ADDITIVE PROPERTY OF GENERALIZED CORE-EP INVERSE IN BANACH *-ALGEBRA

HUANYIN CHEN

ABSTRACT. We present new necessary and sufficient conditions under which the sum of two generalized core-EP invertible elements in a Banach *-algebra has generalized core-EP inverse. As an application, the generalized core-EP invertibility for the matrices with generalized core-EP invertible entries is investigated.

1. INTRODUCTION

Let $\mathcal{A}$ be a Banach algebra with an involution $*$. An element $a \in \mathcal{A}$ has group inverse provided that there exists $x \in \mathcal{A}$ such that

$$xa^2 = a, ax^2 = x, ax = xa.$$

Such x is unique if exists, denoted by $a^{\#}$, and called the group inverse of a. Evidently, a square complex matrix A has group inverse if and only if $rank(A) = rank(A^2)$.

An element $a \in \mathcal{A}$ has core inverse if there exists $x \in \mathcal{A}$ such that

$$xa^2 = a, ax^2 = x, (ax)^* = ax.$$

If such x exists, it is unique, and denote it by $a^{\textcircled{\#}}$. As is well known, an element $a \in \mathcal{A}$ has core inverse if and only if $a \in \mathcal{A}$ has group inverse and it has $(1,3)$-inverse. Here, $a \in \mathcal{A}$ has $(1,3)$ inverse provided that there exists some $x \in \mathcal{A}$ such that $axa = a$ and $(ax)^* = ax$.

In [10], Gao and Chen extended the core inverse and introduced the core-EP inverse (i.e., pseudo core inverse). An element $a \in \mathcal{A}$ has core-EP inverse if there exist $x \in \mathcal{A}$ and $k \in \mathbb{N}$ such that

$$ax^2 = x, (ax)^* = ax, xa^{k+1} = a^k.$$

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If such x exists, it is unique, and denote it by $a^{\textcircled{\dagger}}$. Evidently, $a \in \mathcal{A}$ has core-EP inverse if and only if a^n has core inverse for some $n \in \mathbb{N}$.

Many authors have investigated group, core and core-EP inverses from many different views, e.g., [1, 9, 11, 12, 13, 16, 17, 18, 19, 20, 22]. The additive properties of generalized inverses mentioned above are attractive.

We use $\mathcal{A}^{\#}, \mathcal{A}^{\textcircled{\#}}$ and $\mathcal{A}^{\textcircled{\dagger}}$ to denote the set of all group invertible, core invertible and core-EP invertible elements in $\mathcal{A}$, respectively.

Let $a, b \in \mathcal{A}^{\#}$. In [**?**]B, Benítez, Liu and Zhu proved that $a + b \in \mathcal{A}^{\#}$ if $ab = 0$. The additive property of group invertible was studied in [**?**]ZCZ under the condition $abb^{\#} = baa^{\#}$. Recently, the authors investigated the additive property of group inverses under the wider condition $ab(1 - aa^{\#}) = 0$ (see [6, Theorem 2.3]).

Let $a, b \in \mathcal{A}^{\textcircled{\#}}$. In [20, Theorem 4.3], Xue, Chen and Zhang proved that $a + b \in \mathcal{A}^{\textcircled{\#}}$ if $ab = 0$ and $a^*b = 0$. In [22, Theorem 4.1], Zhou et al. considered the core inverse of $a + b$ under the conditions $a^2a^{\textcircled{\#}}b^{\textcircled{\#}}b = baa^{\textcircled{\#}}, ab^{\textcircled{\#}}b = aa^{\textcircled{\#}}b$. In [7, Theorem 2.5], the authors studied the additive property of core inverses under the conditions $ab = ba$ and $a^*b = ba^*$.

Let $a, b \in \mathcal{A}^{\textcircled{\dagger}}$. In [10, Theorem 4.4], Gao and Chen proved that $a + b$ has core-EP inverse if $ab = ba = 0$ and $a^*b = 0$.

As a natural generalization of core-EP invertibility, the authors introduced the generalized core-EP inverse in Banach algebra with an involution (see [4, 5]). An element $a \in \mathcal{A}$ is generalized core-EP invertible if there exists $x \in \mathcal{A}$ such that

$$ax^2 = x, (ax)^* = ax, \lim_{n\to\infty} ||a^n - xa^{n+1}||^{\frac{1}{n}} = 0.$$

If such x exists, it is unique, and denote it by $a^{\textcircled{d}}$.

Recall that an element $a \in \mathcal{A}$ has generalized Drazin inverse if there exists $x \in \mathcal{A}$ such that

$$ax^2 = x, ax = xa, a - a^2x \in \mathcal{A}^{qnil}.$$

Here, $\mathcal{A}^{qnil} = \{a \in \mathcal{A} \mid 1 + \lambda a \in \mathcal{A}^{-1}\}$. Such x is unique, if exists, and denote it by a^d. The generalized Drazin inverse plays an important role in ring and matrix theory (see [3]).

We use $\mathcal{A}^d, \mathcal{A}^{\textcircled{d}}$ and $\mathcal{A}^{(1,3)}$ to denote the set of all generalized Drazin invertible, generalized core-EP invertible and $(1,3)$-invertible elements in $\mathcal{A}$, respectively. We list several characterizations of generalized core-EP inverse.

Theorem 1.1. *(see [4, 5, 8])Let $\mathcal{A}$ be a Banach *-algebra, and let $a \in \mathcal{A}$. Then the following are equivalent:*

(1) $a \in \mathcal{A}^{\textcircled{d}}$.

(2) There exist $x, y \in \mathcal{A}$ such that
$$a = x + y, x^*y = yx = 0, x \in \mathcal{A}^{\#}, y \in \mathcal{A}^{qnil}.$$
(3) There exists a projection $p \in \mathcal{A}$ such that
$$a + p \in \mathcal{A}^{-1}, pa = pap \in \mathcal{A}^{qnil}.$$
(4) $a \in \mathcal{A}^d$ and $a^d \in \mathcal{A}^{\#}$. In this case, $a^{ⓓ} = (a^d)^2(a^d)^{\#}$.
(5) $a \in \mathcal{A}^d$ and $a^d \in \mathcal{A}^{(1,3)}$. In this case, $a^{ⓓ} = (a^d)^2(a^d)^{(1,3)}$.

Let $a, b \in \mathcal{A}^{ⓓ}$. In [8, Theorem 3.4], the authors proved that $a + b \in \mathcal{A}^{ⓓ}$ provided that $ab = 0, a^*b = 0$ and $ba = 0$. The motivation of this paper is to present new additive results for the generalized core-EP inverses. We shall give necessary and sufficient conditions under which the sum of two generalized core-EP invertible elements has generalized core-EP inverse. As an application, the generalized core-EP invertibility for the matrices with generalized core-EP invertible entries is investigated.

