On diophantine equation $ax^4 + by^4 + cz^4 + dw^4 + eu^4 = 0$

Seiji Tomita

Abstract

In this paper, we prove that there are infinitely many integer solutions of $ax^4 + by^4 + cz^4 + dw^4 + eu^4 = 0$ where $a + b + c + d + e = 0$.

1. Introduction

Richmond[9] showed for $ax^4 + by^4 + cz^4 + dw^4 = 0$ that other solutions could be found using a known solution where abcd is a perfect square.Euler[4] found a parametric solution of $x^4 + y^4 = z^4 + w^4$ and he gave $(x, y, z, w) = (134, 133, 158, 59)$. Elkies[3] showed that $x^4 + y^4 + z^4 = w^4$ has infinitely many integer solutions using elliptic curves. First solution is $(x, y, z, w) = (2682440, 15365639, 18796760, 20615673)$. Adam Logan, David McKinnon, Ronald van Luijk[5] proved that $ax^4+by^4+cz^4+dw^4 = 0$ has infinitely many integer solutions where $abcd$ is a perfect square and $a + b + c + d = 0$. Pinch , Swinnerton-Dyer[8], and Bright[1][2] studied the solubility in integers of the equation $ax^4 + by^4 + cz^4 + dw^4 = 0$. Izadi and Nabardi[6] showed $x^4 + y^4 - 2u^4 - 2v^4 = 0$ has infinitely many integer solutions. They used a specific congruent number elliptic curve namely $y^2 = x^3 - 36x^2$. Janfada and Nabardi[7] showed that a necessary condition for n to have an integral solution for the equation $x^4+y^4-nu^4-nv^4 = 0$ and gave a parametric solution. They gave the numerical solutions for $n = 41, 136, 313, 1028, 1201, 3281, \cdots$. So far, the case $n = 4$ in the equation $a_1x_1^4 + a_2x_2^4 + \cdots + a_nx_n^4 = 0$ has been studied by many researchers.

As far as I know, no previous research has investigated this problem for $n > 4$. The aim of this work is to study the infinity of solutions to the equation for $n = 5$, and we will prove that there are infinitely many integer solutions of $ax^4+by^4+cz^4+dw^4+eu^4 = 0$ where $a + b + c + d + e = 0$. To solve the problem, we will consider the case $c + d = 0$ and $c + d \neq 0$. In addition, we show the numerical solutions for $n = 5$ in each case.

2. Solving the diophantine equation $ax^4 + by^4 + cz^4 + dw^4 + eu^4 = 0$

$$ax^4 + by^4 + cz^4 + dw^4 + eu^4 = 0. \quad (1)$$

We describe two methods to prove that there are infinitely many integer solutions of $ax^4 + by^4 + cz^4 + dw^4 + eu^4 = 0$ where $a + b + c + d + e = 0$. We consider the case $c + d = 0$ and $c + d \neq 0$. The explicit parametric solutions are given for each of these cases.

2.1. Case c+d=0.

Let $x = pt + 1,\ y = qt + 1,\ z = rt + 1,\ w = st + 1,\ u = 1$ and $e = -a - b - c - d$. Then equation (1) becomes to below equation. $a, b, c, d, e, p, q, r, s, t \in \mathbb{Z}$.

$$\begin{aligned}(cr^4 + ap^4 + bq^4 + ds^4)t^4 + (4cr^3 + 4ds^3 + 4bq^3 + 4ap^3)t^3 \\ +(6ds^2 + 6cr^2 + 6bq^2 + 6ap^2)t^2 + (4ap + 4bq + 4ds + 4cr)t = 0.\end{aligned} \quad (2)$$

We will choose r and s, such that coefficient of t^2 and t in equation (2) become 0.

