GENERALIZED GROUP INVERSE OF BLOCK OPERATOR MATRICES

HUANYIN CHEN

Abstract. We derive the generalized group inverse of a triangular block matrix over a Banach algebra. We apply this formula in order to find the generalized group inverse of 2×2 block operators under some conditions. In particular, the weak group inverse of certain block operator matrices are given.

1. Introduction

Let $\mathcal{A}$ be a Banach algebra with an involution $*$. The group inverse of an element a in $\mathcal{A}$ is an element x such that

$$xa^2 = a, ax^2 = x, ax = xa.$$

Such x is unique if exists, denoted by $a^{\#}$. The group inverse is a concept primarily used in the context of matrices and linear operators, particularly in functional analysis and algebra. It is extensively studied by many authors from many different views, e.g., [1, 3, 6, 7, 8, 11, 17].

The involution $*$ is proper if $x^*x = 0 \Longrightarrow x = 0$ for any $x \in \mathcal{A}$, e.g., in a C^*-algebra, the involution is always proper. Let $\mathbb{C}^{n\times n}$ be the Banach algebra of all $n \times n$ complex matrices, with conjugate transpose $*$ as the involution. Then the involution $*$ is proper. The concept of a weak group inverse extends the idea of a group inverse in the context of matrices and operators. An element a in a Banach algebra with proper involution $*$ has weak group inverse provided that there exist $x \in \mathcal{A}$ and $k \in \mathbb{N}$ such that

$$x = ax^2, (a^*a^2x)^* = a^*a^2x, a^k = xa^{k+1}.$$

If such x exists, it is unique, and denote it by $a^{\textcircled{w}}$. The weak group inverse is a valuable tool in linear algebra and functional analysis. It is particularly significant in applications across various fields where traditional group inverses

2020 *Mathematics Subject Classification.* 15A09, 16W10, 16U90.

Key words and phrases. group inverse; weak group inverse; generalized group inverse; block operator matrix; Banach algebra.

fail to exist. We refer the reader for weak group inverse in [5, 9, 10, 12, 13, 15, 16].

In [2], we introduced and studied a new generalized inverse in a Banach *-algebra as a generalization of weak group inverse for complex matrices and linear operators. An element a in a Banach algebra with proper involution has generalized group inverse if there exists $x \in \mathcal{A}$ such that

$$x = ax^2, (a^*a^2x)^* = a^*a^2x, \lim_{n\to\infty} ||a^n - xa^{n+1}||^{\frac{1}{n}} = 0.$$

Such x is unique if it exists. We call the preceding x the generalized group inverse of a, and denote it by $a^{\textcircled{g}}$. Here, we list some characterizations of generalized group inverse.

The generalized Drazin inverse generalizes the group inverse. An element a in $\mathcal{A}$ has generalized Drazin inverse if there exists $x \in \mathcal{A}$ such that $ax^2 = x, ax = xa, a - a^2x \in \mathcal{A}^{qnil}$ (see [4, 18]). Here, $\mathcal{A}^{qnil} = \{a \in \mathcal{A} \mid 1 + \lambda a \in \mathcal{A}^{-1}\}$. Such x is unique, if exists, and denote it by a^d. We use $\mathcal{A}^d$ and $\mathcal{A}^{\#}$ to denote the sets of all generalized Drazin invertible and group invertible elements in $\mathcal{A}$, respectively. Here, we list some characterizations of generalized group inverse.

Theorem 1.1. *(see* [2, Theorem 2.2, Theorem 4.1 and Theorem 5.1]*) Let $\mathcal{A}$ be a Banach *-algebra, and let $a \in \mathcal{A}$. Then the following are equivalent:*

(1) *$a \in \mathcal{A}$ has generalized group inverse.*
(2) *There exist $x, y \in \mathcal{A}$ such that*

$$a = x + y, x^*y = yx = 0, x \in \mathcal{A}^{\#}, y \in \mathcal{A}^{qnil}.$$

In this case, $a^{\textcircled{g}} = x^{\#}$.
(3) $a \in \mathcal{A}^d$ and there exists $x \in \mathcal{A}$ such that

$$x = ax^2, (a^d)^*a^2x = (a^d)^*a, \lim_{n\to\infty} ||a^n - xa^{n+1}||^{\frac{1}{n}} = 0.$$

(4) *There exists an idempotent $p \in \mathcal{A}$ such that*

$$a + p \in \mathcal{A}^{-1}, (a^*ap)^* = a^*ap \text{ and } pa = pap \in \mathcal{A}^{qnil}.$$

For a complex matrix, three generalized inverses mentioned above coincide with one another. The generalized group inverse is particularly useful when dealing with non-weak group invertible elements in algebraic structures, such as linear operator over a Hilbert space. The motivation of this paper is to investigate the generalized group inverse of a block operator matrix over a Banach algebra.

In Section 2, we present some necessary lemmas which will be used in the sequel. In Section 3, we are concerned with when a triangular operator matrix has generalized group inverse and the representation of the generalized group inverse is then given. An additive result of the generalized group inverse is established.

Lex X and Y be Banach spaces. We use $\mathcal{B}(X,Y)$ to stand for the set of all bounded linear operators from X to Y. Set $\mathcal{B}(X)=\mathcal{B}(X,Y)$. Finally, in Section 4, we apply our results and study the generalized group inverse for the block operator matrix $M=\begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where $A\in\mathcal{B}(X), B\in\mathcal{B}(X,Y), C\in\mathcal{B}(Y,X), D\in\mathcal{B}(Y)$. Here, M is a linear operator on Banach space $X\oplus Y$.

Throughout the paper, all Banach algebras are complex with a proper involution $*$. We use $\mathcal{A}^{\textcircled{g}}$ and $\mathcal{A}^{\textcircled{W}}$ to stand for the sets of all generalized group invertible and weak group invertible elements in $\mathcal{A}$, respectively. Let $a\in\mathcal{A}^{\textcircled{g}}$. We denote a^{π} and a^{τ} the idempotents $1-aa^d$ and $a^{\tau}=1-aa^{\textcircled{g}}$, respectively.

2. KEY LEMMAS

In this section, some lemmas are presented. We begin with

Lemma 2.1. *Let $a\in\mathcal{A}^{\textcircled{g}}$ and $b\in\mathcal{A}$. Then the following are equivalent:*

(1) $a^{\pi}b=0$.
(2) $a^{\tau}b=0$.
(3) $(1-a^{\textcircled{g}}a)b=0$.

Proof. (1) $\Rightarrow$ (3) Since $a^{\pi}b=0$, we have that $b=aa^db$. Then $a^{\textcircled{g}}ab=a^{\textcircled{g}}a^2a^db=aa^db=b$. Hence $(1-a^{\textcircled{g}}a)b=0$, as required.