Throughout the paper, all Banach *-algebras are complex with an identity. An element $p \in \mathcal{A}$ is a projection if $p^2 = p = p^*$. Let $a^{\pi} = 1 - aa^d$ and $a^{\sigma} = 1 - aa^{ⓓ}$ for $a \in \mathcal{A}^{ⓓ}$. Let $a, p^2 = p \in \mathcal{A}$. Then a has the Pierce decomposition relative to p, and we denote it by $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}_p$.

2. KEY LEMMAS

To prove the main results, some lemmas are needed. We begin with

Lemma 2.1. *([8, Lemma 3.2])) Let $a, b \in \mathcal{A}^{ⓓ}$. If $ab = ba$ and $a^*b = ba^*$, then $a^{ⓓ}b = ba^{ⓓ}$.*

Lemma 2.2. *([8, Theorem 3.3])) Let $a, b \in \mathcal{A}^{ⓓ}$. If $ab = ba$ and $a^*b = ba^*$, then $ab \in \mathcal{A}^{ⓓ}$ and $(ab)^{ⓓ} = a^{ⓓ}b^{ⓓ}$.*

Lemma 2.3. *Let $a \in \mathcal{A}^{ⓓ}$ and $b \in \mathcal{A}^{qnil}$. If $a^*b = 0$ and $ba = 0$, then $a + b \in \mathcal{A}^{ⓓ}$. In this case,*
$$(a + b)^{ⓓ} = a^{ⓓ}.$$

Proof. Since $a \in \mathcal{A}^{ⓓ}$, by virtue of Theorem 1.1, there exist $x \in \mathcal{A}^{\#}$ and $y \in \mathcal{A}^{qnil}$ such that $a = x + y, x^*y = 0, yx = 0$. As in the proof of [5, Theorem 2.1], $x = aa^{ⓓ}a$ and $y = a - aa^{ⓓ}a$. Then $a = x + (y + b)$. Since $by = b(a - aa^{ⓓ}a) = 0$, it follows by [14, Theorem 2.2] that $y + b \in \mathcal{A}^{qnil}$. We directly verify that
$$\begin{aligned} x^*(y + b) &= x^*y + x^*b = (a^{ⓓ}a)^*(a^*b) = 0, \\ (y + b)x &= yx + (ba)a^{ⓓ}a = 0. \end{aligned}$$

In light of Theorem 1.1, $a+b \in \mathcal{A}^{\textcircled{d}}$. In this case,

$$(a+b)^{\textcircled{d}} = x^{\textcircled{\#}} = a^{\textcircled{d}},$$

as asserted. $\square$

Lemma 2.4. *Let $a \in \mathcal{A}^{\textcircled{d}}$ and $m \in \mathbb{N}$. Then $a^{\textcircled{d}} a^m a^{\textcircled{d}} = a^{m-1} a^{\textcircled{d}}$.*

Proof. Since $a(a^{\textcircled{d}})^2 = a^{\textcircled{d}}$, we see that $a^{\textcircled{d}} = a^{n-m+1}(a^{\textcircled{d}})^{n-m}$ for any $n \geq m+1$. Then

$$(a^{m-1} - a^{\textcircled{d}} a^m) a^{\textcircled{d}} = (a^n - a^{\textcircled{d}} a^{n+1})(a^{\textcircled{d}})^{n-m}.$$

Hence,

$$||(a^{m-1} - a^{\textcircled{d}} a^m) a^{\textcircled{d}}||^{\frac{1}{n}} \leq ||a^n - a^{\textcircled{d}} a^{n+1}||^{\frac{1}{n}} ||a^{\textcircled{d}}||^{\frac{n-m}{n}}.$$

Since $\lim\limits_{n\to\infty} ||a^n - a^{\textcircled{d}} a^{n+1}||^{\frac{1}{n}} = 0$, we deduce that

$$\lim_{n\to\infty} ||(a^{m-1} - a^{\textcircled{d}} a^m) a^{\textcircled{d}}||^{\frac{1}{n}} = 0.$$

Therefore $a^{m-1} a^{\textcircled{d}} = a^{\textcircled{d}} a^m a^{\textcircled{d}}$. $\square$

Lemma 2.5. *Let $a \in \mathcal{A}^{\textcircled{d}}$ and $b \in \mathcal{A}$. Then the following are equivalent:*

(1) $(1 - a^{\textcircled{d}} a) b = 0$.
(2) $(1 - a a^{\textcircled{d}}) b = 0$.
(3) $a^{\pi} b = 0$.

Proof. (1) $\Rightarrow$ (3) Since $(1-a^{\textcircled{d}}a)b = 0$, we have $b = a^{\textcircled{d}}ab$. In view of Theorem 1.1, $a^{\textcircled{d}} = (a^d)^2(a^d)^{\textcircled{\#}}$. Thus, $a^{\pi}b = (1-aa^d)b = (1-aa^d)(a^d)^2(a^d)^{\textcircled{\#}}ab = 0$.

(3) $\Rightarrow$ (2) Since $a^d = (a^d)^2 a = a^d[a^d(a^d)^{\textcircled{\#}}a^d]a = [(a^d)^2(a^d)^{\textcircled{\#}}]aa^d = a^{\textcircled{d}}aa^d$. Then $b = aa^d b = a^{\textcircled{d}}a^2a^d b$; and so $(1-aa^{\textcircled{d}})b = (1-aa^{\textcircled{d}})a^{\textcircled{d}}a^2a^d b = 0$, as desired.

(2) $\Rightarrow$ (1) In view of Lemma 2.4, $aa^{\textcircled{d}} = a^{\textcircled{d}}a^2a^{\textcircled{d}}$. Since $(1-aa^{\textcircled{d}})b = 0$, we get $b = aa^{\textcircled{d}}b$. Therefore $(1-a^{\textcircled{d}}a)b = (1-a^{\textcircled{d}}a)aa^{\textcircled{d}}b = (a-a^{\textcircled{d}}a^2)a^{\textcircled{d}}b = 0$, as asserted. $\square$

Let $\mathcal{A}$ be a Banach *-algebra. Then $M_2(\mathcal{A})$ is a Banach *-algebra with *-transpose as the involution. We come now to generalized EP-inverse of a triangular matrix over $\mathcal{A}$.

Lemma 2.6. *Let $p \in \mathcal{A}$ be a projection and $x = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}_p$.*

(1) If $a, d \in \mathcal{A}^d$, then $x \in M_2(\mathcal{A})_p^d$ and $x^d = \begin{pmatrix} a^d & z \\ 0 & d^d \end{pmatrix}_p$, where

$$z = \sum_{i=0}^{\infty}(a^d)^{i+2}bd^id^\pi + \sum_{i=0}^{\infty} a^i a^\pi b(d^d)^{i+2} - a^dbd^d.$$

(2) If $a, d \in \mathcal{A}^{\textcircled{\#}}$ and $a^\pi b = 0$, then $x \in M_2(\mathcal{A})_p^{\textcircled{\#}}$ and

$$x^{\textcircled{\#}} = \begin{pmatrix} a^{\textcircled{\#}} & -a^{\textcircled{\#}}bd^{\textcircled{\#}} \\ 0 & d^{\textcircled{\#}} \end{pmatrix}_p.$$

Proof. See [23, Lemma 2.1] and [19, Theorem 2.5]. □

We are ready to prove the following lemma which is repeatedly used in the sequel.

