GENERALIZED w-CORE INVERSE IN BANACH ALGEBRAS WITH INVOLUTION

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ABSTRACT. In this paper, we introduce the generalized w-core inverse in a Banach *-algebra. We characterize this new generalized inverse by using the generalized weighted core decomposition and present the representations by the weighted g-Drazin inverse. The generalized w-core orders are investigated as well. These extend the weighted core inverse and pseudo core inverse for complex matrices and linear bounded operators to more general setting.

1. INTRODUCTION

A Banach algebra $\mathcal{A}$ is called a Banach *-algebra if there exists an involution $* : x \to x^*$ satisfying $(x+y)^* = x^* + y^*, (\lambda x)^* = \overline{\lambda} x^*, (xy)^* = y^* x^*, (x^*)^* = x$. Rakic et al. generalized the core inverse of a complex matrix to the case of an element in a ring (see [21]). An element a in a Banach *-algebra $\mathcal{A}$ has core inverse if and only if there exist $x \in \mathcal{A}$ such that

$$ax^2 = x, (ax)^* = ax, xa^2 = a.$$

If such x exists, it is unique, and denote it by $a^{\oplus}$ (see [1, 6]).

Zhu et al. introduced and studied w-core inverse for a ring element (see [30]). Let $a, w \in \mathcal{A}$. An element $a \in \mathcal{A}$ has w-core inverse if there exist $x \in \mathcal{A}$ such that

$$awx^2 = x, (awx)^* = awx, xawa = a.$$

If such x exists, it is unique, and denote it by $a_w^{\oplus}$. Let $\mathcal{A}_w^{\oplus}$ denote the set of all w-core invertible elements in $\mathcal{A}$. The w-core inverse was studied by many authors, e.g., [7, 10, 11, 12, 24, 25, 30, 32]. As is well known,

$$a \in \mathcal{A}_w^{\oplus} \Leftrightarrow awx^2 = x, (awx)^* = awx, xawa = a, awxa = a, xawx = x$$

(see [30, Theorem 2.13]).

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Prasad et al. extended the concept of the core inverse and introduced the notion of core-EP inverse (i.e., pseudo core inverse) (see [19, 8]). An element $a \in \mathcal{A}$ has core-EP inverse (i.e., pseudo core inverse) if there exist $x \in \mathcal{A}$ and $k \in \mathbb{N}$ such that

$$ax^2 = x, (ax)^* = ax, xa^{k+1} = a^k.$$

If such x exists, it is unique, and denote it by $a^{\textcircled{D}}$. The core-EP inverse has been investigated from many different views, e.g., [2, 8, 9, 14, 15, 16, 17, 19, 22, 27, 28].

The motivation of this paper is to introduce and study a new kind of generalized weighted inverse as a natural generalization of generalized inverses mentioned above. Let

$$\mathcal{A}^{qnil} = \{x \in \mathcal{A} \mid \lim_{n\to\infty} \parallel x^n \parallel^{\frac{1}{n}} = 0\}.$$

As is well known, $x \in \mathcal{A}^{qnil}$ if and only if $1 + \lambda x \in \mathcal{A}$ is invertible for any $\lambda \in \mathbb{C}$. Set $\mathcal{A}^{qnil}_w = \{x \in \mathcal{A} \mid xw \in \mathcal{A}^{qnil}\}$. In Section 2, we introduce generalized weighted core inverse in terms of a new kind of generalized weighted core decomposition. Many new properties of the w-core inverse and core-EP inverse are thereby obtained.