(3) $\Rightarrow$ (2) Since $(1-a^{\textcircled{g}}a)b=0$, we have $b=a^{\textcircled{g}}ab$. Thus, $(1-aa^{\textcircled{g}})b=(1-aa^{\textcircled{g}})a^{\textcircled{g}}ab=0$.

(2) $\Rightarrow$ (1) Since $(1-aa^{\textcircled{g}})b=0$, we have $b=aa^{\textcircled{g}}b$. Then

$$aa^db=a^2a^da^{\textcircled{g}}b=aa^{\textcircled{g}}b=b.$$

This implies that $a^{\pi}b=0$, as required. $\square$

Lemma 2.2. *Let $a\in\mathcal{A}^{\textcircled{g}}$ and $b\in\mathcal{A}^{qnil}$. If $a^*b=0$ and $ba=0$, then $a+b\in\mathcal{A}^{\textcircled{g}}$. In this case,*

$$(a+b)^{\textcircled{g}}=a^{\textcircled{g}}.$$

Proof. Since $a\in\mathcal{A}^{\textcircled{g}}$, by virtue of Theorem 1.1, there exist $x\in\mathcal{A}^{\#}$ and $y\in\mathcal{A}^{qnil}$ such that $a=x+y, x^*y=0, yx=0$. As in the proof of [2, Theorem 2.2], $x=a^2a^{\textcircled{g}}$ and $y=a-a^2a^{\textcircled{g}}$. Then $a=x+(y+b)$. Since

$by = b(a - a^2a^{ⓖ}) = 0$, it follows by [18, Lemma 4.1] that $y + b \in \mathcal{A}^{qnil}$. Obviously, $a^*(y + b) = a^*y + a^*b = 0$. In light of Theorem 1.1, $a + b \in \mathcal{A}^{ⓖ}$. In this case,

$$(a + b)^{ⓖ} = x^{\#} = a^{ⓖ},$$

as asserted. □

Lemma 2.3.

(1) Let $M = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$. If $a, d \in \mathcal{A}^{\#}$, then $M \in M_2(\mathcal{A})^{\#}$ if and only if $a^{\pi}bd^{\pi} = 0$. In this case, $M^{\#} = \begin{pmatrix} a^{\#} & z \\ 0 & d^{\#} \end{pmatrix}$, where

$$z = (a^{\#})^2b(1 - dd^{\#}) + (1 - aa^{\#})b(d^{\#})^2 - a^{\#}bd^{\#}.$$

(2) Let $p \in \mathcal{A}$ be an idempotent and let $x = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}_p$. If $a \in (pAp)^{\#}, d \in (p^{\pi}Ap^{\pi})^{\#}$, then $x \in \mathcal{A}^{\#}$ if and only if $a^{\pi}bd^{\pi} = 0$. In this case, $x^{\#} = \begin{pmatrix} a^{\#} & z \\ 0 & d^{\#} \end{pmatrix}_p$, where

$$z = (a^{\#})^2b(1 - dd^{\#}) + (1 - aa^{\#})b(d^{\#})^2 - a^{\#}bd^{\#}.$$

Proof. See [6, Theorem 1] and [7, Theorem 2.1]. □

For further use, we now extend [1, Theorem 3.4 and Theorem 3.7] as follows.

Lemma 2.4. *Let* $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathcal{A})$ *with* $a, d \in \mathcal{A}^{\#}$.

(1) If $bd = 0, a^{\pi}b = 0$ and $d^{\pi}c = 0$, then M has group inverse. In this case,

$$M^{\#} = \begin{pmatrix} a^{\#} & (a^{\#})^2b \\ -d^{\#}ca^{\#} + (d^{\#})^2ca^{\pi} & d^{\#} - d^{\#}c(a^{\#})^2b - (d^{\#})^2ca^{\#}b \end{pmatrix}.$$

(2) If $ab = 0, ca^{\pi} = 0$ and $bd^{\pi} = 0$, then M has group inverse. In this case,

$$M^{\#} = \begin{pmatrix} a^{\#} - bd^{\#}c(a^{\#})^2 - b(d^{\#})^2ca^{\#} & b(d^{\#})^2 \\ d^{\pi}c(a^{\#})^2 - d^{\#}ca^{\#} & d^{\#} \end{pmatrix}.$$

Proof. (1) Write $M = P + Q$, where

$$P = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}, Q = \begin{pmatrix} 0 & 0 \\ c & d \end{pmatrix}.$$

In view of Lemma 2.3, P has group inverse and $P^{\#} = \begin{pmatrix} a^{\#} & (a^{\#})^2 b \\ 0 & 0 \end{pmatrix}$. Similarly, Q has group inverse and $Q^{\#} = \begin{pmatrix} 0 & 0 \\ (d^{\#})^2 c & d^{\#} \end{pmatrix}$. Since $d^{\pi} c = 0$, we see that $c = dd^d c$; hence, $bc = b(dd^d c) = (bd)d^d c = 0$. It is easy to verify that

$$PQ = \begin{pmatrix} bc & bd \\ 0 & 0 \end{pmatrix} = 0.$$

According to [1, Theorem 2.1], $M = P + Q$ has group inverse and

$$\begin{aligned} M^{\#} &= Q^{\pi} P^{\#} + Q^{\#} P^{\pi} \\ &= \begin{pmatrix} 1 & 0 \\ -d^{\#} c & d^{\pi} \end{pmatrix} \begin{pmatrix} a^{\#} & (a^{\#})^2 b \\ 0 & 0 \end{pmatrix} \\ &+ \begin{pmatrix} 0 & 0 \\ (d^{\#})^2 c & d^{\#} \end{pmatrix} \begin{pmatrix} a^{\pi} & -a^{\#} b \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} a^{\#} & (a^{\#})^2 b \\ -d^{\#} c a^{\#} + (d^{\#})^2 c a^{\pi} & d^{\#} - d^{\#} c (a^{\#})^2 b - (d^{\#})^2 c a^{\#} b \end{pmatrix}, \end{aligned}$$

as required.