$$6ds^2 + 6cr^2 + 6bq^2 + 6ap^2 = 0. \quad (3)$$

$$4ap + 4bq + 4ds + 4cr = 0. \quad (4)$$

First, we solve equation (4) for s. We get:

$$s=\frac{-ap+bq+cr}{d}. \tag{5}$$

Substitute s into equation (3). Then we obtain:

$$(\frac{6c^2}{d}+6c)r^2+\frac{12(ap+bq)cr}{d}+\frac{6(ap+bq)^2}{d}+6ap^2+6bq^2=0. \tag{6}$$

Let $d=-c$. Then we obtain

$$r=\frac{-b^2q^2-a^2p^2-2apbq+ap^2c+bq^2c}{2c(ap+bq)}. \tag{7}$$

Hence, we obtain

$$\begin{aligned}t=&-2(b^5q^5-a^3c^2p^5-b^3q^5c^2-4ab^2q^2c^2p^3+6ap^2b^2q^3c^2\\&-5apc^2b^2q^4+6a^2p^3c^2bq^2-5a^2p^4bqc^2-4a^2p^2bq^3c^2\\&+10a^3p^3b^2q^2+10a^2p^2b^3q^3+5apb^4q^4+a^5p^5+5a^4p^4bq)\\&/(b^5q^6+a^5p^6-4apb^2q^5c^2+ap^2b^4q^4+3ap^2c^2b^2q^4\\&-2ab^2q^2c^2p^4+4apb^4q^5+4a^2p^3b^3q^3-2a^2p^2bq^4c^2\\&+6a^2p^2b^3q^4-4a^2bqc^2p^5+3a^2p^4c^2bq^2+a^4p^4bq^2\\&+6a^3b^2q^2p^4+4a^3p^3b^2q^3+4a^4bqp^5-a^3c^2p^6-c^2b^3q^6).\end{aligned}$$

Finally, we get a parametric solution for equation (1) as follows:

$$\begin{aligned}x=&c(-a^3c^2+a^5)p^6+c(6a^4b-6a^2bc^2)qp^5+c(-a^4b+14a^3b^2+9a^2bc^2-6ab^2c^2)q^2p^4\\&+c(12ab^2c^2-4a^3b^2-8a^2bc^2+16a^2b^3)q^3p^3+c(9ab^4-6a^2b^3+2a^2bc^2-13ab^2c^2)q^4p^2\\&+c(-2b^3c^2-4ab^4+2b^5+4ab^2c^2)q^5p+c(b^3c^2-b^5)q^6,\end{aligned}$$

$$\begin{aligned}y=&c(-a^5+a^3c^2)p^6+c(-4a^4b-2a^3c^2+4a^2bc^2+2a^5)qp^5\\&+c(9a^4b-6a^3b^2+2ab^2c^2-13a^2bc^2)q^2p^4\\&+c(-8ab^2c^2+12a^2bc^2-4a^2b^3+16a^3b^2)q^3p^3\\&+c(-6a^2bc^2+14a^2b^3+9ab^2c^2-ab^4)q^4p^2+c(6ab^4-6ab^2c^2)q^5p+c(b^5-b^3c^2)q^6,\end{aligned}$$

$$\begin{aligned}z=&(a^6-a^4c^2)p^6+(6ba^5-6a^3bc^2)qp^5+(15a^4b^2+6a^3bc^2-2ac^3b^2-2a^2bc^3-9a^2c^2b^2)q^2p^4\\&+(-4a^3bc^2+4a^2bc^3+20a^3b^3-4ab^3c^2+4ac^3b^2+12a^2c^2b^2)q^3p^3\\&+(-2a^2bc^3-9a^2c^2b^2+15a^2b^4-2ac^3b^2+6ab^3c^2)q^4p^2+(-6ab^3c^2+6ab^5)q^5p\\&+(-b^4c^2+b^6)q^6,\end{aligned}$$

