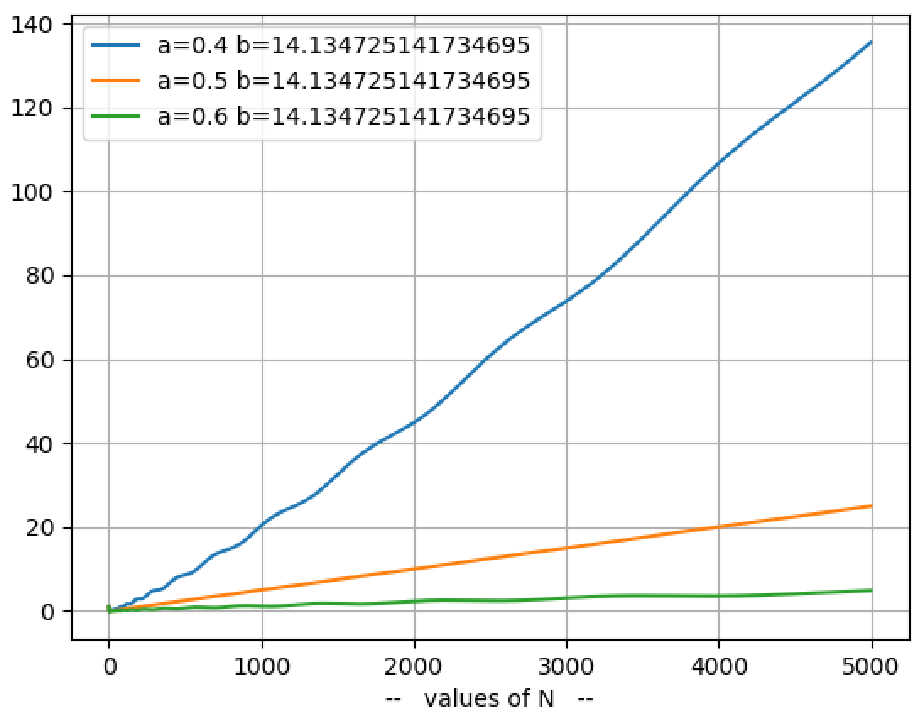


Figure. 19. For a=0.5, b=ß₁, $|X(z,n)|^2$ *collapses to a line*

10.2. $|X(z,n)|^2$ converges when $n \to \infty$ and α>1 to $|\zeta(\alpha,ß)|^2$

The limit of $|X(z,n)|^2$ outside the critical strip [0,1] can be calculated using [16]:

$$\lim_{n\to\infty} |X(z,n)|^2 = \lim_{n\to\infty} \sum_{k=1}^{n} \sum_{j=1}^{n} k^{-\alpha} * j^{-\alpha} * \cos\left(ß \left(\ln\left(\frac{k}{j}\right)\right)\right)$$

As one can see in some examples in the following table where z=α+iß:

α	ß	$\lim_{n\to\infty} \|X(z,n)\|^2$	$\|\zeta(\alpha,ß)\|^2$
1.0	7	1.074711506185445	1.074756
1.0	10	1.4413521753699579	1.441430
2.5	7	1.0093487944300192	1.009349
2.5	10	1.0507402208589398	1.050740

Table 7

$$\lim_{n\to\infty} |X(z,n)|^2 = |\zeta(z)|^2 = \zeta(\alpha + ßi) * \zeta(\alpha - ßi) \quad for\ \alpha > 1$$

And also, in the following Figure. 20:

|X(z,n)|² with α and ß variable

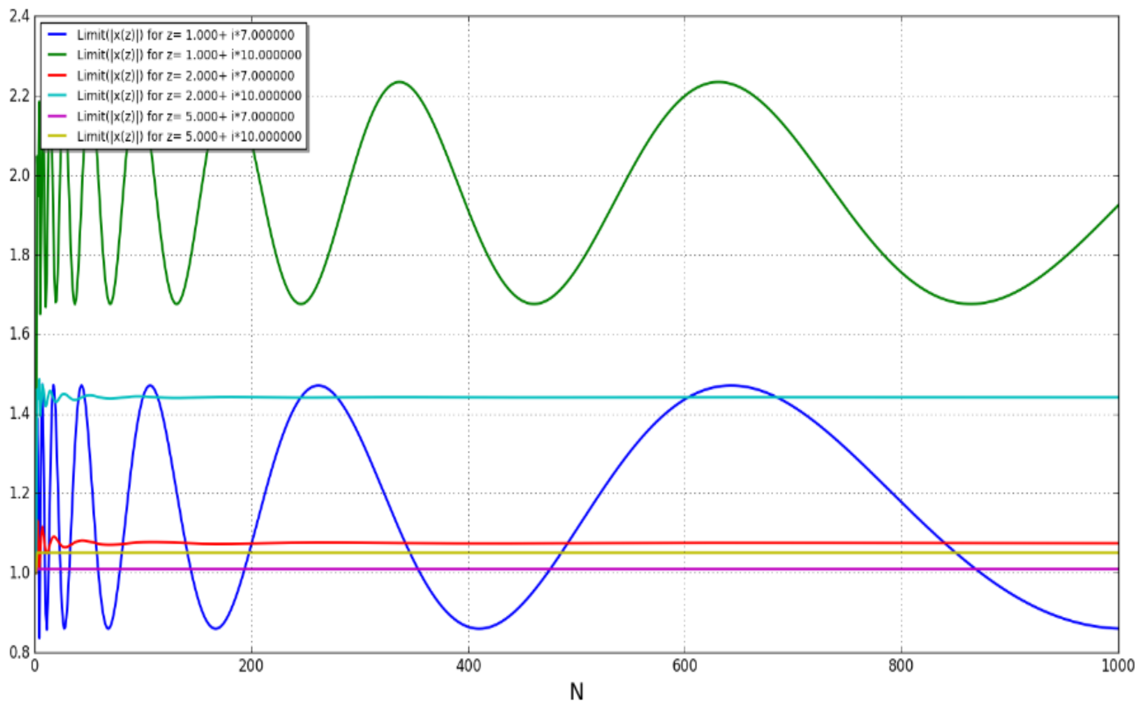


Figure. 20. $|X(z,n)|^2$ *converges when* $n \to \infty$ *and α>1*

One can observe that the graphs for α=1 do not converge while graphs for α>1 they all converge. This observation can be used to prove that there are no zero values of ζ(z) for z with Re(z)>1.