(2) Write $M = P + Q$, where

$$P = \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix}, Q = \begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix}.$$

By virtue of Lemma 2.3, we have

$$P^{\#} = \begin{pmatrix} a^{\#} & 0 \\ c(a^{\#})^2 & 0 \end{pmatrix}, Q^{\#} = \begin{pmatrix} 0 & b(d^{\#})^2 \\ 0 & d^{\#} \end{pmatrix}.$$

As $ca^{\pi} = 0$, we have $c = ca^d a$; whence, $cb = ca^d(ab) = 0$. It is easy to verify that

$$PQ = \begin{pmatrix} 0 & ab \\ 0 & cb \end{pmatrix} = 0.$$

By virtue of [1, Theorem 2.1], $M = P + Q$ has group inverse and

$$\begin{aligned} M^{\#} &= Q^{\pi} P^{\#} + Q^{\#} P^{\pi} \\ &= \begin{pmatrix} 1 & -bd^{\#} \\ 0 & d^{\pi} \end{pmatrix} \begin{pmatrix} a^{\#} & 0 \\ c(a^{\#})^2 & 0 \end{pmatrix} \\ &+ \begin{pmatrix} 0 & b(d^{\#})^2 \\ 0 & d^{\#} \end{pmatrix} \begin{pmatrix} a^{\#} & 0 \\ -ca^{\#} & 1 \end{pmatrix} \\ &= \begin{pmatrix} a^{\#} - bd^{\#} c (a^{\#})^2 - b(d^{\#})^2 c a^{\#} & b(d^{\#})^2 \\ d^{\pi} c (a^{\#})^2 - d^{\#} c a^{\#} & d^{\#} \end{pmatrix}, \end{aligned}$$

as asserted. □

Recall that $a \in \mathcal{A}$ has Drazin inverse provided that there exists $x \in \mathcal{A}$ such that $ax^2 = x, ax = xa, a^k = xa^{k+1}$ for some $k \in \mathbb{N}$.

Lemma 2.5. *Let $a \in \mathcal{A}$. Then $a \in \mathcal{A}^{\textcircled{W}}$ if and only if*

(1) $a \in \mathcal{A}^{\textcircled{g}}$;
(2) $a \in \mathcal{A}$ has Drazin inverse.

In this case, $a^{\textcircled{W}} = a^{\textcircled{g}}$.

Proof. Straightforward by choosing $w = 1$ in [4, Lemma 4.5]. □

3. MAIN RESULTS

We come now to the demonstration for which this section has been developed.

Theorem 3.1. *Let $\mathcal{A}$ be a Banach algebra and $M = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ with $a, d \in \mathcal{A}^{\textcircled{g}}$. If*

$$dd^{\tau}b = 0, b^*dd^{\tau} = 0, a^{\tau}bd^{\tau} = 0,$$

then $M \in \mathcal{A}^{\textcircled{g}}$ and

$$x^{\textcircled{g}} = \begin{pmatrix} a^{\textcircled{g}} & z \\ 0 & d^{\textcircled{g}} \end{pmatrix},$$

where $z = (a^{\textcircled{g}})^2bd^{\tau} + a^{\tau}b(d^{\textcircled{g}})^2 - a^{\textcircled{g}}bd^{\textcircled{g}}$.

Proof. By hypothesis, we have generalized group decompositions:

$$a = x + y, d = s + t,$$

where

$$x, s \in \mathcal{A}^{\#}, y, t \in \mathcal{A}^{qnil}$$

and

$$x^*y = 0, yx = 0; s^*t = 0, ts = 0.$$

As in the proof of [2, Theorem 2.2],

$$\begin{array}{rcl} x & = & a^2a^{\textcircled{g}}, y = d - d^2d^{\textcircled{g}}, \\ s & = & d^2d^{\textcircled{g}}, t = d - d^2d^{\textcircled{g}}. \end{array}$$

Then we have $M = P + Q$, where

$$P = \begin{pmatrix} x & b \\ 0 & s \end{pmatrix}, Q = \begin{pmatrix} y & 0 \\ 0 & t \end{pmatrix}.$$

We verify that

$$\begin{array}{rcl} x^{\pi}bs^{\pi} & = & [1-(a^2a^{ⓖ})a^{ⓖ}]b[1-(d^2d^{ⓖ})d^{ⓖ}] \\ & = & (1-aa^{ⓖ})b(1-dd^{ⓖ})=0. \end{array}$$

In view of Lemma 2.3, P has group inverse. Since $y,t\in \mathcal{A}^{qil}$, we directly verify that Q is quasinilpotent. One easily checks that

$$\begin{array}{rcl} P^*Q & = & \begin{pmatrix} x^* & 0 \\ b^* & s^* \end{pmatrix}\begin{pmatrix} y & 0 \\ 0 & t \end{pmatrix}=\begin{pmatrix} 0 & 0 \\ b^*y & 0 \end{pmatrix}=0, \\ QP & = & \begin{pmatrix} y & 0 \\ 0 & t \end{pmatrix}\begin{pmatrix} x & b \\ 0 & s \end{pmatrix}=\begin{pmatrix} 0 & yb \\ 0 & 0 \end{pmatrix}=0. \end{array}$$

In light of Theorem 1.1 and Lemma 2.3, we derive that

$$M^{ⓖ}=P^{\#}=\begin{pmatrix} x^{\#} & z \\ 0 & s^{\#} \end{pmatrix},$$

where $z=(x^{\#})^2b(1-ss^{\#})+(1-xx^{\#})b(s^{\#})^2-x^{\#}bs^{\#}$. Therefore

$$M^{ⓖ}=\begin{pmatrix} a^{ⓖ} & z \\ 0 & d^{ⓖ} \end{pmatrix},$$

where $z=(a^{ⓖ})^2b[1-dd^{ⓖ}]+[1-aa^{ⓖ}]b(d^{ⓖ})^2-a^{ⓖ}bd^{ⓖ}$. □

Corollary 3.2. *Let $\mathcal{A}$ be a Banach algebra and $M=\begin{pmatrix} a & 0 \\ c & d \end{pmatrix}$ with $a,d\in \mathcal{A}^{ⓖ}$. If*

$$aa^{\tau}c=0, c^*aa^{\tau}=0, d^{\tau}ca^{\tau}=0,$$

then $M\in \mathcal{A}^{ⓖ}$ and

$$x^{ⓖ}=\begin{pmatrix} a^{ⓖ} & z \\ z & d^{ⓖ} \end{pmatrix},$$

where $z=(d^{ⓖ})^2ca^{\tau}+d^{\tau}c(a^{ⓖ})^2-d^{ⓖ}ca^{ⓖ}$.