Lemma 2.7. *Let $p \in \mathcal{A}$ be a projection and $x = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}_p \in M_2(\mathcal{A})_p$ with $a, d \in \mathcal{A}^{\textcircled{d}}$. If*

$$\sum_{i=0}^{\infty} a^i a^\pi b(d^d)^{i+2} = 0,$$

then $x \in M_2(\mathcal{A})_p^{\textcircled{d}}$ and

$$x^{\textcircled{d}} = \begin{pmatrix} a^{\textcircled{d}} & z \\ 0 & d^{\textcircled{d}} \end{pmatrix}_p,$$

where $z = -a^dbd^{\textcircled{d}}$.

Proof. In view of Theorem 1.1, $a, d \in \mathcal{A}^d$ and $a^d, d^d \in \mathcal{A}^{\textcircled{\#}}$. By virtue of Lemma 2.6, we have

$$x^d = \begin{pmatrix} a^d & s \\ 0 & d^d \end{pmatrix},$$

where

$$s = \sum_{i=0}^{\infty}(a^d)^{i+2}bd^id^\pi + \sum_{i=0}^{\infty} a^i a^\pi b(d^d)^{i+2} - a^dbd^d.$$

By hypothesis, we get $s = \sum\limits_{i=0}^{\infty}(a^d)^{i+2}bd^id^\pi - a^dbd^d$. Since $(a^d)^\pi s = (1 - a^da^2a^d)s = p^\pi s = a^\pi[\sum\limits_{i=0}^{\infty}(a^d)^{i+2}bd^id^\pi - a^dbd^d] = 0$. In view of [19, Lemma

2.4], we have $[1-a^d(a^d)^{\oplus}]s=0$. Then it follows by Lemma 2.6 that

$$(x^d)^{\oplus}=\begin{pmatrix} (a^d)^{\oplus} & t \\ 0 & (d^d)^{\oplus} \end{pmatrix},$$

where $t=-(a^d)^{\oplus}s(d^d)^{\oplus}$. Hence, $t=-(a^d)^{\oplus}[\sum\limits_{i=0}^{\infty}(a^d)^{i+2}bd^id^{\pi}-a^dbd^d](d^d)^{\oplus}=(a^d)^{\oplus}a^dbd^d(d^d)^{\oplus}$. Then we have

$$(x^d)^2=\begin{pmatrix} (a^d)^2 & w \\ 0 & (d^d)^2 \end{pmatrix},$$

where $w=\sum\limits_{i=0}^{\infty}(a^d)^{i+3}bd^id^{\pi}-(a^d)^2bd^d-a^db(d^d)^2$. Therefore

$$\begin{array}{rcl} x^{ⓓ} & = & (x^d)^2(x^d)^{\oplus} \\ & = & \begin{pmatrix} (a^d)^2 & w \\ 0 & (d^d)^2 \end{pmatrix}\begin{pmatrix} (a^d)^{\oplus} & t \\ 0 & (d^d)^{\oplus} \end{pmatrix} \\ & = & \begin{pmatrix} a^{ⓓ} & z \\ 0 & d^{ⓓ} \end{pmatrix}, \end{array}$$

where

$$\begin{array}{rcl} z & = & (a^d)^2t+w(d^d)^{\oplus} \\ & = & (a^d)^2[(a^d)^{\oplus}a^dbd^d(d^d)^{\oplus}]-[(a^d)^2bd^d+a^db(d^d)^2](d^d)^{\oplus} \\ & = & (a^d)^2bd^d(d^d)^{\oplus}-a^d(a^db+bd^d)d^d(d^d)^{\oplus} \\ & = & (a^d)^2bd^d(d^d)^{\oplus}-(a^d)^2bd^d(d^d)^{\oplus}-a^d[b(d^d)^2(d^d)^{\oplus}] \\ & = & -a^dbd^{ⓓ} \end{array}$$

This completes the proof. □

Lemma 2.8. *Let* $\alpha=\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}_p\in M_2(\mathcal{A})_p$ *with* $a,d\in\mathcal{A}^{ⓓ}$. *If* $a^{\pi}bd^{ⓓ}=0$, *then* $\alpha\in M_2(\mathcal{A})^{ⓓ}$ *and*

$$\alpha^{ⓓ}=\begin{pmatrix} a^{ⓓ} & -a^{ⓓ}bd^{ⓓ} \\ 0 & d^{ⓓ} \end{pmatrix}_p.$$

Proof. Since $a^{\pi}bd^{ⓓ}=0$, it follows by Theorem 1.1 that $a^{\pi}b(d^d)^2(d^d)^{\oplus}=0$; hence,

$$a^{\pi}bd^d=[a^{\pi}b(d^d)^2(d^d)^{\oplus}]b^db=0.$$

By using Lemma 2.5, we have $(1-aa^{ⓓ})bd^{ⓓ}=0$, and so $bd^{ⓓ}=aa^{ⓓ}bd^{ⓓ}$. Then

$$a^dbd^{ⓓ}=aa^da^{ⓓ}bd^{ⓓ}=a^{ⓓ}bd^{ⓓ}.$$

In light of Lemma 2.7,

$$\alpha^{ⓓ} = \begin{pmatrix} a^{ⓓ} & -a^{ⓓ}bd^{ⓓ} \\ 0 & d^{ⓓ} \end{pmatrix},$$

as asserted. □

3. MAIN RESULTS

This section is devoted to investigate the generalized core-EP inverse of the sum of two generalized core-EP invertible elements in a Banach *-algebra. We come now to establish additive property of generalized core-EP inverse under orthogonal conditions.