10.3. $|X(z,n)|^2$ *diverges when* $n \to \infty$ *for* $\alpha \leq 1$

$|X(z,n)|^2$diverges when $n \to \infty$ for α<1 because of [16] and [17]:

$$|\cos\left(ß\left(\ln\left(\frac{k}{j}\right)\right)\right| < 1$$

And:

$$\sum_{k=1}^{n}\sum_{j=1}^{n} k^{-\alpha} * j^{-\alpha} \ diverges\ for\ \alpha < 1$$

Therefore:

$$\lim_{n\to\infty} |X(z,n)|^2 = \sum_{k=1}^{n}\sum_{j=1}^{n} k^{-\alpha} * j^{-\alpha} * cos\left(ß\left(ln\left(\frac{k}{j}\right)\right)\right) \ diverges\ for\ \alpha < 1$$

10.4. $|X(z,n)|^2$ *does not collapse to any polynomial function* $|X(z,n)|^2 = C * n^t\ for\ t > 1, and\ C\ constant$

One can prove it with a reduction to absurd.

Let's assume that $|X(z,n)|^2 = C * n^t\ for\ t > 1$ where C and t integers C>0 and t>0

$$\text{If } |X(z,n)|^2 = C * n^t \text{ then:}$$

$$\lim_{n\to\infty} |X(z,n)|^2/n^t = C$$

But:

$$\lim_{n\to\infty} |X(z,n)|^2 /n^t = \frac{1}{n^t} * \lim_{n\to\infty} \sum_{k=1}^{n} k^{-2\alpha} + \frac{1}{n^t} * \sum_{k=1}^{n}\sum_{j\neq k}^{n} k^{-\alpha} * j^{-\alpha} * \cos\left(ß\left(\ln\left(\frac{k}{j}\right)\right)\right)$$

And:

$$\frac{1}{n^t} * \lim_{n\to\infty} \sum_{k=1}^{n} k^{-2\alpha} = 0 \quad for\ t > 1$$

$$\frac{1}{n^t} * \sum_{k=1}^{n}\sum_{j\neq k}^{n} k^{-\alpha} * j^{-\alpha} * \cos\left(ß\left(\ln\left(\frac{k}{j}\right)\right)\right) = 0\ for\ t > 1$$

So, C must be 0 which is an absurd [QED]

10.5. $|X(z,n)|^2$ *collapses to a straight-line* $|X(z,n)|^2 = Cn \quad if\ Re(z) = 1/2$

The proposition says that the following limit exists only for Re(z) = 1/2

$$\lim_{n\to\infty} (|X(z,n)|^2/\,n) = S$$

Using the expression:

$$\lim_{n\to\infty}(|X(z,n)|^2 / n) = \lim_{n\to\infty}\frac{1}{n}(\sum_{k=1}^{n} k^{-2\alpha} + \sum_{k=1}^{n}\sum_{j\neq k}^{n} k^{-\alpha} * j^{-\alpha} * cos\left(ß\,(ln\left(\frac{k}{j}\right)\right))$$

10.5.1 For α>1/2, one can see that $\lim_{n\to\infty}(|x(z,n)|^2 / n) = 0$:

$\lim_{n\to\infty}\frac{1}{n}(\sum_{k=1}^{n} k^{-2\alpha}) = 0$ because 2α>1 and the series is convergent

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\sum_{j\neq k}^{n} k^{-\alpha} * j^{-\alpha} * cos\left(ß\,(ln\left(\frac{k}{j}\right)\right)) < \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\sum_{j\neq k}^{n}(k^{-\alpha} * j^{-\alpha}) < \lim_{n\to\infty}\frac{1}{n}(\sum_{k=1}^{n} k^{-2\alpha})$$

So:

$$\lim_{n\to\infty}\left(\frac{1}{n}\sum_{k=1}^{n}\sum_{j\neq k}^{n} k^{-\alpha} * j^{-\alpha} * \cos\left(ß\,(\ln\left(\frac{k}{j}\right)\right)\right) = 0$$

10.5.2. For α<1/2, one can see that $\lim_{n\to\infty}(|X(z,n)|^2/n) = \infty$ as:

$$\lim_{n\to\infty}\frac{1}{n}(\sum_{k=1}^{n} k^{-2\alpha}) < \lim_{n\to\infty}\frac{1}{n}\left(n * \frac{1}{n}\right) = \lim_{n\to\infty}\frac{1}{n} = 0$$

And:

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\sum_{j\neq k}^{n} k^{-\alpha} * j^{-\alpha} * \cos\left(ß\,(\ln\left(\frac{k}{j}\right)\right)) > \lim_{n\to\infty}(\frac{1}{n} * n^2 * \frac{1}{n^{2\alpha}}) = \infty$$

Where the summations are replaced by the number of elements in the matrix (n x n) times the smallest value in each row (1/n) then 1>(2-1-2α)>0 when α<1/2

10.5.3. Limit for α=1/2.

When α=1/2, one can express ($|X(z,n)|^2$/n) as:

$$\lim_{n\to\infty}(|X(z,n)|^2 /n) =$$

$$= \lim_{n\to\infty}\frac{1}{n}(\sum_{k=1}^{n} k^{-1} + \sum_{k=1}^{n}\sum_{j\neq k}^{n} k^{-1/2} * j^{-1/2} * \cos\left(ß\,(\ln\left(\frac{k}{j}\right)\right))$$

$$= \lim_{n\to\infty}\frac{1}{n}\left(\sum_{k=1}^{n} k^{-1}\right) + \lim_{n\to\infty}\frac{1}{n}(\sum_{k=1}^{n}\sum_{j\neq k}^{n} k^{-1/2} * j^{-1/2} * cos\left(ß\,(ln\left(\frac{k}{j}\right)\right)) =$$

$$= 0 + \lim_{n\to\infty} \frac{1}{n} (\sum_{k=1}^{n} \sum_{j\neq k}^{n} k^{-1/2} * j^{-1/2} * cos\left(ß\,(ln\left(\frac{k}{j}\right)\right)) =$$

$$= \lim_{n\to\infty} \frac{2n}{n} (\sum_{j=1}^{n-1} n^{-1/2} * j^{-1/2} * cos\left(ß\,(ln\left(\frac{n}{j}\right)\right)) =$$

$$= \lim_{n\to\infty} 2(\, n^{-\frac{1}{2}} \sum_{j=1}^{n-1} * j^{-\frac{1}{2}} * cos\left(ß\,(ln\left(\frac{n}{j}\right)\right)) =$$

Using the integral approximation of the infinite series

$$= 2 * \lim_{n\to\infty} \frac{2 * \sqrt{n} * \cos\left(ß * ln\left(\frac{n}{n}\right)\right) - 2 * ß * \sin\left(ß * \ln\left(\frac{n}{n}\right)\right)}{4 * ß^2 + 1} * n^{-\frac{1}{2}}$$

$$= 2 * \frac{2 * \sqrt{n}}{4 * ß^2 + 1} n^{-\frac{1}{2}} = 2 * \frac{2}{4 * ß^2 + 1} = \frac{1}{ß^2 + 1/4}$$