$$\begin{aligned}w=&(a^6-a^4c^2)p^6+(6ba^5-6a^3bc^2)qp^5+(15a^4b^2+6a^3bc^2+2ac^3b^2+2a^2bc^3-9a^2c^2b^2)q^2p^4\\&+(20a^3b^3-4a^2bc^3-4a^3bc^2-4ac^3b^2-4ab^3c^2+12a^2c^2b^2)q^3p^3\\&+(2a^2bc^3+15a^2b^4-9a^2c^2b^2+6ab^3c^2+2ac^3b^2)q^4p^2+(-6ab^3c^2+6ab^5)q^5p+(-b^4c^2+b^6)q^6,\end{aligned}$$

$$\begin{aligned}u=&c(-a^3c^2+a^5)p^6+(4a^4b-4a^2bc^2)cqp^5+(6a^3b^2+a^4b+3a^2bc^2-2ab^2c^2)cq^2p^4\\&+(4a^3b^2+4a^2b^3)cq^3p^3+(-2a^2bc^2+6a^2b^3+ab^4+3ab^2c^2)cq^4p^2+(-4ab^2c^2+4ab^4)cq^5p\\&+c(b^5-b^3c^2)q^6.\end{aligned}$$

a,b,c,p,q are arbitrary integers.

Thus, we obtain infinitely many integer solutions for equation (1).

We give the numerical examples.

$$(a,b,c,d,e)=(1,2,3,-3,-3)$$

Table 1: Solutions of $x^4+2y^4+3z^4-3w^4-3u^4=0$

p	q	x	y	z	w	u
2	1	633	3	586	262	639
3	1	2001	489	1402	673	1734
3	2	4467	1149	5329	2413	5487
4	1	57	21	34	18	47
4	2	633	3	586	262	639
4	3	16041	6531	21682	10018	21999
5	1	1797	807	962	557	1458
5	2	47	7	37	17	43

$$(a,b,c,d,e)=(4,5,6,-6,-9)$$

Table 2: Solutions of $4x^4+5y^4+6z^4-6w^4-9u^4=0$

p	q	x	y	z	w	u
2	1	319818	103758	171857	296273	112302
3	1	285954	656934	1227751	108007	1128378
3	2	157062	71196	148673	166169	100536
4	1	10062	6654	12823	6679	12226
4	3	79175154	44858526	86411801	90890777	58091358
5	1	1398426	873654	1432207	934543	1441674
5	2	18	12	7	17	8
5	3	6177774	2341158	5035193	6154937	3413766
5	4	28041	18021	32914	33886	22059

2.2. Case c+d $\neq$ 0.

We consider the quadratic equation (6) for r. Since r must be a rational number, the discriminant must be a perfect square as follows:

$$v^2=-cd(ca+a^2+ad)p^2-2pcdbqa-cd(cbq^2+b^2q^2+bq^2d). \tag{8}$$

Taking $p=0$, then we get $v^2=-cdbq^2(c+b+d)$.

Hence, if $-cdb(c+b+d)$ is a perfect square, we get the rational number v.

We know a parametric solution (b,c,d) by simple calculation.

$(b,c,d)=(m(m-1),(m-1)(m^2-m-n^2),-(m^2-m-n^2)m)$ where m and n are any integer.

Let $V=\frac{v}{q}$ and $U=\frac{p}{q}$. Then we get

$$\begin{aligned}V^2&=m(m-1)(-m^2+m+n^2)^2(-am^2+am+an^2+a^2)U^2+m(m-1)(-m^2+m+n^2)^2(-2am+2am^2)U\\&\quad+m(m-1)(-m^2+m+n^2)^2(m^2n^2-mn^2).\end{aligned} \tag{9}$$

Since the equation (9) has a rational solution $(U,V)=(0,(m-1)(-m^2+m+n^2)mn)$, we can get a parametric solution.

Since the solution is cumbersome to write, and thus we will omit to describe everything. An explicit formula for x is given in Appendix.