Theorem 3.1. *Let $a, b, a^{\sigma}b \in \mathcal{A}^{ⓓ}$. If*

$$a^{\pi}ab = 0, a^{\pi}ba = 0 \text{ and } a^{\pi}b^*a = 0,$$

then the following are equivalent:

(1) $a + b \in \mathcal{A}^{ⓓ}$ and $a^{\pi}(a + b)^{ⓓ}aa^{ⓓ} = 0$.
(2) $(a + b)aa^{ⓓ} \in \mathcal{A}^{ⓓ}$ and

$$\sum_{i=0}^{\infty}(a + b)^i(a + b)^{\pi}aa^{ⓓ}(a + b)a^{\sigma}(b^d)^{i+2} = 0.$$

In this case,

$$(a + b)^{ⓓ} = [(a + b)aa^{ⓓ}]^{ⓓ} + (a^{\sigma}b)^{ⓓ} - (a + b)^d aa^{ⓓ}(a + b)(a^{\sigma}b)^{ⓓ}.$$

Proof. (1) ⇒ (2) Let $p = aa^{ⓓ}$. By hypothesis and Lemma 2.5, we have $p^{\pi}ab = 0, p^{\pi}ba = 0$ and $p^{\pi}b^*a = 0$. Hence, $p^{\pi}bp = (p^{\pi}ba)a^{ⓓ} = 0$,

$$p^{\pi}ap = (1 - aa^{ⓓ})a^2a^{ⓓ} = 0$$

and

$$pap^{\pi} = aa^{ⓓ}a(1 - aa^{ⓓ}) = aa^{ⓓ}a - a^2a^{ⓓ}.$$

Then we have

$$a = \begin{pmatrix} a_1 & a_2 \\ 0 & a_4 \end{pmatrix}_p, b = \begin{pmatrix} b_1 & b_2 \\ 0 & b_4 \end{pmatrix}_p.$$

Hence

$$a + b = \begin{pmatrix} a_1 + b_1 & a_2 + b_2 \\ 0 & a_4 + b_4 \end{pmatrix}_p.$$

Here, $a_1 = aa^{ⓓ}a^2a^{ⓓ} = a^2a^{ⓓ}$ and $b_1 = aa^{ⓓ}baa^{ⓓ} = baa^{ⓓ}$.

Since $a^{\pi}(a+b)^{\textcircled{d}}aa^{\textcircled{d}}=0$, it follows by Lemma 2.5 that $p^{\pi}(a+b)^{\textcircled{d}}aa^{\textcircled{d}}=0$. Write

$$(a+b)^{\textcircled{d}}=\left(\begin{array}{cc}\alpha & \gamma\\ 0 & \beta\end{array}\right)_p.$$

Then

$$(a_1+b_1)\alpha^2=\alpha, [(a_1+b_1)\alpha]^*=(a_1+b_1)\alpha, \lim\limits_{n\to\infty}||(a_1+b_1)^n-\alpha(a_1+b_1)^{n+1}||^{\frac{1}{n}}=0.$$

We infers that $(a_1+b_1)^{\textcircled{d}}=\alpha$, as required.

(2) $\Rightarrow$ (1) Let $p=aa^{\textcircled{d}}$. Construct $a_i,b_i(i=1,2,4)$ as in (1) $\Rightarrow$ (2). Then

$$a+b=\left(\begin{array}{cc}a_1+b_1 & a_2+b_2\\ 0 & a_4+b_4\end{array}\right)_p.$$

Hence $a_1+b_1=(a+b)aa^{\textcircled{d}}$. Since $p^{\pi}(a+b)=a^{\pi}a+p^{\pi}b$ and $(p^{\pi}b)(p^{\pi}a)=0$, it follows by [3, Lemma 15.2.2] that $p^{\pi}(a+b)\in \mathcal{A}^d$. As $p^{\pi}(a+b)aa^{\textcircled{d}}=0$, by using [21, Lemma 2.2],

$$(a_1+b_1)^d=[(a+b)aa^{\textcircled{d}}]^d=(a+b)^daa^{\textcircled{d}}.$$

Moreover, we have

$$\begin{array}{rll}(a_1+b_1)^{\pi} &=& aa^{\textcircled{d}}-(a+b)^daa^{\textcircled{d}}(a+b)aa^{\textcircled{d}}\\ &=& aa^{\textcircled{d}}-(a+b)^d(a+b)aa^{\textcircled{d}}\\ &=& (a+b)^{\pi}aa^{\textcircled{d}}.\end{array}$$

We see that

$$a_1+b_1=(a+b)aa^{\textcircled{d}}\in \mathcal{A}^{\textcircled{d}}.$$

Also we have $a_4=p^{\pi}ap^{\pi}=p^{\pi}a$ and $b_4=p^{\pi}bp^{\pi}=p^{\pi}b$, and so

$$a_4+b_4=p^{\pi}a+p^{\pi}b.$$

We claim that

$$\begin{array}{rll}(p^{\pi}a)(p^{\pi}b) &=& p^{\pi}ab=0,\\ (p^{\pi}b)^*(p^{\pi}a) &=& (p^{\pi}bp^{\pi})^*(p^{\pi}a)\\ &=& (1-aa^{\textcircled{d}})b^*(1-aa^{\textcircled{d}})(p^{\pi}a)\\ &=& p^{\pi}b^*(p^{\pi}a)=0.\end{array}$$

As in the proof of [5, Theorem 2.1], $a-a^{\textcircled{d}}a^2\in \mathcal{A}^{qnil}$. By using Cline's formula, $p^{\pi}a=a-aa^{\textcircled{d}}\in \mathcal{A}^{qnil}$. Thus, $a_4+b_4\in \mathcal{A}^{\textcircled{d}}$ and $(a_4+b_4)^{\textcircled{d}}=(p^{\pi}b)^{\textcircled{d}}$ by Lemma 2.3.

We check that

$$\begin{array}{rll}(a_4+b_4)^d &=& p^{\pi}b^d,\\ (a_4+b_4)^{\pi} &=& p^{\pi}b^{\pi}.\end{array}$$

Moreover, we see that

$$\begin{array}{rl} & \sum\limits_{i=0}^{\infty}(a_1+b_1)^i(a_1+b_1)^{\pi}(a_2+b_2)[(a_4+b_4)^d]^{i+2} \\ = & \sum\limits_{i=0}^{\infty}(a+b)^i(a+b)^{\pi}aa^{ⓓ}(a+b)(1-aa^{ⓓ})(b^d)^{i+2} \\ = & 0. \end{array}$$

According to Lemma 2.7, $a+b\in\mathcal{A}^{ⓓ}$. Furthermore, we have

$$\begin{array}{rcl} (a+b)^{ⓓ} & = & (a_1+b_1)^{ⓓ}+(a_4+b_4)^{ⓓ}+z \\ & = & [(a+b)aa^{ⓓ}]^{ⓓ}+[(1-aa^{ⓓ})b]^{ⓓ}+z, \end{array}$$

where

$$\begin{array}{rcl} z & = & -(a_1+b_1)^d(a_2+b_2)(a_4+b_4)^{ⓓ} \\ & = & -(a+b)^d aa^{ⓓ}(a+b)[(1-aa^{ⓓ})b]^{ⓓ}, \end{array}$$

as asserted. □

Corollary 3.2. *([8, Theorem 3.4]) Let $a,b\in\mathcal{A}^{ⓓ}$. If $a^*b=0$ and $ab=ba=0$, then $a+b\in\mathcal{A}^{ⓓ}$. In this case,*

$$(a+b)^{ⓓ}=a^{ⓓ}+b^{ⓓ}.$$

Proof. This is immediate from Theorem 3.1. □

Corollary 3.3. *Let $a,b\in\mathcal{A}^{ⓓ}$. If $a^{\pi}b=0$ and $a^{\pi}b^*=0$, then the following are equivalent:*

(1) $a+b\in\mathcal{A}^{ⓓ}$ and $a^{\pi}(a+b)^{ⓓ}aa^{ⓓ}=0$.
(2) $(a+b)aa^{ⓓ}\in\mathcal{A}^{ⓓ}$.