Definition 1.1. *An element $a \in \mathcal{A}$ has generalized w-core decomposition if there exist $x, y \in \mathcal{A}$ such that*

$$a = x + y, x^*y = ywx = 0, x \in \mathcal{A}^{\textcircled{\#}}_w, y \in \mathcal{A}^{qnil}_w.$$

Let $a, w \in \mathcal{A}$. We prove that $a \in \mathcal{A}$ has generalized w-core decomposition if and only if there exists unique $x \in \mathcal{A}$ such that

$$x = awx^2, (awx)^* = awx, x(aw)^2x = awx, \lim_{n\to\infty} ||(aw)^n - awx(aw)^n||^{\frac{1}{n}} = 0.$$

Recall that $a \in \mathcal{A}$ has g-Drazin inverse (i.e., generalized Drazin inverse) if there exists $x \in \mathcal{A}$ such that

$$ax^2 = x, ax = xa, a - a^2x \in \mathcal{A}^{qnil}.$$

Such x is unique, if exists, and denote it by a^d. The g-Drazin inverse plays an important role in matrix and operator theory (see [3]). An element $a \in \mathcal{A}$ has generalized w-Drazin inverse x if there exists unique $x \in \mathcal{A}$ such that

$$awx = xwa, xwawx = x \text{ and } a - awxwa \in \mathcal{A}^{qnil}.$$

We denote x by $a^{d,w}$ (see [13]). Evidently, $a^{d,w} = x$ if and only if $x = [(aw)^d]^2a = a[(wa)^d]^2 = (aw)^da(wa)^d$. In Section 3, we establish equivalences between generalized w-core inverse and weighted g-Drazin inverse for a Banach algebra element by using involved images. We prove that $a \in \mathcal{A}_w^{\textcircled{d}}$ if and only if $a \in \mathcal{A}^{d,w}$ and there exists $x \in \mathcal{A}$ such that $xawx = x, x\mathcal{A} = a^{d,w}\mathcal{A}, \mathcal{A}x = \mathcal{A}(a^{d,w})^*$. The aim of Section 4 is to characterize generalized weighted core inverse of an element in a Banach *-algebra by other related generalized inverses, e.g., weighted core. It is shown that $a \in \mathcal{A}_w^{\textcircled{d}}$ if and only if $a \in \mathcal{A}^{d,w}$ and $a^{d,w} \in \mathcal{A}_w^{\textcircled{\#}}$.

Finally, in Section 5, the generalized w-core order for a Banach *-algebra element was introduced. Let $w \in \mathcal{A}$ and $a, b \in \mathcal{A}_w^{\textcircled{d}}$. We say that $a \leq_w^{\textcircled{d}} b$ provided that

$$awa_w^{\textcircled{d}} = bwa_w^{\textcircled{d}}, a_w^{\textcircled{d}}a = a_w^{\textcircled{d}}b.$$

The characterizations of the generalized weighted core order are present.

Throughout the paper, all Banach *-algebras are complex with an identity. $\mathcal{A}^{d,w}$ and $\mathcal{A}_w^{\textcircled{\#}}$ denote the sets of all weighted g-Drazin and w-core invertible elements in $\mathcal{A}$, respectively. Let $\mathbb{C}^{n\times n}$ be the Banach algebra of all $n \times n$ complex matrices with conjugate transpose $*$.

2. Generalized w-core decomposition

The aim of this section is to introduce the notion of the generalized w-core inverse in a Banach *-algebra. We begin with

Theorem 2.1. *Let $a, w \in \mathcal{A}$. Then the following are equivalent:*

(1) $a \in \mathcal{A}$ has generalized w-core decomposition.
(2) There exists $x \in \mathcal{A}$ such that

$$x = awx^2, (awx)^* = awx, x(aw)^2x = awx, \lim_{n\to\infty} ||(aw)^n - awx(aw)^n||^{\frac{1}{n}} = 0.$$

Proof. (1) $\Rightarrow$ (2) By hypothesis, there exist $z, y \in \mathcal{A}$ such that

$$a = z + y, z^*y = ywz = 0, z \in \mathcal{A}_w^{\textcircled{\#}}, y \in \mathcal{A}_w^{qnil}.$$

Set $x = z_w^{\textcircled{\#}}$. Then

$$\begin{array}{rcl} awx & = & (z+y)wz_w^{\textcircled{\#}} = zwz_w^{\textcircled{\#}}, \\ (awx)^* & = & awx, \\ awx^2 & = & (awx)x = zwz_w^{\textcircled{\#}}(z+y) = zwz_w^{\textcircled{\#}}z = x. \end{array}$$