Proof. Clearly, we have $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}M\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}=\begin{pmatrix} d & c \\ b & a \end{pmatrix}$. Applying Theorem 3.1 to the matrix $\begin{pmatrix} d & c \\ b & a \end{pmatrix}$, we see that $\begin{pmatrix} d & c \\ b & a \end{pmatrix}$ has generalized group inverse. This implies that M has generalized group inverse. In this case,

$$M^{ⓖ}=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} d & c \\ b & a \end{pmatrix}^{ⓖ}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Therefore we complete the proof by Theorem 3.1. □

Corollary 3.3. *Let $\mathcal{A}$ be a Banach algebra and $M = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$ with $a \in \mathcal{A}^{\circledR}$. If $a^\pi b = 0$, then $M \in \mathcal{A}^{\circledR}$ and*

$$M^{\circledR} = \begin{pmatrix} a^{\circledR} & (a^{\circledR})^2 b \\ 0 & 0 \end{pmatrix}.$$

Proof. Since $a^\pi b = 0$, it follows by Lemma 2.1 that $a^\tau b = 0$. This completes the proof by Theorem 3.1. □

Corollary 3.4. *Let $\mathcal{A}$ be a Banach algebra and $M = \begin{pmatrix} 0 & 0 \\ c & d \end{pmatrix}$ with $d \in \mathcal{A}^{\circledR}$. If $d^\pi c = 0$, then $x \in \mathcal{A}^{\circledR}$ and*

$$M^{\circledR} = \begin{pmatrix} 0 & 0 \\ (d^{\circledR})^2 c & d^{\circledR} \end{pmatrix},$$

Proof. In view of Lemma 2.1, $d^\tau c = 0$. We are through by Corollary 3.2. □

Let $p = p^2 \in \mathcal{A}$. We can represent $a \in \mathcal{A}$ as $a = \begin{pmatrix} pap & pap^\pi \\ p^\pi ap & p^\pi ap^\pi \end{pmatrix}_p$. We next use the matrix approach to establish an additive result of generalized group inverse.

Theorem 3.5. *Let $a, b \in \mathcal{A}^d, a^\pi b \in \mathcal{A}^{\circledR}$. If $a^\pi ba = 0, aa^\pi b = 0, a^* a^\pi b = 0$ and $(a+b)^\pi aa^d ba^\pi = 0$, the following are equivalent:*

(1) $a + b \in \mathcal{A}^{\circledR}$ and $a^d (a+b)^{\circledR} a^\pi = 0$.
(2) $(a+b)aa^d \in \mathcal{A}^{\circledR}$.

In this case,

$$(a+b)^{\circledR} = [(a+b)aa^d]^{\circledR} + (a^\pi b)^{\circledR} - [(a+b)aa^d]^{\circledR} aa^d ba^\pi (a^\pi b)^{\circledR}.$$

Proof. Let $p = aa^d$. By hypothesis, we have $p^\pi ba = 0$. Hence, $p^\pi bp = (a^\pi ba)a^d = 0$. Moreover, we have $p^\pi ap = (1 - aa^d)a^2 a^d = 0, pap^\pi = aa^d a(1 - aa^d) = 0, p^\pi ap^\pi = 0$. Then

$$a = \begin{pmatrix} a^2 a^d & 0 \\ 0 & a^\pi a \end{pmatrix}_p, b = \begin{pmatrix} b_1 & b_2 \\ 0 & a^\pi b \end{pmatrix}_p.$$

Hence

$$a + b = \begin{pmatrix} (a+b)aa^d & aa^d ba^\pi \\ 0 & a^\pi (a+b) \end{pmatrix}_p.$$

By hypothesis, we verify that

$$\begin{array}{rcl} (a^\pi b)^*(a^\pi a) & = & (a^*a^\pi b)^*a^\pi = 0, \\ (a^\pi a)(a^\pi b) & = & aa^\pi b = 0, \\ a^\pi a & \in & \mathcal{A}^{qnil}. \end{array}$$

In view of Lemma 2.2, $a_4 + b_4 = a^\pi(a+b) = a^\pi a + a^\pi b \in \mathcal{A}^{ⓖ}$. Additionally,

$$a^\pi b \in \mathcal{A}^{ⓖ}, (a^\pi b)^d = p^\pi b^d, (a^\pi b)^\pi = p^\pi b^\pi.$$

By virtue of Lemma 2.2, we derive that $(a_4 + b_4)^{ⓖ} = b_4^{ⓖ} = (a^\pi b)^{ⓖ}$.

Since $p^\pi(a+b)aa^d = 0$ and $p^\pi(a+b) \in \mathcal{A}^d$, it follows by [14, Lemma 2.2] that

$$(a_1 + b_1)^d = [(a+b)aa^d]^d = (a+b)^d aa^d.$$

Moreover, we have

$$\begin{array}{rcl} (a_1 + b_1)^\pi & = & aa^d - (a+b)^d aa^d(a+b)aa^d \\ & = & aa^d - (a+b)^d(a+b)aa^d \\ & = & (a+b)^\pi aa^d. \end{array}$$

(1) $\Rightarrow$ (2) Since $a^d(a+b)^{ⓖ}a^\pi = 0$, we have $p(a+b)^{ⓖ}p^\pi = 0$. Then we write

$$(a+b)^{ⓖ} = \begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix}_p.$$

Then

$$\begin{gathered} (a+b)((a+b)^{ⓖ})^2 = \alpha, \\ \big((a+b)^*(a+b)^2(a+b)^{ⓖ}\big)^* = (a+b)^*(a+b)^2(a+b)^{ⓖ}, \\ \lim_{n\to\infty} ||(a+b)^n - (a+b)^{ⓖ}(a+b)^{n+1}||^{\frac{1}{n}} = 0. \end{gathered}$$

We infers that

$$\begin{gathered} (a_1 + b_1)\alpha^2 = \alpha, [(a_1+b_1)^*(a_1+b_1)^2\alpha]^* = (a_1+b_1)^*(a_1+b_1)^2\alpha, \\ \lim_{n\to\infty} ||(a_1+b_1)^n - \alpha(a_1+b_1)^{n+1}||^{\frac{1}{n}} = 0. \end{gathered}$$

Therefore $(a_1 + b_1)^{ⓖ} = \alpha$, as desired.

(2) $\Rightarrow$ (1) By hypothesis, we have

$$a_1 + b_1 = (a+b)aa^d \in \mathcal{A}^{ⓖ}.$$

By the preceding discussion, $a_1 + b_1, a_4 + b_4 \in \mathcal{A}^{ⓖ}$, and so we have generalized group decompositions: $a_1 + b_1 = x + y, a_4 + b_4 = s + t$, where $x, s \in \mathcal{A}^{\#}, y, t \in$

$\mathcal{A}^{qnil}$ and $x^*y=0, yx=0; s^*t=0, ts=0$. As in the proof of [2, Theorem 2.2],

$$\begin{array}{rcl}
x &=& (a_1+b_1)^2(a_1+b_1)^{ⓖ},\\
y &=& (a_1+b_1)-(a_1+b_1)^2(a_1+b_1)^{ⓖ},\\
s &=& (a_4+b_4)^2(a_4+b_4)^{ⓖ}\\
&=& a^{\pi}(a+b)^2(a^{\pi}b)^{ⓖ},\\
t &=& (a_4+b_4)-(a_4+b_4)^2(a_4+b_4)^{ⓖ}\\
&=& a^{\pi}[(a+b)-(a+b)^2(a^{\pi}b)^{ⓖ}].
\end{array}$$