In this case,

$$(a+b)^{ⓓ}=[(a+b)aa^{ⓓ}]^{ⓓ}.$$

Proof. By hypothesis, we see that $a^{\pi}ab=a(a^{\pi}b)=0, a^{\pi}ba=(a^{\pi}b)a=0, a^{\pi}b^*a=(a^{\pi}b^*)a=0$. Since $a^{\pi}b=0$, it follows by Lemma 2.5 that $a^{\sigma}b^d=[(1-aa^{ⓓ})b](b^d)^2=0$. In light of Theorem 3.1, $a+b\in\mathcal{A}^{ⓓ}$ and $a^{\pi}(a+b)^{ⓓ}aa^{ⓓ}=0$ if and only if $(a+b)aa^{ⓓ}\in\mathcal{A}^{ⓓ}$. In this case, $a^{\sigma}=0$, and therefore $(a+b)^{ⓓ}=[(a+b)aa^{ⓓ}]^{ⓓ}$. □

Corollary 3.4. *Let $a,b\in\mathcal{A}^{ⓓ}$. If $a^{\pi}b=0, a^{\pi}b^*=0$ and $ba^d=0$, then $a+b\in\mathcal{A}^{ⓓ}$. In this case, $(a+b)^{ⓓ}=a^{ⓓ}$.*

Proof. We easily verify that $(a^2a^{\textcircled{d}})a^{\textcircled{d}} = aa^{\textcircled{d}}$; hence, $[(a^2a^{\textcircled{d}})a^{\textcircled{d}}]^* = (a^2a^{\textcircled{d}})a^{\textcircled{d}}$. Moreover, we have $(a^2a^{\textcircled{d}})a^{\textcircled{d}}(a^{\textcircled{d}})^2 = a^{\textcircled{d}}$. By induction, we prove that $(a^2a^{\textcircled{d}})^n = a^{n+1}a^{\textcircled{d}}$ and $(a^2a^{\textcircled{d}})^{n+1} = a^{n+2}a^{\textcircled{d}}$. Therefore

$$(a^2a^{\textcircled{d}})^n - a^{\textcircled{d}}(a^2a^{\textcircled{d}})^{n+1} = [a^{n)} - a^{\textcircled{d}}a^{n+1}]aa^{\textcircled{d}}.$$

Since $\lim\limits_{n\to\infty} ||a^n - a^{\textcircled{d}}a^{n+1}||^{\frac{1}{n}} = 0$, we deduce that

$$\lim_{n\to\infty} ||(a^2a^{\textcircled{d}})^n - a^{\textcircled{d}}(a^2a^{\textcircled{d}})^{n+1}||^{\frac{1}{n}} = 0.$$

Hence, $(a^2a^{\textcircled{d}})^{\textcircled{d}} = a^{\textcircled{d}}$. Therefore we complete the proof by Corollary 3.3. □

We next present the additive property of generalized core-EP inverse under commutative conditions. For the detailed formula of the generalized core-EP inverse of the sum, we leave to the readers as it can be derived by the straightforward computation according to our proof.

Theorem 3.5. *Let $a, b \in \mathcal{A}^{\textcircled{d}}$. If $ab = ba$ and $a^*b = ba^*$, then the following are equivalent:*

(1) $a + b \in \mathcal{A}^{\textcircled{d}}$ and $a^\pi(a+b)^{\textcircled{d}}aa^{\textcircled{d}} = 0$.
(2) $1 + a^{\textcircled{d}}b \in \mathcal{A}^{\textcircled{d}}$ and

$$\sum_{i=0}^{\infty}(1+a^{\textcircled{d}}b)^i a^i a^{\textcircled{d}}(1+a^{\textcircled{d}}b)^\pi aa^{\textcircled{d}}a\big[(1-aa^{\textcircled{d}})b^{\textcircled{d}}\big(1+(1-aa^{\textcircled{d}})ab^d]^{-1}\big)\big]^{i+2} = 0.$$

Proof. Since $ab = ba$ and $a^*b = ba^*$, it follows by Lemma 2.1 that $a^{\textcircled{d}}b = ba^{\textcircled{d}}$. Let $p = aa^{\textcircled{d}}$. Then $p^\pi bp = (1-aa^{\textcircled{d}})baa^{\textcircled{d}} = (1-aa^{\textcircled{d}})aa^{\textcircled{d}}b = 0$. Moreover, we have $pbp^\pi = aa^{\textcircled{d}}b(1-aa^{\textcircled{d}}) = aba^{\textcircled{d}}(1-aa^{\textcircled{d}}) = 0$. In light of Lemma 2.4, we have

$$\begin{array}{rcl} p^\pi ap & = & (1-aa^{\textcircled{d}})aaa^{\textcircled{d}} \\ & = & a^2a^{\textcircled{d}} - aa^{\textcircled{d}}a^2a^{\textcircled{d}} \\ & = & 0. \end{array}$$

So we get

$$a = \begin{pmatrix} a_1 & a_2 \\ 0 & a_4 \end{pmatrix}_p, b = \begin{pmatrix} b_1 & 0 \\ 0 & b_4 \end{pmatrix}_p.$$

Hence

$$a + b = \begin{pmatrix} a_1 + b_1 & a_2 \\ 0 & a_4 + b_4 \end{pmatrix}_p.$$

Moreover,

$$a_1 = aa^{\textcircled{d}}a^2a^{\textcircled{d}} = a^2a^{\textcircled{d}}.$$

Obviously, $(1-aa^{ⓓ})baa^{ⓓ}=b(1-aa^{ⓓ})aa^{ⓓ}=0$. It follows by Lemma 2.5 that $(1-a^{ⓓ}a)baa^{ⓓ}=0$. Hence we have $b_1=aa^{ⓓ}baa^{ⓓ}=baa^{ⓓ}=a^{ⓓ}abaa^{ⓓ}=a^{ⓓ}ba^2a^{ⓓ}$, and then