It is easy to verify that

$$\begin{array}{rcl} xawx & = & z_w^{\textcircled{\#}}(zwz_w^{\textcircled{\#}}) = z_w^{\textcircled{\#}} = x, \\ x(aw)^2x & = & (xaw)(awx) = z_w^{\textcircled{\#}}(z+y)wzwz_w^{\textcircled{\#}} = z_w^{\textcircled{\#}}zwzwz_w^{\textcircled{\#}} \\ & = & z_w^{\textcircled{\#}}zwzwz_w^{\textcircled{\#}} = zwz_w^{\textcircled{\#}} = awx. \end{array}$$

Moreover, we have

$$awxa = (awx)a = zwz_w^{\oplus}(z+y) = zwz_w^{\oplus}z = z,$$

and so

$$a - awxa = a - z = y \in \mathcal{A}_w^{qnil}.$$

Then

$$\begin{array}{rcl} ||(aw)^n - awx(aw)^n|| & = & ||(a-awxa)w(aw)^{n-1}|| \\ & = & ||yw(aw)^{n-1}|| = ||ywaw(aw)^{n-2}|| \\ & = & ||yw(z+y)w(aw)^{n-2}|| = ||(yw)^2(aw)^{n-2}|| \\ & = & \cdots = ||(yw)^n||. \end{array}$$

Since $y \in \mathcal{A}_w^{qnil}$, we see that $\lim\limits_{n\to\infty} ||(yw)^n||^{\frac{1}{n}} = 0$. Therefore

$$\lim_{n\to\infty} ||(aw)^n - awx(aw)^n||^{\frac{1}{n}} = 0,$$

as required.

(2) $\Rightarrow$ (1) By hypotheses, there exists $x \in \mathcal{A}$ such that

$$x = awx^2, (awx)^* = awx, x(aw)^2x = awx, \lim_{n\to\infty} ||(aw)^n - awx(aw)^n||^{\frac{1}{n}} = 0.$$

Then we check that

$$xawx = xaw(awx^2) = [x(aw)^2x]x = awx^2 = x.$$

Set $z = awxa$ and $y = a - awxa$. We verify that

$$\begin{array}{rcl} ywz & = & (a-awxa)wawxa = awawxa - awx(aw)^2xa \\ & = & awawxa - aw(awx)a = 0, \\ z^*y & = & (awxa)^*y = a^*(awx)y = a^*(awx)(a-awxa) \\ & = & a^*aw(xa - xawxa) = 0. \end{array}$$

We claim that $z \in \mathcal{A}_w^{\oplus}$ and $z_w^{\oplus} = x$.

Claim 1. $x = zwx^2$. We verify that

$$zwx^2 = awx(awx^2) = awx^2 = x.$$

Claim 2. $(zwx)^* = zwx$. Clearly, we have $zwx = aw(xawx) = awx$, and then $(zwx)^* = (awx)^* = awx = zwx$.

Claim 3. $xzwz = z$. One checks that

$$xzwz = (xawx)awawxa = x(aw)^2xa = awxa = z.$$

Therefore $z \in \mathcal{A}_w^{\oplus}$. Moreover, we see that

$$\begin{array}{rcl} ||(aw)^n - awx(aw)^n|| & = & ||(a-awxa)w(aw)^{n-1}|| \\ & = & ||yw(aw)^{n-1}|| = ||ywaw(aw)^{n-2}|| \\ & = & ||yw(z+y)w(aw)^{n-2}|| = ||(yw)^2(aw)^{n-2}|| \\ & = & \cdots = ||(yw)^n||. \end{array}$$

Therefore

$$\lim_{n\to\infty} ||(yw)^n||^{\frac{1}{n}} = 0,$$

and then $y \in \mathcal{A}_w^{qnil}$. This completes the proof. □

Corollary 2.2. *Let $a, w \in \mathcal{A}$. Then the following are equivalent:*

(1) $a \in \mathcal{A}$ has generalized w-core decomposition.
(2) There exist unique $x, y \in \mathcal{A}$ such that

$$a = x + y, x^*y = ywx = 0, x \in \mathcal{A}_w^{\oplus}, y \in \mathcal{A}_w^{qnil}.$$