So, if $\lim_{n\to\infty} (|X(z,n)|^2 /n)$ exists will be equal to:

$$\lim_{n\to\infty} (|X(z,n)|^2/n) = \frac{1}{ß^2 + 1/4} \qquad [18]$$

if z=1/2+iß

And this limit can only exist when $|\mathrm{X(z,n)}|^2$ is monotonous which means that the curve will cross the x-axis only once.

$$|X(z,n)|^2 = \left(\sum_{k=1}^{n} \sum_{j=k}^{n} k^{-\frac{1}{2}} * j^{-\frac{1}{2}} * cos\left(ß\,(ln\left(\frac{k}{j}\right)\right)\right)$$

$$= 2 * n^{-a} * \left(\sum_{j=1}^{n-1} j^{-a} * \cos\left(ß * (\ln\left(\frac{x}{j}\right))\right)\right)$$

10.5.4. Calculating the zeros of $|\boldsymbol{X}(\boldsymbol{z},\boldsymbol{n})|^2$

Let's introduce the function $C_2(n,a,b) = |X(z,n)|^2$ in R (where z=a+bi) such that:

$$C_2(n,a,b) = 2 * n^{-a} * \left(\sum_{j=1}^{n-1} j^{-a} * \cos\left(b * (\ln\left(\frac{n}{j}\right))\right)\right) \qquad [19]$$

With the following wave representation for $C_2(n,a,b)$:

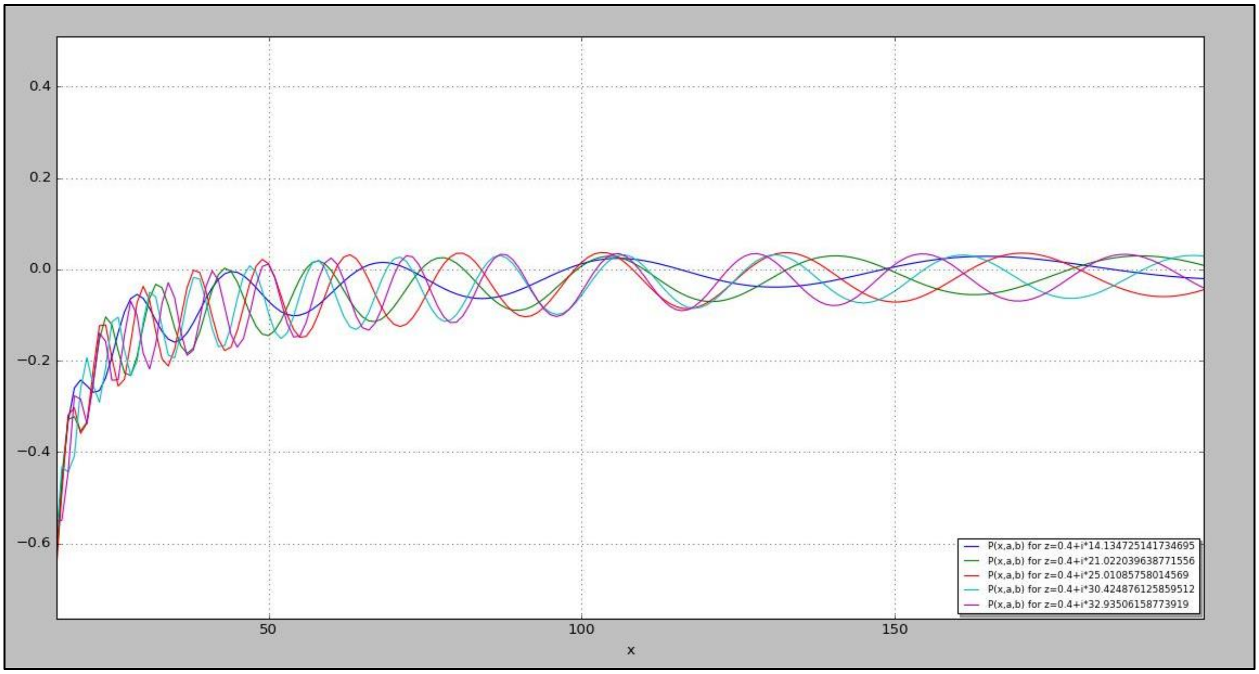


Figure. 21. $C_2(x, a, b)$ *for a=0.4 and variable b*

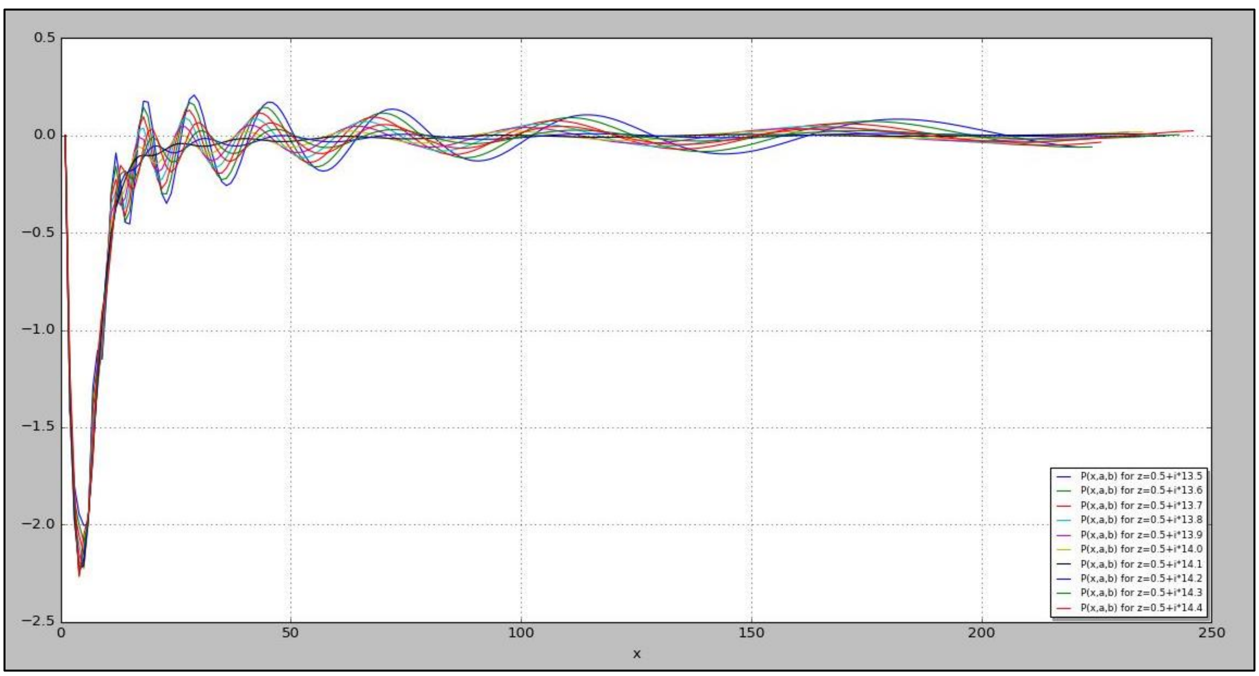


Figure. 22. $C_2(n, a, b)$ *for a=0.5 and variable b*

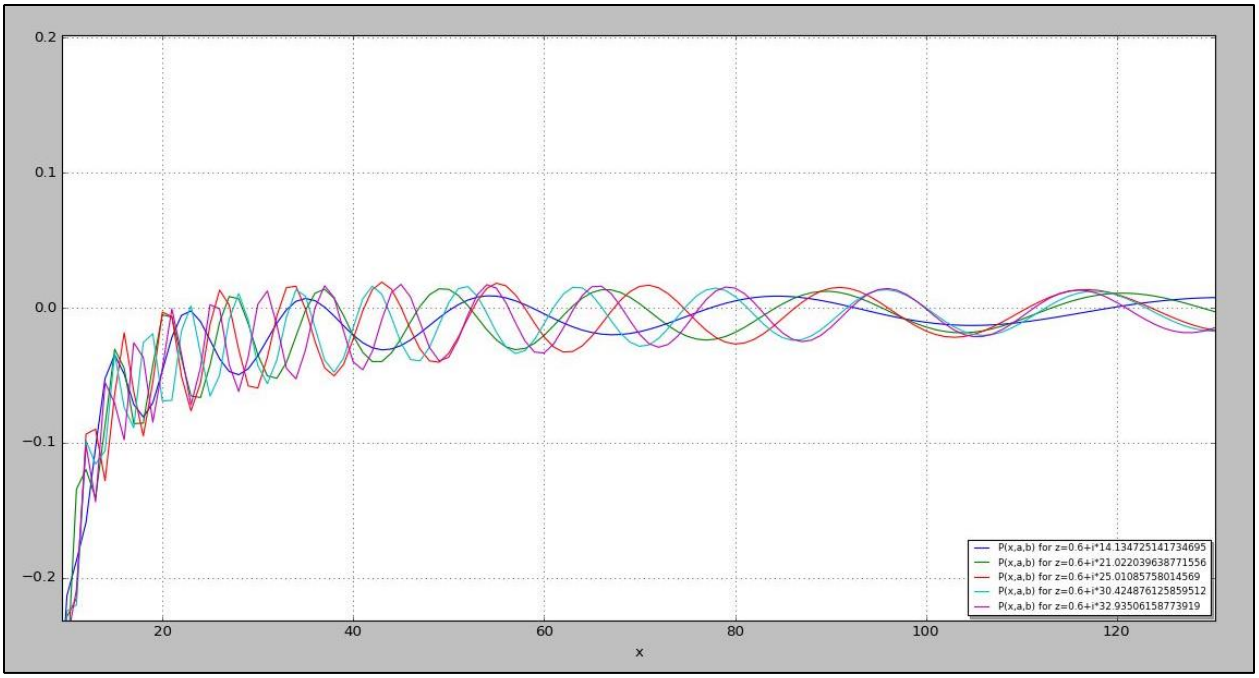


Figure. 23. $C_2(n, a, b)$ for a=0.6 and variable b

As a wave, $C_2(n, a, b)$ can have one or more zeros. For $C_2(n, a, b)$ to have only one zero, it must cross the axis y=0 only once, which means that the wave collapses to a polynomial line. A numeric method has been created and coded (Python) to find the values of (n, a, b) such that $C_2(n, a, b)=0$. The following table shows an example of those calculated values, where x=n, a=Alfa, and b=Beta:

Alfa	Beta	Number of Zeros	Zero at X=
0.4	14.1	5	
0.4	14.2	5	
0.4	14.3	5	
0.4	14.4	5	
0.5	14.07	5	
0.5	14.08	5	
0.5	14.09	5	
0.5	14.1	4	
0.5	14.11	4	
0.5	14.12	3	
0.5	14.13	1	200
0.5	14.14	3	
0.5	20.97	11	
0.5	20.98	11	
0.5	20.99	11	
0.5	21	9	
0.5	21.01	5	
0.5	21.02	1	442
0.5	21.03	3	

Table 8. Number of Zeros of $C_2(x, a, b)$ for different values of a=Alfa, and b=Beta

The calculations for $a \in (0,1)$ and $b \in [1, 100]$ only found single zeros for $C_2(x, a, b)$ for values of $a = 0.5$ as shown in the following table that summarizes the single zeros found in those intervals:

Values (x,a,b) | C2(x,a,b)=0 SINGLE		
x*	a*	b*
200.1000	0.5000	14.1368
442.2000	0.5000	21.0226
625.8000	0.5000	25.0110
926.0000	0.5000	30.4261
1085.0000	0.5000	32.9355
1413.0000	0.5000	37.5866
1674.6000	0.5000	40.9188
1877.5000	0.5000	43.3272
2304.8000	0.5000	48.0057

Table 9. List of first Zeros of $C_2(x, a, b)$

One can observe that:

$$if \;\; C_2(x, a, b) = 0 \rightarrow$$

$$a = 1/2$$

$$b = \; Im(z) \quad | \quad \zeta(z) = 0$$

(a, b) are the Nontrivial Zeros of ζ(z) in the critical line.

$$x = b^2 + \frac{1}{4}$$

And the calculated values of $\lim_{x \to \infty} C_2(x, a, b)$ for the values of (a,b) from Table 9 are:

Values (x,a,b) | C2)x,a,b=0			Limit (C2(x,a,b))
x	a	b	when x->∞
200.1000	0.5000	14.1368	0.0050
442.2000	0.5000	21.0226	0.0023
625.8000	0.5000	25.0110	0.0016
926.0000	0.5000	30.4261	0.0011
1085.0000	0.5000	32.9355	0.0009
1413.0000	0.5000	37.5866	0.0007
1674.6000	0.5000	40.9188	0.0006
1877.5000	0.5000	43.3272	0.0005

Table 10. Limit of $C_2(x, a, b)$ *for b in Table 10 and x->∞*

Values (x,a,b) | C2(x,a,b)=0			Limit (C2(x,a,b))	
x	a	b	when x->∞	Known Zero
200.1000	0.5000	14.1368	0.0050	14.1347
442.2000	0.5000	21.0226	0.0023	21.0220
625.8000	0.5000	25.0110	0.0016	25.0109
926.0000	0.5000	30.4261	0.0011	30.4249
1085.0000	0.5000	32.9355	0.0009	32.9351
1413.0000	0.5000	37.5866	0.0007	37.5862
1674.6000	0.5000	40.9188	0.0006	40.9187
1877.5000	0.5000	43.3272	0.0005	43.3271
2304.8000	0.5000	48.0057	0.0004	48.0052
2477.7000	0.5000	49.7740	0.0004	49.7738