Then we have $a+b=\beta+\gamma$, where

$$\begin{array}{rcl}
\beta &=& \left(\begin{array}{cc} x & xx^db_2\\ 0 & s \end{array}\right)_p,\\
\gamma &=& \left(\begin{array}{cc} y & x^{\pi}b_2\\ 0 & t \end{array}\right)_p.
\end{array}$$

Clearly, we have $x^{\pi}(xx^db_2)=0$, and so $\beta\in\mathcal{A}^{\#}$ by Lemma 2.3. Since $y,t\in\mathcal{A}^{qnil}$, we see that $\gamma\in\mathcal{A}^{qnil}$. By hypothesis, we easily check that

$$\begin{array}{rcl}
x^*x^{\pi}b_2 &=& x^*(a_1+b_1)^{\pi}b_2=x^*(a+b)^{\pi}aa^dba^{\pi}=0,\\
x^*t &=& x^*a^{\pi}[(a+b)-(a+b)^2(a^{\pi}b)^{ⓖ}\\
&=& (aa^dx)^*a^{\pi}[a+b-(a^2+ab+ba+b^2)a^{\pi}ba^{\pi}b^d(a^{\pi}b)^{ⓖ}\\
&=& (a^dx)^*a^{\pi}[a^*a^{\pi}b][(1-a-b)a^{\pi}ba^{\pi}b^d(a^{\pi}b)^{ⓖ}\\
&+& (aa^dx)^*a^{\pi}[1-a)[aa^{\pi}b]a^{\pi}b^d(a^{\pi}b)^{ⓖ}\\
&-& (aa^dx)^*[aa^{\pi}b]a^{\pi}ba^{\pi}b^d(a^{\pi}b)^{ⓖ}=0,\\
s^*y &=& [a^{\pi}(a+b)^2(a^{\pi}b)^{ⓖ}]^*aa^dz\\
&=& [a^{\pi}(a^2+ab+ba+b^2)(a^{\pi}b)(a^{\pi}b^d)(a^{\pi}b)^{ⓖ}]^*aa^dz\\
&=& [a^{\pi}(a^2+b^2)(a^{\pi}b)(a^{\pi}b^d)(a^{\pi}b)^{ⓖ}]^*aa^dz\\
&=& [a^{\pi}b^2(a^{\pi}b)(a^{\pi}b^d)(a^{\pi}b)^{ⓖ}]^*aa^dz\\
&=& [b(a^{\pi}b)(a^{\pi}b^d)(a^{\pi}b)^{ⓖ}]^*[a^*a^{\pi}b]^*a^dz=0,\\
s^*x^{\pi}b_2 &=& [a^{\pi}(a+b)^2(a^{\pi}b)^{ⓖ}]^*aa^dz'\\
&=& [b(a^{\pi}b)(a^{\pi}b^d)(a^{\pi}b)^{ⓖ}]^*[a^*a^{\pi}b]^*a^dz'=0.
\end{array}$$

Moreover, we derive that

$$x^{\pi}b_2s=(a+b)^{\pi}aa^dba^{\pi}b^2a^{\pi}(a+b)^2(a^{\pi}b)^{ⓖ}=0.$$

We directly check that

$$\begin{array}{rcl}\beta^*\gamma & = & [x^* + (b_2)^*(xx^d)^* + s^*]\begin{pmatrix} y & x^\pi b_2 \\ 0 & t \end{pmatrix}_p \\ & = & 0, \\ \gamma\beta & = & \begin{pmatrix} y & x^\pi b_2 \\ 0 & t \end{pmatrix}_p \begin{pmatrix} x & xx^d b_2 \\ 0 & s \end{pmatrix}_p \\ & = & 0. \end{array}$$

According to Lemma 2.2, $a + b = \beta + \gamma \in M_2(\mathcal{A})^{ⓖ}$. In this case, we have

$$\begin{array}{rcl}(a+b)^{ⓖ} & = & \beta^{\#} \\ & = & \begin{pmatrix} x^{\#} & z \\ 0 & s^{\#} \end{pmatrix} \\ & = & \begin{pmatrix} (a_1+b_1)^{ⓖ} & z \\ 0 & (a^\pi b)^{ⓖ} \end{pmatrix}_p, \end{array}$$

where

$$\begin{array}{rcl} z & = & -x^{\#} z s^{\#} \\ & = & -(a_1+b_1)^{ⓖ} a a^d b a^\pi (a^\pi b)^{ⓖ}. \end{array}$$

Obviously, $a^d(a+b)^{ⓖ}a^\pi = a^d[aa^d(a+b)^{ⓖ}a^\pi] = 0$, as required. □

Corollary 3.6. *Let $a, b \in \mathcal{A}^d, a^\pi b \in \mathcal{A}^{ⓖ}$. If $a^\pi ba = 0, a^*a^\pi b = 0$ and $aba^\pi = 0$, then the following are equivalent:*

(1) $a + b \in \mathcal{A}^{ⓖ}$.
(2) $(a + b)aa^d \in \mathcal{A}^{ⓖ}$.

In this case, $(a + b)^{ⓖ} = [(a + b)aa^d]^{ⓖ} + (a^\pi b)^{ⓖ}$.

Proof. This is obvious by Theorem 3.4. □

Corollary 3.7. *Let $a, b \in \mathcal{A}^{ⓖ}$. If $ab = 0, ba = 0$ and $a^*b = 0$, then $a+b \in \mathcal{A}^{ⓖ}$. In this case,*

$$(a + b)^{ⓖ} = a^{ⓖ} + b^{ⓖ}.$$

Proof. Since $ab = 0$, we see that $a^\pi ba = 0, a^*a^\pi b = 0$ and $aba^\pi = 0$. Moreover, we see that $(a + b)aa^d = a^2a^d \in \mathcal{A}^{ⓖ}$. This completes the proof by Corollary 3.6. □

4. APPLICATIONS

Lex X and Y be Banach spaces, and let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where $A \in \mathcal{B}(X)^{\textcircled{g}}, B \in \mathcal{B}(X,Y), C \in \mathcal{B}(Y,X), D \in \mathcal{B}(Y)^{\textcircled{g}}$. Choose $p = \begin{pmatrix} I_X & 0 \\ 0 & I_Y \end{pmatrix}$. Then $M = \begin{pmatrix} pMp & pMp^{\pi} \\ p^{\pi}Mp & p^{\pi}Mp^{\pi} \end{pmatrix}_p$. Here, every subblock matrices can be seen as the bounded linear operators on Banach space $X \oplus Y$. Throughout this section, without loss the generality, we consider M as the block operator matrix in a specifical case $X = Y$. Evidently, $\mathcal{B}(X \oplus X)$ is indeed an Banach algebra with the adjoint operation as the involution.