Moreover, let $n=1$ to simplify the equation and we get,

$p=-2m(m-1)(m^2-m-1)(m^4a-2m^3a+ma+k)$,

$q=(-m-m^2+3m^3+m^4+m^6-3m^5)a^2+(-3m^6-5m^5+3m^3-2m^2-m+5m^4-m^8+4m^7)a-k^2$,

$r=-m((m^5+3m^4-m^2+m^7-m^3-3m^6)a^2+(-2m^3k+11m^7+8m^3-2m-3m^2+7m^4+m^9-6m^8$
$-m^6+2m^4k-4m^2k+2mk-15m^5+2k)a+mk^2)/(m^2-m-1)$,

$x=(m-1)(5m+3m^2+2)k^8+(m-1)(8m^5-24m^2+16m^2a-40m^4a+16m^6a-16m^3a+8ma-8m^6-8m^3+24m^4-8m)k^7$
$\cdots(m-1)(13m^5a^8+4m^5a^7+2m^4a^8+\cdots 35m^5a^4+13294m^{11}a^4-19723m^{14}a^4)$.

$a,k,m\in\mathbb{Z}$
Thus, we obtain infinitely many integer solutions for equation (1).

We give the numerical examples. $(a,b,c,d,e,m)=(3,2,1,-2,4,2)$ and $(a,b,c,d,e,m)=(5,6,10,-15,-6,3)$.

Table 3: Solutions of $3x^4+2y^4+z^4-2w^4-4u^4=0$

k	x	y	z	w	u
1	11786	21809	9802	3377	18982
2	19	34	29	14	31
3	16858	26569	27962	13769	25822
4	6068	8489	10072	4961	8758
5	864634	1093321	1407482	691913	1189222
6	377	438	599	294	499
7	645298	698053	1000658	490693	828182
8	87988	89569	133352	65369	110122
9	1038298	1003129	1540762	755321	1272758

Table 4: Solutions of $5x^4+6y^4+10z^4-15w^4-6u^4=0$

k	x	y	z	w	u
1	15484078977889	32027173756201	16437555713407	16121827993505	32032418402711
2	246332601439	527262861226	277493664832	267242421755	527602306361
3	65965152867	146212586203	78833243421	74627168515	146420807733
4	63574855174	146028962941	80590394962	75063980330	146392542851
5	150861367	359386063	202848841	186069815	360761393
6	1074051417	2655912478	1532016096	1385121765	2670313883
7	16882871846161	43375924041049	25552267707343	22788503917745	43691011118039
8	6049793801	16165910534	9719065613	8556374545	16316909374
9	70624305643	196495678787	120490091509	104783131435	198783100157
10	17257279	50053546	31286272	26893355	50762081

3. Appendix

x is given by:

$$
\begin{aligned}
x = {}& (m-1)(11m^{6} + 13m^{5} - 90m^{7} - 31m^{25} + 2285m^{17} - 130m^{23} + 3m^{26} + 1415m^{18} - 130m^{20} + 895m^{21} + 116m^{24} - 1920m^{19} - 295m^{22} - 1670m^{12} - 632m^{11} + 689m^{10} \\
& + 291m^{9} + 2591m^{14} + 1145m^{13} - 2572m^{16} - 1826m^{15} - 160m^{8} + 2m^{4})a^{8} \\
& + (m-1)(-7328m^{14} - 12948m^{13} - 32204m^{21} - 10956m^{20} - 4184m^{18} + 26632m^{15} - 41236m^{17} + 10532m^{16} + 45856m^{19} + 2380m^{12} + 4608m^{11} - 1052m^{9} - 104m^{10} \\
& + 3148m^{25} - 2272m^{27} - 13924m^{24} + 20256m^{22} + 9416m^{23} + 2776m^{26} + 740m^{28} - 120m^{29} - 968m^{10}k - 1368m^{9}k + 176m^{8}k + 344m^{7}k + 16m^{6}k - 5784m^{17}k \\
& + 7232m^{15}k + 536m^{16}k - 2288m^{14}k - 5920m^{13}k + 2144m^{12}k + 2504m^{19}k + 3408m^{11}k - 1472m^{20}k + 1408m^{18}k - 40m^{5}k - 8m^{4}k - 232m^{21}k + 440m^{22}k - 172m^{8} \\
& + 104m^{7} - 144m^{23}k + 4m^{5} + 40m^{6} + 16m^{24}k + 8m^{3}0)a^{7} \\
& + (m-1)*(-35832m^{14} - 56028m^{13} - 116772m^{21} - 66450m^{20} + 984m^{18} + 113970m^{15} - 172596m^{17} + 38814m^{16} + 182706m^{19} + 8m^{2}k^{2} + 16842m^{12} + 20742m^{11} \\
& - 5496m^{9} - 4260m^{10} + 17760m^{25} - 9174m^{27} - 55758m^{24} + 93288m^{22} + 24438m^{23} + 8796m^{26} + 3306m^{28} - 582m^{29} + 272m^{25}k + 16m^{22}k^{2} - 960m^{10}k - 10544m^{9}k \\
& - 1200m^{8}k + 2496m^{7}k + 2364m^{8}k^{2} + 720m^{6}k + 808m^{7}k^{2} - 5328m^{10}k^{2} - 50768m^{17}k + 58496m^{15}k + 1776m^{15}k^{2} - 464m^{16}k - 1056m^{9}k^{2} - 8144m^{14}k \\
& - 7368m^{14}k^{2} - 46656m^{13}k + 6304m^{12}k - 992m^{13}k^{2} + 7696m^{12}k^{2} + 26800m^{19}k - 1896m^{17}k^{2} + 848m^{11}k^{2} + 26704m^{11}k - 14496m^{20}k + 4332m^{16}k^{2} \\
& + 12736m^{18}k + 48m^{3}k^{2} + 36m^{4}k^{2} - 224m^{5}k - 572m^{6}k^{2} - 128m^{4}k - 312m^{5}k^{2} - 32m^{26}k + 872m^{19}k^{2} - 5632m^{21}k + 6352m^{22}k - 1156m^{18}k^{2} - 44m^{20}k^{2} \\
& - 80m^{21}k^{2} + 366m^{8} + 906m^{7} - 992m^{23}k - 66m^{5} + 66m^{6} - 12m^{4} - 624m^{24}k + 42m^{3}0 - 16km^{3})a^{6} \\
& + (m-1)(-44572m^{14} - 83864m^{13} - 246840m^{21} - 196824m^{20} + 48972m^{18} + 185268m^{15} - 305912m^{17} + 37880m^{16} + 354516m^{19} + 24m^{2}k^{2} + 21744m^{12} \\
& + 28860m^{11} - 7272m^{9} - 5676m^{10} + 83072m^{25} - 63708m^{27} - 198968m^{24} + 278076m^{22} + 42204m^{23} + 50660m^{26} + 18768m^{28} + 8316m^{29} + 48m^{28}k - 3696m^{25}k \\