Table 11. Comparing "b" calculated with known zeros of ζ(z)

Therefore, $|X(z,n)|^2 = C(n,a,b)$ has the following special properties for all (a,b) such that ζ(a+bi)=0.

if $\quad S = \frac{1}{b^2+1/4}$

$C_2(n,a,b) = 0 \; when \; x = \frac{1}{S}, \; a = \frac{1}{2}, \; b = Im(z^*)$ with z^* a nontrivial zero of $\zeta(z)$

$\lim_{x\to\infty} C_2\left(n, \frac{1}{2}, b\right) = S$

Graphically:

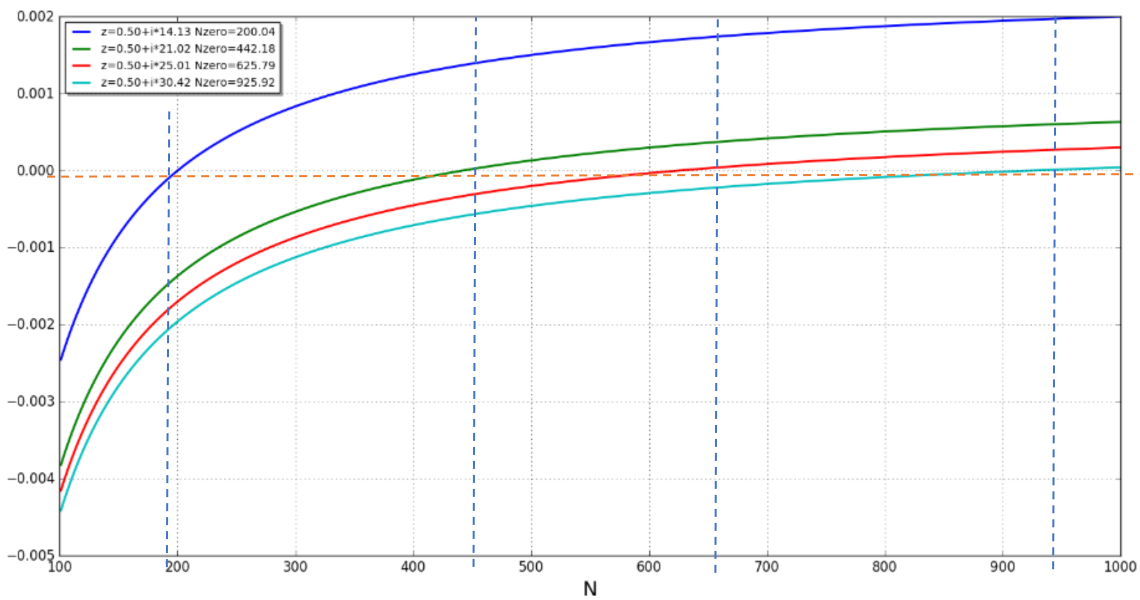


Figure. 24. $C_2(n, 1/2, b)$ *such that ζ(1/2+b*i)=0*

11. Conclusion PART 3

Using the defined C-transformation, one can write the Riemann Zeta function as the difference of two functions X(z) and Y(z) for $Re(z) \geq 0, z \neq 1$. This will provide a new way of analyzing the zeros of the Zeta function, and a new approach to the Riemann Hypothesis.

The decomposition is as follows:

$$\zeta(z) = X(z) - Y(z), \text{ where:} \qquad [20]$$

$$X(z,n) = \sum_{k=1}^{n} k^{-\alpha} * \cos(ß \ln(k)) + \frac{1}{2} n^{-\alpha} \cos(ß \ln(n)) + + i * \left(\sum_{k=1}^{n} k^{-\alpha} * \sin(ß \ln(k)) + \frac{1}{2} n^{-\alpha} \sin(ß \ln(n))\right) \qquad [21]$$

and: $X(z) = \lim_{n\to\infty} X(z,n)$

$$Y(z,n) = n^{(1-\alpha)} \frac{1}{[(1-\alpha)^2 + ß^2]} \left[\left((1-\alpha) * \cos(ß \ln(n)) + ß * \sin(ß \ln(n))\right) + + i \ (ß \cos(ß \ln(n)) - (1-\alpha) * \sin(ß \ln(n)))\right] \qquad [22]$$

and: $Y(z) = \lim_{n\to\infty} Y(z,n)$

Using the fact that:

a. $|X(z,n)|$ has a wave representation
b. $|X(z,n)|$ becomes a parable when z is a nontrivial zero of Riemann Zeta
c. $|X(z,n)|^2$ becomes a line when z is a nontrivial zero of RZF with slope equal $1/(ß^2 + ¼)$

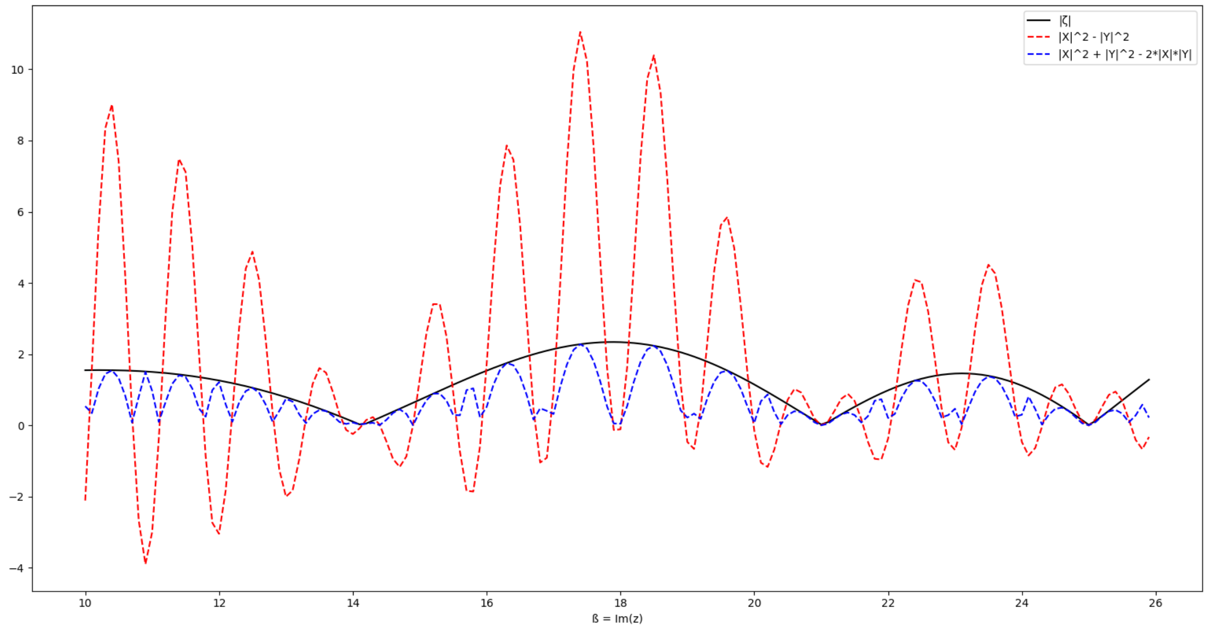
d. $|Y(z,n)|$ has a polynomial representation
e. $|Y(z,n)|$ becomes a parable when z is a nontrivial zero of Riemann Zeta
f. $|Y(z,n)|^2$ becomes a line when Re(z)=1/2 with slope equal $1/(ß^2 + ¼)$