$$a_1+b_1=(1+a^{ⓓ}b)a^2a^{ⓓ}\in\mathcal{A}^{ⓓ}.$$

This implies that

$$(a_1+b_1)^i=(1+a^{ⓓ}b)^i(a^2a^{ⓓ})^i=(1+a^{ⓓ}b)^ia^{i+1}a^{ⓓ}.$$

Furthermore,

$$(a_1+b_1)^d=(1+a^{ⓓ}b)^da^{ⓓ}.$$

Thus

$$(a_1+b_1)^{\pi}=1-(1+a^{ⓓ}b)(1+a^{ⓓ}b)^daa^{ⓓ}.$$

Clearly, we have $(1-aa^{ⓓ})aaa^{ⓓ}=a^2a^{ⓓ}-aa^{ⓓ}aaa^{ⓓ}=0$. Then

$$a_4=(1-aa^{ⓓ})a(1-aa^{ⓓ})=a-aa^{ⓓ}a.$$

As in the proof of [5, Theorem 2.1], $a-a^{ⓓ}a^2\in\mathcal{A}^{qnil}$. By using Cline's formula, $a_4\in\mathcal{A}^{qnil}$. Moreover,

$$b_4=(1-aa^{ⓓ})b(1-aa^{ⓓ})=(1-aa^{ⓓ})b.$$

Since $bp^{\pi}=p^{\pi}b, b^*p^{\pi}=(p^{\pi}b)^*=(bp^{\pi})^*=p^{\pi}b^*$. In light of Lemma 2.2, $b_4=p^{\pi}b\in\mathcal{A}^{ⓓ}$ and $b_4^{ⓓ}=p^{\pi}b^{ⓓ}$. Furthermore,

$$a_4+b_4=(1-aa^{ⓓ})(a+b)$$

$$(a_4+b_4)^i=(1-aa^{ⓓ})(a+b)^i.$$

(1) ⇒ (2) We have

$$(a+b)^{ⓓ}=\begin{pmatrix}\alpha & \beta\\ 0 & \gamma\end{pmatrix}_p.$$

As in the proof of Theorem 3.1, $[p(a+b)p]^{ⓓ}=\alpha$. That is, $(a+b)aa^{ⓓ}\in\mathcal{A}^{ⓓ}$.

We observe that

$$\begin{aligned}1+a^{ⓓ}b &= [1-aa^{ⓓ}]+[aa^{ⓓ}+a^{ⓓ}b]\\ &= [1-aa^{ⓓ}]+[aa^{ⓓ}+ba^{ⓓ}]\\ &= [1-aa^{ⓓ}]+[a+b]a^{ⓓ}\end{aligned}$$

We easily check that $[(a+b)aa^{ⓓ}]a^{ⓓ}=a^{ⓓ}[(a+b)aa^{ⓓ}]$. In view of [3, Theorem 15.2.16], $(a+b)a^{ⓓ}=[(a+b)aa^{ⓓ}]a^{ⓓ}\in\mathcal{A}^d$ and

$$[a+b)a^{ⓓ}]^d=[(a+b)aa^{ⓓ}]^d[a^{ⓓ}]^d.$$

In view of Theorem 1.1, $[(a+b)aa^{\textcircled{d}}]^d$ has $(1,3)$-inverse. Then there exists $y\in\mathcal{A}$ such that

$$[(a+b)aa^{\textcircled{d}}]^d=[(a+b)aa^{\textcircled{d}}]^d y[(a+b)aa^{\textcircled{d}}]^d,$$
$$\big([(a+b)aa^{\textcircled{d}}]^d y\big)^*=[(a+b)aa^{\textcircled{d}}]^d y.$$

We verify that

$$\begin{array}{rl} & [(a+b)a^{\textcircled{d}}]^d[(a^2a^{\textcircled{d}})y][(a+b)a^{\textcircled{d}}]^d[a^2a^{\textcircled{d}}]\\ = & [(a+b)aa^{\textcircled{d}}]^d y[(a+b)aa^{\textcircled{d}}]^d\\ = & [(a+b)aa^{\textcircled{d}}]^d\\ = & [(a+b)a^{\textcircled{d}}]^d[a^2a^{\textcircled{d}}]. \end{array}$$

Clearly, $[a^2a^{\textcircled{d}}](a^{\textcircled{d}})^d=aa^{\textcircled{d}}$. Then

$$\begin{array}{rl} & [(a+b)a^{\textcircled{d}}]^d[(a^2a^{\textcircled{d}})y][(a+b)a^{\textcircled{d}}]^d\\ = & [(a+b)a^{\textcircled{d}}]^d,\\ & [(((a+b)a^{\textcircled{d}})^d(a^2a^{\textcircled{d}})y)]^*\\ = & [((a+b)aa^{\textcircled{d}})y]^*\\ = & ((a+b)aa^{\textcircled{d}})y\\ = & [(a+b)a^{\textcircled{d}}]^d(a^2a^{\textcircled{d}})y. \end{array}$$

Therefore $[(a+b)a^{\textcircled{d}}]^d$ has $(1,3)$-inverse $(a^2a^{\textcircled{d}})y$. In light of Theorem 1.1, $(a+b)a^{\textcircled{d}}\in\mathcal{A}^{\textcircled{d}}$.

Obviously, we have

$$\begin{array}{rcl} [1-aa^{\textcircled{d}}](a+b)a^{\textcircled{d}} & = & [1-aa^{\textcircled{d}}]^*[a+b]a^{\textcircled{d}}\\ & = & [a+b]a^{\textcircled{d}}[1-aa^{\textcircled{d}}]=0. \end{array}$$

According to Corollary 3.2, $1+a^{\textcircled{d}}b\in\mathcal{A}^{\textcircled{d}}$.

In view of Lemma 2.6,

$$(a+b)^d=\left(\begin{array}{cc}(a_1+b_1)^d & z\\ 0 & (a_4+b_4)^d\end{array}\right)_p,$$

where

$$\begin{array}{rcl} z & = & \sum\limits_{i=0}^{\infty}[(a_1+b_1)^d]^{i+2}a_2(a_4+b_4)^i(a_4+b_4)^{\pi}\\ & + & \sum\limits_{i=0}^{\infty}(a_1+b_1)^i(a_1+b_1)^{\pi}a_2[(a_4+b_4)^d]^{i+2}\\ & - & (a_1+b_1)^da_2(a_4+b_4)^d. \end{array}$$

By virtue of Theorem 1.1,

$$(a+b)^{\textcircled{d}}=[(a+b)^d]^2[(a+b)^d]^{\textcircled{\#}}$$

Hence,

$$\begin{array}{rcl}[(a+b)^d]^{\textcircled{\#}} & = & (a+b)(a+b)^d[(a+b)^d]^{\textcircled{\#}}\\ & = & (a+b)^2(a+b)^{\textcircled{d}}.\end{array}$$

Since $p^\pi(a+b)^2p=p^\pi(a+b)^dp=0$, we see that $p^\pi[(a+b)^d]^{\textcircled{\#}}p=0$. As in the proof of [19, Theorem 2.5], $[(a_1+b_1)^d]^\pi z=0$. Thus, we have $(a_1+b_1)^\pi z=0$; hence,

$$\sum_{i=0}^{\infty}(a_1+b_1)^i(a_1+b_1)^\pi a_2[(a_4+b_4)^d]^{i+2}=0.$$

Thus,

$$(a_4+b_4)^d=(1-aa^{\textcircled{d}})b^{\textcircled{d}}[1+(1-aa^{\textcircled{d}})ab^d]^{-1}].$$