& - 528m^{27}k - 368m^{18}k^{3} - 48m^{24}k^{2} - 1020m^{22}k^{2} + 384m^{23}k^{2} + 10176m^{10}k - 22512m^{9}k - 6384m^{8}k + 5136m^{7}k + 7824m^{8}k^{2} + 2256m^{6}k + 4644m^{7}k^{2} \\
& - 18876m^{10}k^{2} - 74064m^{17}k + 102480m^{15}k + 15384m^{15}k^{2} - 22512m^{16}k - 10188m^{9}k^{2} + 16176m^{14}k - 21312m^{14}k^{2} - 92640m^{13}k - 12000m^{12}k \\
& - 18888m^{13}k^{2} - 4376m^{11}k^{3} - 1272m^{10}k^{3} + 26376m^{12}k^{2} + 33792m^{19}k - 6060m^{17}k^{2} + 15912m^{11}k^{2} + 56448m^{11}k + 88m^{5}k^{3} + 4104m^{13}k^{3} \\
& - 3744m^{20}k - 616m^{6}k^{3} + 8580m^{16}k^{2} + 2824m^{9}k^{3} + 1192m^{8}k^{3} - 968m^{7}k^{3} + 1112m^{12}k^{3} + 18816m^{18}k - 1192m^{14}k^{3} + 144m^{3}k^{2} + 128m^{4}k^{3} \\
& + 24m^{3}k^{3} + 36m^{4}k^{2} - 384m^{5}k - 1584m^{6}k^{2} - 336m^{4}k - 1260m^{5}k^{2} + 2208m^{26}k - 588m^{19}k^{2} - 13920m^{21}k - 2208m^{15}k^{3} - 3696m^{22}k + 1016m^{16}k^{3} \\
& - 1464m^{18}k^{2} + 64m^{19}k^{3} + 448m^{17}k^{3} + 1416m^{20}k^{2} + 564m^{21}k^{2} + 580m^{8} + 1188m^{7} + 9744m^{23}k - 92m^{5} + 64m^{6} - 16m^{4} - 816m^{24}k - 9732m^{3}0 \\
& + 4192m^{3}1 + 136m^{3}3 - 1012m^{3}2 - 8m^{3}4 - 48km^{3})a^{5} \\
& + (m-1)(-19723m^{14} - 41569m^{13} - 170769m^{21} - 127172m^{20} + 33597m^{18} + 98664m^{15} - 175269m^{17} + 15838m^{16} + 220314m^{19} + 24m^{2}k^{2} + 9168m^{12} \\
& + 13294m^{11} - 3114m^{9} - 2238m^{10} + 55011m^{25} - 51786m^{27} - 152922m^{24} + 192077m^{22} + 41904m^{23} + 49077m^{26} + 11358m^{28} + 9578m^{29} + 96m^{25}k^{2} \\
& - 2416m^{28}k - 8176m^{25}k + 6800m^{27}k - 3624m^{18}k^{3} - 64m^{21}k^{3} + 288m^{20}k^{3} - 900m^{24}k^{2} - 2280m^{22}k^{2} + 2928m^{23}k^{2} + 26368m^{10}k - 20528m^{9}k \\
& - 9792m^{8}k + 4240m^{7}k + 11940m^{8}k^{2} + 2544m^{6}k + 6576m^{7}k^{2} - 34488m^{10}k^{2} - 7920m^{17}k + 83856m^{15}k + 27984m^{15}k^{2} - 161504m^{16}k - 15168m^{9}k^{2} \\
& + 112944m^{14}k - 68184m^{14}k^{2} - 90944m^{13}k - 59440m^{12}k - 28656m^{13}k^{2} - 24360m^{11}k^{3} - 1646m^{11}k^{4} - 2280m^{10}k^{3} + 59988m^{12}k^{2} - 82592m^{19}k \\
& + 318m^{5}k^{4} - 20832m^{17}k^{2} + 23664m^{11}k^{2} + 274m^{6}k^{4} + 55008m^{11}k + 528m^{5}k^{3} + 26440m^{13}k^{3} - 57168m^{20}k - 1184m^{6}k^{3} + 52572m^{16}k^{2} \\