Theorem 4.1. *If $BD = 0, CA = 0, A^{\pi}B = 0$ and $D^{\pi}C = 0$, then M has generalized group inverse. In this case,*

$$M^{\textcircled{g}} = \begin{pmatrix} A^{\textcircled{g}} & (A^{\textcircled{g}})^2B \\ (D^{\textcircled{g}})^2C & D^{\textcircled{g}} \end{pmatrix}.$$

Proof. Since $A^{\pi}B = 0$, we verify that $CB = C(AA^dB) = (CA)A^dB = 0$. It follows from $D^{\pi}C = 0$ that $BC = B(DD^dC) = (BD)D^dC = 0$.

Write $M = P + Q$, where

$$P = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}, Q = \begin{pmatrix} 0 & 0 \\ C & D \end{pmatrix}.$$

It is easy to verify that

$$\begin{aligned} P^*Q &= \begin{pmatrix} A^* & 0 \\ B^* & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ C & D \end{pmatrix} = 0, \\ PQ &= \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ C & D \end{pmatrix} = \begin{pmatrix} BC & BD \\ 0 & 0 \end{pmatrix} = 0, \\ QP &= \begin{pmatrix} 0 & 0 \\ C & D \end{pmatrix}\begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ CA & CB \end{pmatrix} = 0, \end{aligned}$$

In view of Corollary 3.3, P has generalized group inverse. By virtue of Corollary 3.4, Q has generalized group inverse. Therefore M has generalized group inverse by Corollary 3.7. Moreover, we have

$$\begin{aligned} M^{\textcircled{g}} &= P^{\textcircled{g}} + Q^{\textcircled{g}} \\ &= \begin{pmatrix} A^{\textcircled{g}} & (A^{\textcircled{g}})^2B \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ (D^{\textcircled{g}})^2C & D^{\textcircled{g}} \end{pmatrix} \\ &= \begin{pmatrix} A^{\textcircled{g}} & (A^{\textcircled{g}})^2B \\ (D^{\textcircled{g}})^2C & D^{\textcircled{g}} \end{pmatrix}, \end{aligned}$$

as asserted. □

Corollary 4.2. *If $BD = 0, CA = 0, A^*(A^\pi B) = 0, D^*(D^\pi C) = 0$ and $(A^\pi B)(D^\pi C)$ is quasinilpotent, then M has generalized group inverse. In this case,*

$$M^{ⓖ} = \begin{pmatrix} A^{ⓖ} & (A^{ⓖ})^2 AA^d B \\ (D^{ⓖ})^2 DD^d C & D^{ⓖ} \end{pmatrix}.$$

Proof. Write $M = P + Q$, where

$$P = \begin{pmatrix} A & AA^d B \\ DD^d C & D \end{pmatrix}, Q = \begin{pmatrix} 0 & A^\pi B \\ D^\pi C & 0 \end{pmatrix}.$$

Since $(AA^dB)D = 0, (DD^dC)A = 0, A^\pi(AA^dB) = 0$ and $D^\pi(DD^dC) = 0$, it follows by Theorem 4.1 that P has generalized group inverse and

$$P^{ⓖ} = \begin{pmatrix} A^{ⓖ} & (A^{ⓖ})^2 AA^d B \\ (D^{ⓖ})^2 DD^d C & D^{ⓖ} \end{pmatrix}.$$

Since $(A^\pi B)(D^\pi C)$ is quasinilpotent, then so is $(D^\pi C)(A^\pi B)$. Then $Q^2 = \begin{pmatrix} (A^\pi B)(D^\pi C) & 0 \\ 0 & (D^\pi C)(A^\pi B) \end{pmatrix}$ is quasinilpotent, and then so is Q.

One easily checks that

$$\begin{aligned} P^*Q &= \begin{pmatrix} A^* & (DD^dC)^* \\ (AA^dB)^* & D^* \end{pmatrix}\begin{pmatrix} 0 & A^\pi B \\ D^\pi C & 0 \end{pmatrix} \\ &= \begin{pmatrix} (D^dC)^*(D^*D^\pi C) & A^*A^\pi B \\ D^*D^\pi C & (A^dB)^*(A^*A^\pi B) \end{pmatrix} = 0, \\ QP &= \begin{pmatrix} 0 & A^\pi B \\ D^\pi C & 0 \end{pmatrix}\begin{pmatrix} A & AA^dB \\ DD^dC & D \end{pmatrix} \\ &= \begin{pmatrix} A^\pi BDD^dC & A^\pi BD \\ D^\pi CA & D^\pi CAA^dB \end{pmatrix} = 0, \end{aligned}$$

Therefore M has generalized group inverse by Lemma 2.2. Moreover, we have

$$M^{ⓖ} = P^{ⓖ} = \begin{pmatrix} A^{ⓖ} & (A^{ⓖ})^2 AA^d B \\ (D^{ⓖ})^2 DD^d C & D^{ⓖ} \end{pmatrix}.$$

□

Theorem 4.3. *If $BD = 0, A^*B = 0, D^*C = 0, A^\pi B = 0$ and $D^\pi C = 0$, then M has generalized group inverse. In this case,*

$$M^{ⓖ} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

where

$$\begin{array}{rcl}\alpha & = & A^{ⓖ},\\ \beta & = & (A^{ⓖ})^2B,\\ \gamma & = & -D^{ⓖ}CA^{ⓖ}+(D^{ⓖ})^2C[I-AA^{ⓖ}],\\ \delta & = & D^{ⓖ}-D^{ⓖ}C(A^{ⓖ})^2B-(D^{ⓖ})^2CA^{ⓖ}B.\end{array}$$

Proof. Since A and D has generalized group inverse, it follows by Theorem 1.1 that we have

$$A=A_1+A_2, A_1^*A_2=0, A_2A_1=0,$$
$$D=D_1+D_2, D_1^*D_2=0, D_2D_1=0,$$
$$A_1, D_1 \text{ has group inverse}, A_2, D_2 \text{ is quasinilpotent}.$$

Evidently, $A_1=A^2A^{ⓖ}, A_2=A-A^2A^{ⓖ}$ and $D_1=D^2D^{ⓖ}, D_2=D-D^2D^{ⓖ}$. Write $M=P+Q$, where

$$P=\left(\begin{array}{cc}A_1 & B\\ C & D_1\end{array}\right), Q=\left(\begin{array}{cc}A_2 & 0\\ 0 & D_2\end{array}\right).$$