Therefore

$$\begin{array}{l}\sum\limits_{i=0}^{\infty}(1+a^{\textcircled{d}}b)^ia^{i+1}a^{\textcircled{d}}[1-(1+a^{\textcircled{d}}b)(1+a^{\textcircled{d}}b)^daa^{\textcircled{d}}]a\\ \qquad\qquad \big[(1-aa^{\textcircled{d}})b^{\textcircled{d}}\big(1+(1-aa^{\textcircled{d}})ab^d]^{-1}\big)\big]^{i+2}=0.\end{array}$$

Accordingly,

$$\sum_{i=0}^{\infty}(1+a^{\textcircled{d}}b)^ia^ia^{\textcircled{d}}(1+a^{\textcircled{d}}b)^\pi aa^{\textcircled{d}}a\big[(1-aa^{\textcircled{d}})b^{\textcircled{d}}\big(1+(1-aa^{\textcircled{d}})ab^d]^{-1}\big)\big]^{i+2}=0.$$

(2) $\Rightarrow$ (1) Step 1. Since $(1+a^{\textcircled{d}}b)aa^{\textcircled{d}}=aa^{\textcircled{d}}(1+a^{\textcircled{d}}b)$ and $(aa^{\textcircled{d}})^*=aa^{\textcircled{d}}$, it follows by Lemma 2.2 that

$$(1+a^{\textcircled{d}}b)aa^{\textcircled{d}}\in\mathcal{A}^{\textcircled{d}}.$$

Then

$$[(1+a^{\textcircled{d}}b)aa^{\textcircled{d}}]^d=(1+a^{\textcircled{d}}b)^daa^{\textcircled{d}}\in\mathcal{A}^{(1,3)}.$$

Thus, we can find a $y\in\mathcal{A}$ such that

$$\begin{array}{rcl}(1+a^{\textcircled{d}}b)^daa^{\textcircled{d}} & = & (1+a^{\textcircled{d}}b)^daa^{\textcircled{d}}y(1+a^{\textcircled{d}}b)^daa^{\textcircled{d}},\\ \big((1+a^{\textcircled{d}}b)^daa^{\textcircled{d}}y\big)^* & = & (1+a^{\textcircled{d}}b)^daa^{\textcircled{d}}y.\end{array}$$

We easily verify that

$$\begin{array}{rcl}(1+a^{\textcircled{d}}b)^da^{\textcircled{d}} & = & (1+a^{\textcircled{d}}b)^da^{\textcircled{d}}z(1+a^{\textcircled{d}}b)^da^{\textcircled{d}},\\ \big((1+a^{\textcircled{d}}b)^da^{\textcircled{d}}z\big)^* & = & (1+a^{\textcircled{d}}b)^da^{\textcircled{d}}z,\end{array}$$

where $z=a^2a^{\textcircled{d}}y$.

Clearly, $[(1+a^{\textcircled{d}}b)a^2a^{\textcircled{d}}]^d=(1+a^{\textcircled{d}}b)^da^{\textcircled{d}}\in\mathcal{A}^{(1,3)}$. By virtue of Theorem 1.1, $(a+b)aa^{\textcircled{d}}=(1+a^{\textcircled{d}}b)a^2a^{\textcircled{d}}\in\mathcal{A}^{\textcircled{d}}$.

Step 2. Obviously, $a_4b_4 = b_4a_4$. Since $1+a_4^db_4 = 1$, it follows by [23, Theorem 3.3] that $(a_4+b_4)^d = \sum\limits_{i=0}^{\infty}(b^d)^{i+1}(-a_4)^i = b_4^d(1+a_4b_4^d)^{-1}$. Since $b_4 \in \mathcal{A}^{\textcircled{d}}$, by virtue of Theorem 1.1 that $b_4^d \in \mathcal{A}^{(1,3)}$. Then we can find a $y \in \mathcal{A}$ such that

$$b_4^d = b_4^dyb_4^d, (b_4^dy)^* = b_4^dy.$$

Set $z = (1+a_4b_4^d)y$. Then we verify that

$$\begin{array}{rcl} b_4^d(1+a_4b_4^d)^{-1} & = & b_4^d(1+a_4b_4^d)^{-1}zb_4^d(1+a_4b_4^d)^{-1}, \\ (b_4^d(1+a_4b_4^d)^{-1}z)^* & = & (b_4^dy)^* = b_4^dy = b_4^d(1+a_4b_4^d)^{-1}z. \end{array}$$

Hence, $b_4^d(1+a_4b_4^d)^{-1} \in \mathcal{A}^{(1,3)}$. In light of Theorem 1.1., $a_4+b_4 \in \mathcal{A}^{\textcircled{d}}$.

Step 3. By virtue of Theorem 1.1, $a_1+b_1, a_4+b_4 \in \mathcal{A}^d$. By virtue of Lemma 2.6,

$$(a+b)^d = \begin{pmatrix} (a_1+b_1)^d & z \\ 0 & (a_4+b_4)^d \end{pmatrix}_p,$$

where

$$\begin{array}{rcl} z & = & \sum\limits_{i=0}^{\infty}[(a_1+b_1)^d]^{i+2}a_2(a_4+b_4)^i(a_4+b_4)^{\pi} \\ & + & \sum\limits_{i=0}^{\infty}(a_1+b_1)^i(a_1+b_1)^{\pi}a_2[(a_4+b_4)^d]^{i+2} \\ & - & (a_1+b_1)^da_2(a_4+b_4)^d \end{array}$$

By hypothesis, we have

$$\sum_{i=0}^{\infty}(1+a^{\textcircled{d}}b)^ia^ia^{\textcircled{d}}(1+a^{\textcircled{d}}b)^{\pi}aa^{\textcircled{d}}a\big[(1-aa^{\textcircled{d}})b^{\textcircled{d}}\big(1+(1-aa^{\textcircled{d}})ab^d]^{-1}\big)\big]^{i+2} = 0.$$

This implies that

$$\sum_{i=0}^{\infty}(a_1+b_1)^i(a_1+b_1)^{\pi}a_2[(a_4+b_4)^d]^{i+2} = 0.$$

Then $(a_1+b_1)^{\pi}z = 0$; and so $[(a_1+b_1)^d]^{\pi}z = 0$. In light of Lemma 2.8, $a+b \in \mathcal{A}^{\textcircled{d}}$. Moreover, we have $p^{\pi}(a+b)^{\textcircled{d}}p = 0$. In view of Lemma 2.5, $a^{\pi}(a+b)^{\textcircled{d}}aa^{\textcircled{d}} = 0$. This completes the proof. □

Corollary 3.6. *Let $a, b \in \mathcal{A}^{\textcircled{d}}$. If $ab = ba, a^*b = ba^*$ and $1+a^{\textcircled{d}}b \in \mathcal{A}^{-1}$, then $a+b \in \mathcal{A}^{\textcircled{d}}$.*

Proof. Since $1+a^{\textcircled{d}}b \in \mathcal{A}^{-1}$, we have $(1+a^{\textcircled{d}}b)^{\pi} = 0$. This completes the proof by Theorem 3.5. □

4. APPLICATIONS

Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathcal{A})$. The aim of this section is to present the generalized core-EP invertibility of the square matrix M by using the generalized core-EP invertibility of its entries.