& + 1838m^{10}k^{4} + 13576m^{9}k^{3} + 1578m^{9}k^{4} + 2088m^{8}k^{3} - 1166m^{8}k^{4} - 4264m^{7}k^{3} - 994m^{7}k^{4} + 4296m^{12}k^{3} + 147712m^{18}k - 8520m^{14}k^{3} \\
& + 144m^{3}k^{2} + 248m^{4}k^{3} + 24m^{3}k^{3} + 18m^{4}k^{4} - 26m^{3}k^{4} - 12m^{4}k^{2} - 240m^{5}k - 1944m^{6}k^{2} - 320m^{4}k - 1584m^{5}k^{2} - 4k^{4}m^{2} - 7776m^{26}k \\
& + 12336m^{19}k^{2} + 96512m^{21}k - 16096m^{15}k^{3} - 28720m^{22}k + 8688m^{16}k^{3} - 1346m^{12}k^{4} - 29064m^{18}k^{2} + 1186m^{13}k^{4} + 352m^{19}k^{3} - 448m^{15}k^{4} \\
& + 322m^{14}k^{4} + 3864m^{17}k^{3} + 96m^{16}k^{4} + 12108m^{20}k^{2} - 7248m^{21}k^{2} + 432m^{29}k + 204m^{8} + 476m^{7} - 36336m^{23}k - 32m^{3}0k - 35m^{5} + 27m^{6} - 6m^{4} \\
& + 37536m^{24}k - 8718m^{3}0 + 3400m^{3}1 + 93m^{3}3 - 754m^{3}2 - 5m^{3}4 - 48km^{3})a^{4} \\
& + (m-1)*(8m^{2}k^{2} + 964m^{25}k^{2} - 992m^{28}k + 4416m^{25}k - 56m^{4}k^{5} + 336m^{5}k^{5} - 104m^{3}k^{5} + 3912m^{27}k - 24m^{2}k^{5} - 84m^{26}k^{2} - 10512m^{18}k^{3} \\
& - 192m^{21}k^{3} - 840m^{20}k^{3} - 4268m^{24}k^{2} + 1836m^{22}k^{2} + 7752m^{23}k^{2} + 15752m^{10}k - 7392m^{9}k - 4744m^{8}k + 1272m^{7}k + 7068m^{8}k^{2} + 992m^{6}k \\
& + 2600m^{7}k^{2} - 24132m^{10}k^{2} - 11152m^{17}k + 48824m^{15}k + 3176m^{15}k^{2} - 133272m^{16}k - 4692m^{9}k^{2} + 85256m^{14}k - 82380m^{14}k^{2} - 44528m^{13}k \\
& - 41096m^{12}k + 1140m^{13}k^{2} - 31320m^{11}k^{3} - 12076m^{11}k^{4} - 23448m^{10}k^{3} + 52788m^{12}k^{2} + 432m^{6}k^{5} - 51208m^{19}k + 1924m^{5}k^{4} - 21708m^{17}k^{2} \\
& + 3056m^{11}k^{2} - 316m^{6}k^{4} + 23144m^{11}k + 192m^{5}k^{3} + 32760m^{13}k^{3} - 88792m^{20}k - 3192m^{6}k^{3} + 93236m^{16}k^{2} + 2844m^{10}k^{4} + 15960m^{9}k^{3} \\
& + 10980m^{9}k^{4} + 10704m^{8}k^{3} - 1348m^{8}k^{4} - 4008m^{7}k^{3} - 6260m^{7}k^{4} + 38664m^{12}k^{3} + 143304m^{18}k - 46344m^{14}k^{3} + 48m^{3}k^{2} + 432m^{4}k^{3} \\
& + 72m^{3}k^{3} + 400m^{4}k^{4} - 212m^{3}k^{4} - 12m^{4}k^{2} - 40m^{5}k - 988m^{6}k^{2} - 104m^{4}k - 636m^{5}k^{2} + 568m^{10}k^{5} - 64k^{4}m^{2} - 744m^{8}k^{5} + 232m^{9}k^{5} \\
& - 8168m^{26}k + 37008m^{19}k^{2} + 77616m^{21}k - 13152m^{15}k^{3} + 12056m^{22}k + 34104m^{16}k^{3} - 1372m^{12}k^{4} - 70812m^{18}k^{2} + 8356m^{13}k^{4} \\