We easily check that $C^*D_2=C^*D(I-DD^{ⓖ})=[D^*C]^*[I-DD^{ⓖ}]=0$. Analogously, we have $B^*A_2=0$. Since $A^{\pi}B=0$, we verify that

$$\begin{array}{rcl}A_2B & = & (A-A^2A^{ⓖ})B\\ & = & (A-A^2A^{ⓖ})AA^dB\\ & = & A^2A^dB-A^2A^{ⓖ}AA^dB\\ & = & A^2A^dB-A^2A^dB=0.\end{array}$$

Likewise, we have $D_2C=0$. Then

$$\begin{array}{rcl}P^*Q & = & \left(\begin{array}{cc}A_1^* & C^*\\ B^* & D_1^*\end{array}\right)\left(\begin{array}{cc}A_2 & 0\\ 0 & D_2\end{array}\right)=\left(\begin{array}{cc}0 & C^*D_2\\ B^*A_2 & 0\end{array}\right)=0,\\ QP & = & \left(\begin{array}{cc}A_2 & 0\\ 0 & D_2\end{array}\right)\left(\begin{array}{cc}A_1 & B\\ C & D_1\end{array}\right)=\left(\begin{array}{cc}0 & A_2B\\ D_2C & 0\end{array}\right)=0.\end{array}$$

Obviously, $BD_1=(BD)DD^{ⓖ}=0$. Since $A^{\pi}B=0$, it follows by Lemma 2.1 that $A_1^{\pi}B=[I-A^2A^{ⓖ})A^{ⓖ}]B=(I-AA^{ⓖ})B=0$. Similarly, $D_1^{\pi}C=0$. In view of Lemma 2.4, P has group inverse. In this case,

$$P^{\#}=\left(\begin{array}{cc}\alpha & \beta\\ \gamma & \delta\end{array}\right),$$

where

$$\begin{array}{rcl}\alpha & = & A^{ⓖ},\\ \beta & = & (A^{ⓖ})^2B,\\ \gamma & = & -D^{ⓖ}CA^{ⓖ}+(D^{ⓖ})^2C[I-AA^{ⓖ}],\\ \delta & = & D^{ⓖ}-D^{ⓖ}C(A^{ⓖ})^2B-(D^{ⓖ})^2CA^{ⓖ}B.\end{array}$$

Obviously, Q is quasinilpotent. According to Theorem 1.1, M has generalized group inverse and $M^{ⓖ} = P^{\#}$, as required. □

Corollary 4.4. *If $CA = 0, A^*B = 0, D^*C = 0, A^{\pi}B = 0$ and $D^{\pi}C = 0$, then M has generalized group inverse. In this case,*

$$M^{ⓖ} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

where

$$\begin{aligned} \alpha &= A^{ⓖ} - A^{ⓖ}B(D^{ⓖ})^2C - (A^{ⓖ})^2BD^{ⓖ}C, \\ \beta &= -A^{ⓖ}BD^{ⓖ} + (A^{ⓖ})^2B[I - DD^{ⓖ}], \\ \gamma &= (D^{ⓖ})^2C, \\ \delta &= D^{ⓖ}. \end{aligned}$$

Proof. Applying Theorem 4.3 to the matrix $\begin{pmatrix} D & C \\ B & A \end{pmatrix}$, we prove that $\begin{pmatrix} D & C \\ B & A \end{pmatrix}$ has generalized group inverse. Thus M has generalized group inverse and

$$M^{ⓖ} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} D & C \\ B & A \end{pmatrix}^{ⓖ} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The proof is true by Theorem 4.3. □

Lemma 4.5. *If $A^*B = 0, BD = 0$, then $N = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$ has generalized group inverse. In this case,*

$$N^{ⓖ} = \begin{pmatrix} A^{ⓖ} & 0 \\ 0 & D^{ⓖ} \end{pmatrix}.$$

Proof. Write $N = P + Q$, where

$$P = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}, Q = \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}.$$

We easily check that

$$\begin{aligned} P^*Q &= \begin{pmatrix} A^* & 0 \\ 0 & D^* \end{pmatrix} \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & A^*B \\ 0 & 0 \end{pmatrix} = 0, \\ QP &= \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} = \begin{pmatrix} 0 & BD \\ 0 & 0 \end{pmatrix} = 0, \end{aligned}$$

Since A and D have generalized group inverse, P has generalized group inverse and

$$P^{ⓖ} = \begin{pmatrix} A^{ⓖ} & 0 \\ 0 & D^{ⓖ} \end{pmatrix}.$$

Obviously, Q is nilpotent, and so it is quasinilpotent. According to Lemma 2.2, $M^{\circledg}=P^{\circledg}$, as required. □

Lemma 4.6. *If $D^*C=0, CA=0$, then $N=\begin{pmatrix} A & 0\\ C & D\end{pmatrix}$ has generalized group inverse. In this case,*

$$N^{\circledg}=\begin{pmatrix} A^{\circledg} & 0\\ 0 & D^{\circledg}\end{pmatrix}.$$

Proof. Write $N=P+Q$, where

$$P=\begin{pmatrix} A & 0\\ 0 & D\end{pmatrix}, Q=\begin{pmatrix} 0 & 0\\ C & 0\end{pmatrix}.$$

We easily check that

$$\begin{aligned} P^*Q &= \begin{pmatrix} A^* & 0\\ 0 & D^*\end{pmatrix}\begin{pmatrix} 0 & 0\\ C & 0\end{pmatrix}=0,\\ QP &= \begin{pmatrix} 0 & 0\\ C & 0\end{pmatrix}\begin{pmatrix} A & 0\\ 0 & D\end{pmatrix}=0, \end{aligned}$$

By virtue of Lemma 2.2, M has generalized group inverse and $M^{\circledg}=P^{\circledg}$, as desired. □

We are ready to prove:

Theorem 4.7. *If $BC=0, BD=0, CA=0, CB=0, A^*B=0, D^*C=0$, then M has generalized group inverse. In this case,*

$$M^{\circledg}=\begin{pmatrix} A^{\circledg} & 0\\ 0 & D^{\circledg}\end{pmatrix}.$$

Proof. Write $M=P+Q$, where

$$P=\begin{pmatrix} A & B\\ 0 & 0\end{pmatrix}, Q=\begin{pmatrix} 0 & 0\\ C & D\end{pmatrix}.$$