So, the only common representation for |X(z)| and |Y(z)| occurs when Re(z)=1/2, so

The conclusion is that $X(z) - Y(z) = 0$ if and only if Re(z)=1/2

12. Additional graphical representations:

12.1. $\zeta(z), |x(z)|^2 - |y(z)|^2, |x(z)|^2 + |y(z)|^2 - 2|x||y|$



$Fig\ 25$: $\zeta(z), |x(z)|^2 - |y(z)|^2, |x(z)|^2 + |y(z)|^2 - 2|x||y|$

The zeros of $\zeta(z)$, $|x(z)|^2 - |y(z)|^2$, $|x(z)|^2 + |y(z)|^2 - 2|x||y|$ are the same.

PART 4:

Proof of the Riemann Hypothesis using the decomposition

$$\boldsymbol{\zeta(z) = X(z) - Y(z)}$$

Theorem: For Re(z)≥0, if z* is a nontrivial zero of ζ(z), then Re(z*)=1/2

Proof:

- From [10], [11], [12]: $\zeta(z) = X(z) - Y(z)$ for Re(z)>0, z≠1
- From [13]: $|Y(z, n)|^2$ is always a polynomial line.
- From [14]: $|Y(z, n)|^2$ is only straight line if and only if Re(z)= ½

  $|Y(z^*)|^2 = \lim_{n\to\infty} |Y(z^*, n)|^2$ tends to a straight line with slope $\frac{1}{[ß^{*2}+1/4]}$
- From [15]: $|X(z, n)|^2$ is a wave function that has only one polynomial representation in the form of a straight line if and only if Re(z)= ½ [18] and for certain values of Im(z)=ß* calculated using [19]. These values of ß* coincide with the imaginary parts of the nontrivial zeros of Riemann Zeta z*, so:

  $|X(z^*)|^2 = \lim_{n\to\infty} |X(z^*, n)|^2$ tends to a straight line with slope $\frac{1}{[ß^{*2}+1/4]}$

  when Re(z)=1/2 and ß=NTZ of RZF
- Therefore $|X(z^*)|^2 = |Y(z^*)|^2$ and $|X(z)| = |Y(z)|$ only occur when Re(z*) = ½
- As ζ(z)=X(z)-Y(z), therefore all zeros of ζ(z) for z>=0, z≠1 have Re(z)=1/2. [QED]

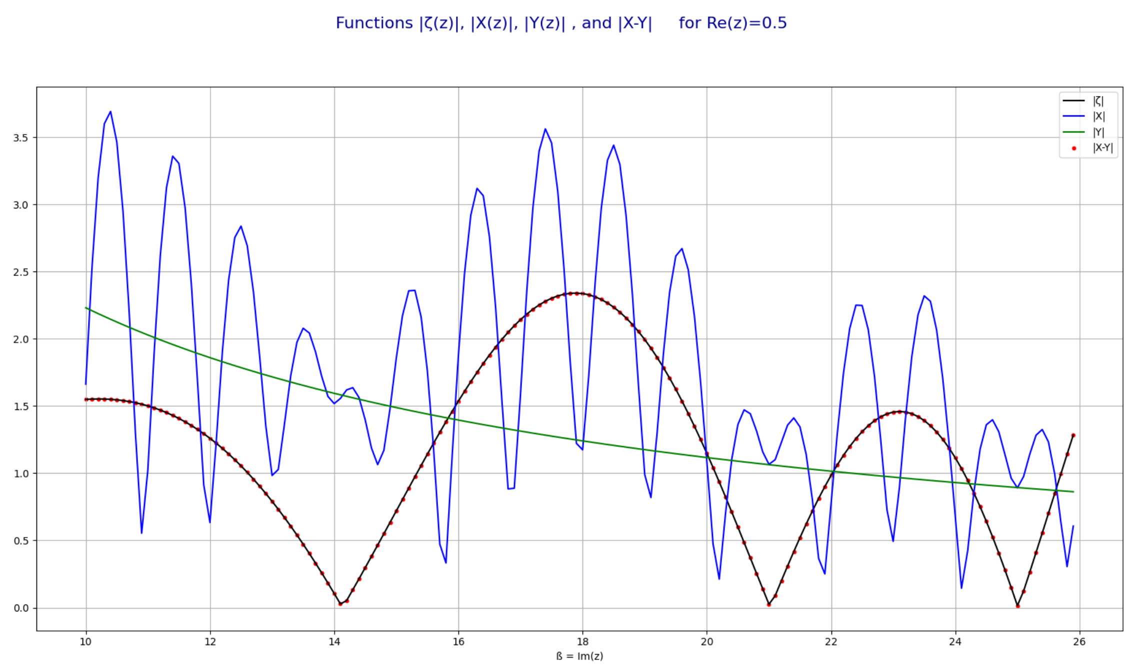


Figure. 26: for ζ(z)=X(z)-Y(z)=0 -> |X(z)| = |Y(z)| for Re(z) = 1/2,

PART 5:

On the distribution of the zeroes of the RZF in the critical line

1. **From [17] one can write:**

$$X(z,n) = (\sum_{k=1}^{n} k^{-2\alpha} + \sum_{k=1}^{n}\sum_{j\neq k}^{n} k^{-\alpha} * j^{-\alpha} * cos\left(ß\,(ln\left(\frac{k}{j}\right))\right))$$

therefore, the limit of $(|X(z,n)|^2/n$

$$\lim_{n\to\infty}(|X(z,n)|^2\,/\,n) = \lim_{n\to\infty}\frac{1}{n}\left(\sum_{k=1}^{n} k^{-2\alpha} + \sum_{k=1}^{n}\sum_{j\neq k}^{n} k^{-\alpha} * j^{-\alpha} * cos\left(ß\,(ln\left(\frac{k}{j}\right))\right)\right)$$