& + 6192m^{19}k^{3} - 384m^{7}k^{5} - 2784m^{15}k^{4} - 1260m^{14}k^{4} + 48m^{22}k^{3} - 6120m^{17}k^{3} - 176m^{12}k^{5} - 144m^{11}k^{5} - 104m^{17}k^{4} + 1316m^{16}k^{4} \\
& - 24m^{18}k^{4} + 27532m^{20}k^{2} - 28500m^{21}k^{2} + 64m^{13}k^{5} + 136m^{29}k - 44792m^{23}k - 8m^{3}0k + 19624m^{24}k - 16km^{3})a^{3} \\
& + (m-1)(8k^{6} + 32mk^{6} + 672m^{4}k^{5} + 1440m^{5}k^{5} - 240m^{3}k^{5} - 112k^{6}m^{3} + 12k^{6}m^{2} - 240m^{2}k^{5} + 2544m^{18}k^{3} + 1272m^{21}k^{3} - 5160m^{20}k^{3} \\
& + 984m^{11}k^{3} - 9738m^{11}k^{4} - 30000m^{10}k^{3} - 432m^{6}k^{5} + 612m^{5}k^{4} - 124m^{8}k^{6} + 16m^{9}k^{6} - 2508m^{6}k^{4} - 360m^{5}k^{3} - 13656m^{13}k^{3} - 96m^{14}k^{5} \\
& - 48m^{7}k^{6} - 2904m^{6}k^{3} - 8778m^{10}k^{4} + 1008m^{9}k^{3} + 8610m^{9}k^{4} + 12048m^{8}k^{3} + 5586m^{8}k^{4} + 240m^{7}k^{3} - 3810m^{7}k^{4} + 48360m^{12}k^{3} - 47976m^{14}k^{3} \\
& + 312m^{4}k^{3} + 72m^{3}k^{3} + 570m^{4}k^{4} + 54m^{3}k^{4} + 120m^{5}k^{6} - 136m^{4}k^{6} + 240m^{10}k^{5} + 16m^{10}k^{6} - 12k^{4}m^{2} - 288m^{8}k^{5} + 2784m^{9}k^{5} + 29520m^{15}k^{3} \\
& + 22512m^{16}k^{3} + 10194m^{12}k^{4} + 4818m^{13}k^{4} + 8424m^{19}k^{3} + 224m^{6}k^{6} - 2736m^{7}k^{5} + 408m^{15}k^{4} - 7350m^{14}k^{4} - 120m^{22}k^{3} - 27120m^{17}k^{3} \\
& + 192m^{12}k^{5} - 1584m^{11}k^{5} - 1002m^{17}k^{4} + 2208m^{16}k^{4} + 138m^{18}k^{4} + 336m^{13}k^{5} - 48mk^{5})a^{2} \\
& + (m-1)(8k^{6} + 8mk^{7} + 32mk^{6} - 16m^{3}k^{7} + 16m^{2}k^{7} + 1008m^{4}k^{5} + 600m^{5}k^{5} - 24m^{3}k^{5} - 252k^{6}m^{3} - 44k^{6}m^{2} - 216m^{2}k^{5} + 8112m^{11}k^{4} - 2208m^{6}k^{5} \\
& - 1344m^{5}k^{4} + 44m^{8}k^{6} + 212m^{9}k^{6} - 1008m^{6}k^{4} + 72m^{14}k^{5} - 552m^{7}k^{6} - 2784m^{10}k^{4} - 6672m^{9}k^{4} + 2688m^{8}k^{4} + 3696m^{7}k^{4} + 48m^{4}k^{4} + 240m^{3}k^{4} \\
& + 568m^{5}k^{6} + 4m^{4}k^{6} + 16k^{7}m^{6} - 2400m^{10}k^{5} - 68m^{10}k^{6} + 48k^{4}m^{2} + 2976m^{8}k^{5} + 648m^{9}k^{5} - 5712m^{13}k^{4} + 56m^{6}k^{6} - 1176m^{7}k^{5} + 1344m^{15}k^{4} \\
& + 2352m^{14}k^{4} + 816m^{12}k^{5} + 408m^{11}k^{5} + 432m^{17}k^{4} - 1392m^{16}k^{4} - 48m^{18}k^{4} - 456m^{13}k^{5} - 40k^{7}m^{4} - 48mk^{5})a \\
& + (m-1)(-8k^{7}m^{6} + 3m^{2}k^{8} + 5k^{8}m + 24k^{7}m^{4} - 8mk^{7} + 8k^{7}m^{5} - 8m^{3}k^{7} + 2k^{8} - 24m^{2}k^{7}).
\end{aligned}
$$

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