It is easy to verify that

$$\begin{aligned} P^*Q &= \begin{pmatrix} A^* & 0\\ B^* & 0\end{pmatrix}\begin{pmatrix} 0 & 0\\ C & D\end{pmatrix}=0,\\ PQ &= \begin{pmatrix} A & B\\ 0 & 0\end{pmatrix}\begin{pmatrix} 0 & 0\\ C & D\end{pmatrix}=\begin{pmatrix} BC & BD\\ 0 & 0\end{pmatrix}=0,\\ QP &= \begin{pmatrix} 0 & 0\\ C & D\end{pmatrix}\begin{pmatrix} A & B\\ 0 & 0\end{pmatrix}=\begin{pmatrix} 0 & 0\\ CA & CB\end{pmatrix}=0, \end{aligned}$$

In view of Lemma 4.5, P has generalized group inverse. In view of Lemma 4.6, Q has generalized group inverse. Moreover, we have

$$\begin{array}{rcl} P^{\textcircled{g}} & = & \begin{pmatrix} A^{\textcircled{g}} & (0 \\ 0 & 0 \end{pmatrix}, \\ Q^{\textcircled{g}} & = & \begin{pmatrix} 0 & 0 \\ 0 & D^{\textcircled{g}}. \end{pmatrix}. \end{array}$$

In light of Corollary 3.7,

$$M^{\textcircled{g}} = P^{\textcircled{g}} + Q^{\textcircled{g}} = \begin{pmatrix} A^{\textcircled{g}} & 0 \\ 0 & D^{\textcircled{g}} \end{pmatrix},$$

as asserted. □

Finally, we present some new formulas for the weak group inverse of a block operator matrix over a Banach space.

Theorem 4.8. *Let* $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, *where* $A \in \mathcal{B}(X)^{\textcircled{w}}, B \in \mathcal{B}(X,Y), C \in \mathcal{B}(Y,X), D \in \mathcal{B}(Y)^{\textcircled{w}}$.

(1) If $BD = 0, CA = 0, A^{\pi}B = 0$ and $D^{\pi}C = 0$, then

$$M^{\textcircled{w}} = \begin{pmatrix} A^{\textcircled{w}} & (A^{\textcircled{w}})^2B \\ (D^{\textcircled{w}})^2C & D^{\textcircled{w}} \end{pmatrix}.$$

(2) $BD = 0, A^*B = 0, D^*C = 0, A^{\pi}B = 0$ and $D^{\pi}C = 0$, then

$$M^{\textcircled{w}} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

where

$$\begin{array}{rcl} \alpha & = & A^{\textcircled{w}}, \\ \beta & = & (A^{\textcircled{w}})^2B, \\ \gamma & = & -D^{\textcircled{w}}CA^{\textcircled{w}} + (D^{\textcircled{w}})^2C[I - AA^{\textcircled{w}}], \\ \delta & = & D^{\textcircled{w}} - D^{\textcircled{w}}C(A^{\textcircled{w}})^2B - (D^{\textcircled{w}})^2CA^{\textcircled{w}}B. \end{array}$$

(3) $BD = 0, CA = 0, BC = 0, CB = 0, A^*B = 0, D^*C = 0$, then

$$M^{\textcircled{w}} = \begin{pmatrix} A^{\textcircled{w}} & 0 \\ 0 & D^{\textcircled{w}} \end{pmatrix}.$$

Proof. They are direct consequences from Lemma 2.5, Theorem 4.1, Theorem 4.3 and Theorem 4.7. □

References

[1] J. Benitez; X. Liu and T. Zhu, Additive results for the group inverse in an algebra with applications to block operators, *Linear Multilinear Algebra*, **59**(2011), 279–289.
[2] H. Chen and M. Sheibani, Generalized group inverse in a Banach *-algebra, preprint, 2023. https://doi.org/10.21203/rs.3.rs-3338906/v1.
[3] H. Chen; D. Liu and M. Sheibani, Group invertibility of the sum in rings and its applications, *Georgian Math. J.*, **31**(2024), https://doi.org/10.1515/gmj-2024-2010.
[4] D.S. Cvetković; D.S. Djordjević and Y. Wei, Additive results for the generalized Drazin inverse in a Banach algebra, *Linear Algebra Appl.*, **418**(2006), 53–61.
[5] D.E. Ferreyra; V. Orquera and N. Thome, A weak group inverse for rectangular matrices, *Rev. R. Acad. Cienc. Exactas Fis. Nat., Ser. A Mat.*, **113**(2019), 3727–3740.
[6] R.E. Hartwig and K. Spindelbock, Group inverses and Drazin inverses of bidiagonal and triangular toeplitz matrices, *J. Austral. Math. Soc.*, **24**(1977), 10–34.
[7] N. Mihajlovic and D.S. Djordjevic, On group invrtibility in rings, *Filomat*, **33**(2019), 6141–6150.
[8] N. Mihajlovic, Group inverse and core inverse in Banach and C^*-algebras, *Comm. Algebra*, **48**(2020), 1803–1818.
[9] D. Mosić and D. Zhang, Weighted weak group inverse for Hilbert space operators, *Front. Math. China*, **15**(2020), 709–726.
[10] D. Mosić; P.S. Stanimirovic, Representations for the weak group inverse, *Appl. Math. Comput.*, **397**(2021), Article ID 125957, 19 p.
[11] D.S Rakic; N.C. Dincic and D.S. Djordjevic, Group, Moore-Penrose, core and dual core inverse in rings with involution, *Linear Algebra Appl.*, **463**(2014), 115–133.
[12] H. Wang; J. Chen, Weak group inverse, *Open Math.*, **16**(2018), 1218–1232.
[13] H. Yan; H. Wang; K. Zuo and Y. Chen, Further characterizations of the weak group inverse of matrices and the weak group matrix, *AIMS Math.*, **6**(2021), 9322–9341.
[14] D. Zhang and D. Mosić, Explicit formulae for the generalized Drazin inverse of block matrices over a Banach algebra, *Filomat*, **32**(2018), 5907–5917.
[15] M. Zhou; J. Chen and Y. Zhou, Weak group inverses in proper *-rings, *J. Algebra Appl.*, **19**(2020), DOI:10.1142/S0219498820502382.
[16] M. Zhou; J. Chen; Y. Zhou and N. Thome, Weak group inverses and partial isometries in proper *-rings, *Linear and Multilinear Algebra*, **70**(2021), 1–16.
[17] M. Zhou; J. Chen and X. Zhu, The group inverse and core inverse of sum of two elements in a ring, *Comm. Algebra*, **48**(2020), 676-690.
[18] H. Zou; D. Mosić snd J. Chen, Generalized Drazin invertibility of the product and sum of two elements in a Banach algebra and its applications, *Turk. J. Math.*, **41**(2017), 548-563.

School of Big Data, Fuzhou University of International Studies and Trade, Fuzhou 350202, China
E-mail address: <huanyinchenfz@163.com>