From [18], if $\lim_{n\to\infty}(|\mathrm{X(z,n)}|^2\,/n)$ exists will be equal to:

$$\lim_{n\to\infty}(|\mathrm{X(z,n)}|^2\,/n) = \frac{1}{ß^2+1/4} \qquad if\;\; z = \frac{1}{2} + iß$$

2. **Calculating the nontrivial zeros of ζ(z) using the Harmonic function**

From the previous equations, and for any $z^* = \frac{1}{2} + ßi$, a nontrivial zero of Zeta in the critical line α=1/2, one can write:

$$\sum_{k=1}^{n} k^{-1} \to \frac{n}{(ß^2+\frac{1}{4})} - \sum_{k=1}^{n}\sum_{j\neq k}^{n} k^{-1/2} * j^{-1/2} * cos\;(ß\,(ln\left(\frac{k}{j}\right)) \qquad \text{when } n \to \infty$$

Where $H_n = \sum_{k=1}^{n} k^{-1}$ is the Harmonic function. One can simplify the expression by creating functions $O(n)$ and $P(n)$:

$$P(n) = -\sum_{k=1}^{n}\sum_{j\neq k}^{n} k^{-1/2} * j^{-1/2} * cos\;(ß\,(ln\left(\frac{k}{j}\right)$$

And

$$O(n) = \frac{n}{(ß^2+\frac{1}{4})}$$

From the definition of limit, one can write that for any ε arbitrarily small, there exists and N such that for any $n>N$:

$$H_n - \big(O(n) + P(n)\big) < \varepsilon \qquad [23]$$

If $H(n) = O(n) + P(n)$, then [20] can be written as:

$$H_n - H(n) < \in$$

The following chart shows the representation of H(n), O(n), and P(n) [O(n) is a straight line with slope $\frac{1}{(ß^2+1/4\,)}$]:

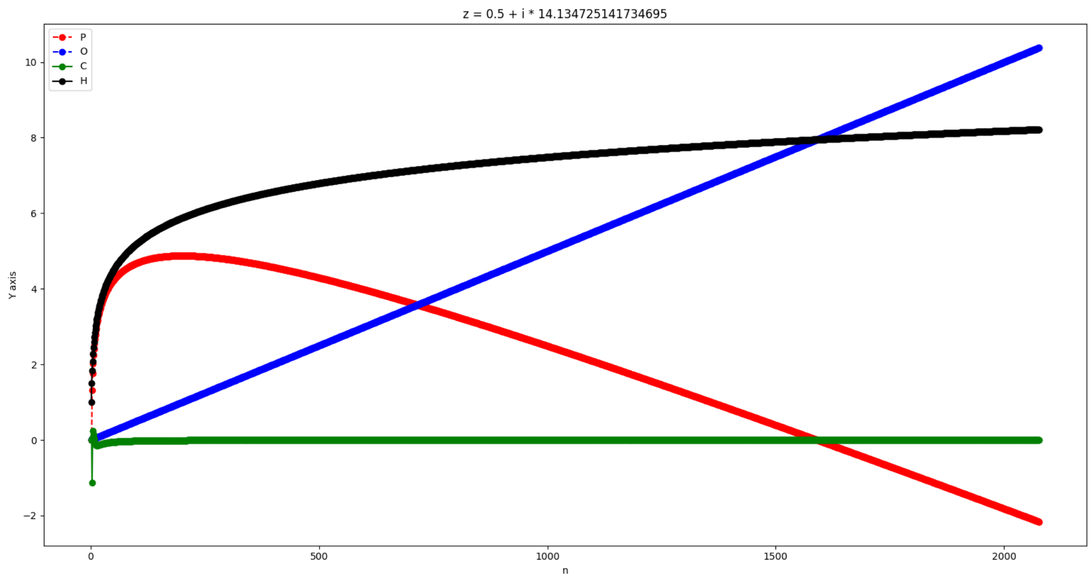


Figure. 27. Graph of H(n), O(n), P(n)

Let's analyze these charts by defining:

N1 value of n for which $O(N1) = P(N1)$

N2 value of n for which $P(N2) = 0$ equal to value of n for which $H(N3) = O(N3)$

The following table shows these values for the first non-trivial zeros of Zeta:

b	N1	N2	1/C2
14.134725	715	1590	201
21.022040	1783	3904	443
25.010858	2647	5782	626
30.424876	4117	8961	926
32.935062	4910	10692	1085
37.586178	6572	14339	1413
40.918719	7916	17309	1675

Table 12: Values of N1 and N2 vs. b

The following correlations exist between these variables:

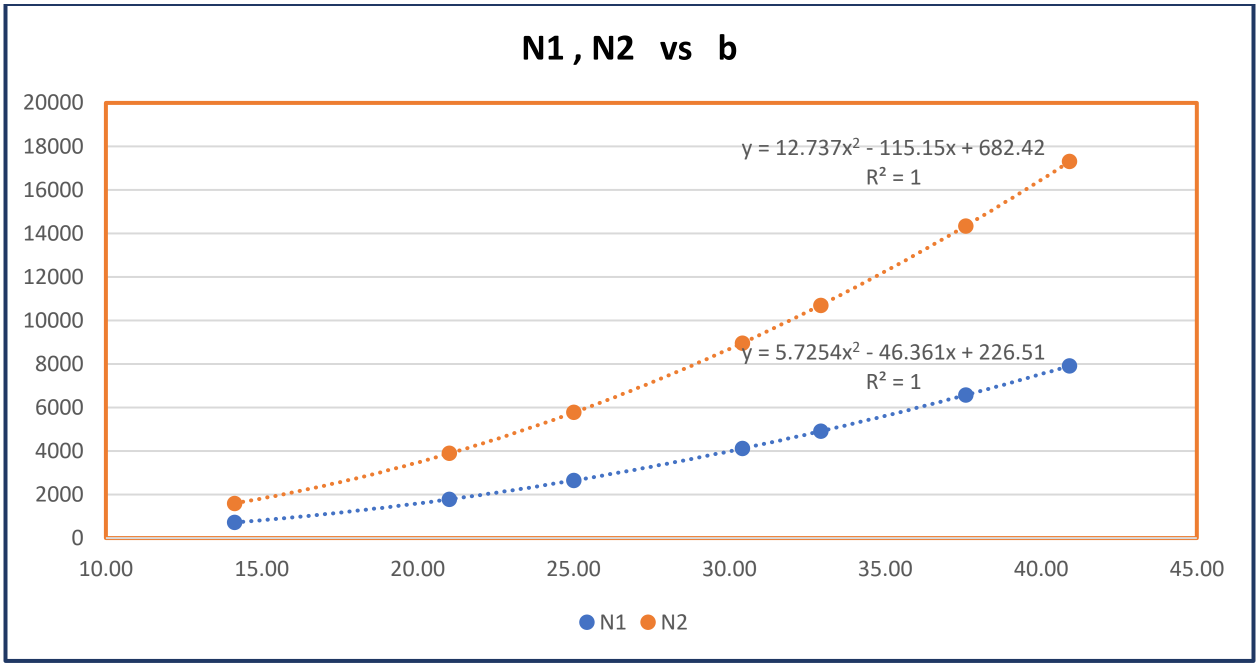


Fig 28: N1, N2 for different values of b = Im(z*) | ζ(z*)=0

The chart shows that correlation values for N1 and N2, which in turn, give the values of O(n) and P(n) and the location of the zeros of ζ(z).