Lemma 4.1. *Let $b, c \in \mathcal{A}$. If $bc, cb \in \mathcal{A}^{\textcircled{d}}$, then $Q := \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}$ has generalized core-EP inverse. In this case,*

$$Q^{\textcircled{d}} = \begin{pmatrix} 0 & b(cb)^{\textcircled{d}} \\ c(bc)^{\textcircled{d}} & 0 \end{pmatrix}.$$

Proof. Since $Q^2 = \begin{pmatrix} bc & 0 \\ 0 & cb \end{pmatrix}$, we see that Q^2 has generalized core-EP inverse and

$$(Q^2)^{\textcircled{d}} = \begin{pmatrix} (bc)^{\textcircled{d}} & 0 \\ 0 & (cb)^{\textcircled{d}} \end{pmatrix}.$$

In light of [4, Lemma 3.4], Q has generalized core-EP inverse and

$$\begin{aligned} Q^{\textcircled{d}} &= Q(Q^2)^{\textcircled{d}} \\ &= \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} \begin{pmatrix} (bc)^{\textcircled{d}} & 0 \\ 0 & (cb)^{\textcircled{d}} \end{pmatrix} \\ &= \begin{pmatrix} 0 & b(cb)^{\textcircled{d}} \\ c(bc)^{\textcircled{d}} & 0 \end{pmatrix}, \end{aligned}$$

as asserted. □

We are now ready to prove:

Theorem 4.2. *Let $a, d, bc, cb \in \mathcal{A}^{\textcircled{d}}$. If*

$$bd^d = 0, ca^d = 0, a^{\pi}b = 0, d^{\pi}c = 0, a^{\pi}c^* = 0, d^{\pi}b^* = 0,$$

then M has generalized core-EP inverse.

Proof. Write $M = P + Q$, where

$$P = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}, Q = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}.$$

Since a and d have generalized core-EP inverses, so has P, and that

$$P^d = \begin{pmatrix} a^d & 0 \\ 0 & d^d \end{pmatrix}, P^{\pi} = \begin{pmatrix} a^{\pi} & 0 \\ 0 & d^{\pi} \end{pmatrix}.$$

In view of Lemma 4.1, Q has generalized core-EP inverse. By hypothesis, we check that

$$\begin{aligned} P^{\pi}Q &= \begin{pmatrix} 0 & a^{\pi}b \\ d^{\pi}c & 0 \end{pmatrix} = 0, \\ P^{\pi}Q^* &= \begin{pmatrix} 0 & a^{\pi}c^* \\ d^{\pi}b^* & 0 \end{pmatrix} = 0, \\ QP^d &= \begin{pmatrix} 0 & bd^d \\ ca^d & 0 \end{pmatrix} = 0. \end{aligned}$$

According to Corollary 3.4, M has generalized core-EP inverse. □

Corollary 4.3. *Let $a, d, bc, cb \in \mathcal{A}^{\textcircled{d}}$. If*

$$a^d b = 0, d^d c = 0, bd^{\pi} = 0, ca^{\pi} = 0, b^* a^{\pi} = 0, c^* d^{\pi} = 0,$$

then M has generalized core-EP inverse.

Proof. Obviously, $M^* = \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}$. By hypothesis, we have

$$c^*(d^*)^d = 0, b^*(a^*)^d = 0, (a^*)^{\pi}c^* = 0, (d^*)^{\pi}b^* = 0, (a^*)^{\pi}b = 0, (d^*)^{\pi}c = 0.$$

Applying Theorem 4.2 to the operator M^*, we prove that M^* has generalized core-EP inverse. Therefore M has generalized core-EP inverse, as asserted. □

Theorem 4.4. *Let $a, d, bc, cb \in \mathcal{A}^{\textcircled{d}}$. If*

$$ab = bd, dc = ca, a^*b = bd^*, d^*c = ca^*$$

and $a^{\oplus}bd^{\oplus}c \in \mathcal{A}^{qnil}$, then M has generalized core-EP inverse.

Proof. Write $M = P + Q$, where

$$P = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}, Q = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}.$$

As in the proof of Theorem 4.2, P and Q have generalized core-EP inverses.

It is easy to verify that

$$PQ = \begin{pmatrix} 0 & ab \\ dc & 0 \end{pmatrix} = \begin{pmatrix} 0 & bd \\ ca & 0 \end{pmatrix} = QP.$$

Likewise, we verify that $P^*Q = QP^*$. Moreover, we check that

$$I_2 + P^{\textcircled{d}}Q = \begin{pmatrix} 1 & a^{\textcircled{d}}b \\ d^{\textcircled{d}}c & 1 \end{pmatrix}.$$

Obviously, we have

$$\begin{pmatrix} 1 & a^{\textcircled{d}}b \\ d^{\textcircled{d}}c & 1 \end{pmatrix} = \begin{pmatrix} 1 - a^{\textcircled{d}}bd^{\textcircled{d}}c & a^{\textcircled{d}}b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ d^{\textcircled{d}}c & 1 \end{pmatrix}.$$

As $a^{\textcircled{d}}bd^{\textcircled{d}}c \in \mathcal{A}^{qnil}$, $1 - a^{\textcircled{d}}bd^{\textcircled{\#}}c \in \mathcal{A}^{-1}$. This implies that $\begin{pmatrix} 1 & a^{\textcircled{d}}b \\ d^{\textcircled{d}}c & 1 \end{pmatrix}$ is invertible. This implies that $I_2 + P^{\textcircled{d}}Q$ is invertible. By using Corollary 3.6, M has generalized core-EP inverse. □

Corollary 4.5. *Let $a, d, bc, cb \in \mathcal{A}^{\textcircled{d}}$. If*

$$ab = bd, ca = dc, a^*b = bd^*, ac^* = c^*d$$

and $bd^{\textcircled{\#}}ca^{\textcircled{\#}} \in \mathcal{A}^{qnil}$, then M has generalized core-EP inverse.

Proof. Analogously to Corollary 4.3, we complete the result by applying Theorem 4.4 to $M^* = \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}$. □

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School of Big Data, Fuzhou University of International Studies and Trade, Fuzhou 350202, China

E-mail address: <huanyinchenfz@163.com>