The previous table also shows that N2/N1 = 2.18

2.1 Algorithm N1:

An algorithm to find the zeros of ζ based on this charts would work as follows:

Given a b
Calculate N1_f with regression formula
Calculate N1_c for O(N1_c) = P(N1_c)
If abs(N1_f – N1_c) < epsilon
Then b is a zero of ζ(z)

Examples:

Ex.1. Given b = 10:
N1_formula = 335
N1_calculated = 213
Then b=10 is not a non-trivial zero of ζ(z)

Ex.2. Given b = 25.0108
N1_formula = 2648
N1_calculated = 2648
Then b=25.0108 is a non-trivial zero of ζ(z)

2.2. Algorithm "H1"

The equation [20] can be used to create an algorithm to find the nontrivial zeros of zeta in the critical line without knowing any of them based on their connection to the Harmonic function. An example of a Python code to calculate the zeros of zeta in the critical line with 1 decimal places accuracy based on [20]:

```
# __Python 3.7
# __Pedro Caceres__ 2020 Feb 17
#Rough code to find zeros of Riemann Zeta using the Harmonic function
harmo   = 0
epsilon = 0.01
nn      = 50

for j in range(1,nn):
     harmo += 1 / j
print('Harmonic(',nn,')=', harmo)

for b in range (1,500):
     b = b / 10
     a1   = nn/((1-alfa)**2 + b**2)
     b1 = 0
     for k in range(1,nn):
         for j in range(1,nn):
             if j!=k:
                 b1 += (k*j)**(-alfa) * m.cos(b * m.log(k/j))
     h1=a1-b1

     if abs(h1-harmo) < epsilon:
         print('------> Solution beta=',b, '  ... and->', h1-harmo)
#end_of_code
```

This code tends the following results:

```
Harmonic( 50 )= 4.4792053383294235

--------------> Solution beta= 14.1   ... and error -> 0.0067952158225219605
--------------> Solution beta= 25.0   ... and error -> -0.008460202279115592
--------------> Solution beta= 30.4   ... and error -> 0.0024237587453344034
--------------> Solution beta= 37.6   ... and error -> 0.0012958863904977136
--------------> Solution beta= 40.9   ... and error -> -0.009083573623293262
--------------> Solution beta= 48.0   ... and error -> -0.0027214317425938717
--------------> Solution beta= 49.6   ... and error -> 0.0024275253143217768
```

These values compared to (Odlyzko):

ß(1) = 14.134725142
ß(3) = 25.010857580
ß(4) = 30.424876126
ß(6) = 37.586178159
ß(7) = 40.918719012
ß(9) = 48.005150881
ß(10) = 49.773832478

Changing the values of "n" and epsilon, one can increase the accuracy of the results.

2.3. Algorithm "H2"

The fact that the Harmonic function, Hn , can be expressed in an infinite number of ways as a function of any ß=Im(z) imaginary part of a nontrivial solution of ζ(z), provides also an algorithm to calculate all nontrivial zeros from any known zero.

Let's define the function:

$$H(\alpha, ß, n) -> \frac{n}{\left(ß^2+\frac{1}{4}\right)} - \sum_{k=1}^{n}\sum_{j\neq k}^{n} k^{-\frac{1}{2}} * j^{-\frac{1}{2}} * \cos\left(ß * \left(ln\left(\frac{k}{j}\right)\right)\right) \text{ when } n \to \infty$$

For α=1/2, and ε arbitrarily small, for any two nontrivial zeros of zeta $(\alpha, ß_1)$ and $(\alpha, ß_2)$, there exists and N such that for any n>N:

$$H\left(\alpha=\frac{1}{2}, ß_1, n\right) - H\left(\alpha=\frac{1}{2}, ß_2, n\right) < \varepsilon \qquad [24]$$

This proposition means that the nontrivial zeros of the Riemann Zeta are not distributed randomly, and they follow a defined structure.

Sample code to show how [21] can be used to find zeros based on a known zero:

```
# Code to find zeros from any known zero
# __Pedro Caceres__ 2020 Feb 17
nn = 60 #Not really high. Used for a rough calculation

epsilon = 0.00002
# Known Zero ß(1)
zero= 14.134725142

#Calculating H(1/2,zero,n) = a - b
a2  = nn/((1-alfa)**2 + zero**2)
b2=0
for k in range(1, nn):
    for j in range(1, nn):
        if j != k:
            b2 += (k * j) ** (-alfa) * m.cos(zero * m.log(k / j))

h2 = a2-b2 #H2 to compare against

# range to find additional zeros of zeta
for b in range (245000,310000): #adding digits increases accuracy
    b = b / 10000

    #Calculating a, b
    a1   = nn/((1-alfa)**2 + b**2)
    b1 = 0
    for k in range(1,nn):
        for j in range(1,nn):
            if j!=k:
                b1 += (k*j)**(-alfa) * m.cos(b * m.log(k/j))

    #Calculating H1
    h1=a1-b1

    #If error < epsilon, then print potential zero
    if abs(h1-h2) < epsilon:
        print('-----------> Solution beta=',b, '  ... and error ->', h1-h2)
```

```
#end_of_code
```

Results:

```
-----> Solution beta= 25.0155   ... and error -> +1.442262027140373 e-05
-----> Solution beta= 30.4385   ... and error -> -1.140533215249206 e-05
```

These values compared to (Odlyzko):

ß(3) = 25.010857580
ß(4) = 30.424876126

Changing the values of the variable "nn" and epsilon in the code, the accuracy can be increased to more decimal places.

3. Conclusion

The distribution of the nontrivial zeros of the Riemann Zeta function in the critical line is not random. They are located in values of $z^* = \frac{1}{2} + ßi$ that verify that for any ß, and ε arbitrarily small, there exists and *N* such that for any *n*>*N*:

$$H(n) - \frac{n}{\left(ß^2 + \frac{1}{4}\right)} + \sum_{k=1}^{n}\sum_{j\neq k}^{n} k^{-\frac{1}{2}} * j^{-\frac{1}{2}} * \cos\left(ß\left(ln\left(\frac{k}{j}\right)\right)\right) < \varepsilon \qquad [25]$$

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