Proof. (1) ⇒ (2) In view of Theorem 2.1, there exists $v \in \mathcal{A}$ such that

$$v = awv^2, (awv)^* = awv, v(aw)^2v = awv, \lim_{n\to\infty} ||(aw)^n - awv(aw)^n||^{\frac{1}{n}} = 0.$$

Set $z = awva$ and $y = a - awva$. As in the proof of Theorem 2.1, we have

$$a = z + y, z^*y = ywz = 0, z \in \mathcal{A}_w^{\oplus}, y \in \mathcal{A}_w^{qnil}.$$

Suppose that there exist $b, c \in \mathcal{A}$ such that

$$a = b + c, b^*c = cwb = 0, b \in \mathcal{A}_w^{\oplus}, c \in \mathcal{A}_w^{qnil}.$$

Obviously, $awv = aw(awv^2) = (aw)^2v^2 = \cdots = (aw)^nv^n$. Since $a = b + c$, we have $aw = bw + cw$. As $(cw)(bw) = (cwb)w = 0$, we have

$$(aw)^n = \sum_{i=0}^{n} (bw)^i(cw)^{n-i} = (cw)^n + \sum_{i=1}^{n} (bw)^i(cw)^{n-i}.$$

Hence,

$$[(aw)^n]^*c = [(cw)^n]^*c.$$

Clearly, $(aw)^nb = (bw)^nb$, and then $(aw)^nbw = (bw)^nbw = (bw)^{n+1}$. This implies that $(aw)^nbw = (bw)^nbw = (bw)^{n+1}$, and so

$$(aw)^nbw[(bw)^{\oplus}]^{n+1} = (bw)^nbw = (bw)^{n+1}[(bw)^{\oplus}]^{n+1}.$$

We infer that

$$(aw)^nbw[(bw)^{\oplus}]^{n+1}bw = bw(bw)^{\oplus}bw = bw.$$

Therefore $b = bwb_w^{\oplus}b = (aw)^n z$, where $z = bw[(bw)^{\oplus}]^{n+1}bw$. Accordingly,

$$\begin{array}{rcl} b-z & = & b-awva = b-awv(b+c) = b-awvb-awvc \\ & = & b-(aw)^n v^n b-[(aw)^n v^n]^* c \\ & = & b-(aw)^n v^n b-[(aw)^n v^n]^* c \\ & = & b-(aw)vb-(v^n)^*((aw)^n)^* c \\ & = & (aw)^n z-(aw)v(aw)^n z-(v^n)^*((cw)^n)^* c \\ & = & [(aw)^n-(aw)v(aw)^n]z-(v^n)^*((cw)^n)^* c \end{array}$$

Hence,

$$||b-z||^{\frac{1}{n}} \leq ||(aw)^n-(aw)v(aw)^n||^{\frac{1}{n}}||z||^{\frac{1}{n}}+||(v^n)^*||^{\frac{1}{n}}||((cw)^n)^*||^{\frac{1}{n}}||c||^{\frac{1}{n}}.$$

Since $cw \in \mathcal{A}^{qnil}$,

then $1-\overline{\lambda}cw \in \mathcal{A}^{-1}$; hence, $1-\lambda(cw)^* \in \mathcal{A}^{-1}$. This implies that $(cw)^* \in \mathcal{A}^{qnil}$. Thus, we prove that $\lim\limits_{n\to\infty}||((cw)^n)^*||^{\frac{1}{n}}=0$. It follows that

$$\lim_{n\to\infty}||b-z||^{\frac{1}{n}}=0.$$

Therefore $b=z$, and then $c=a-b=a-z=y$, as required.

(2) $\Rightarrow$ (1) This is trivial. □

Theorem 2.3. *Let $a, w \in \mathcal{A}$. Then the following are equivalent:*

(1) $a \in \mathcal{A}$ has generalized w-core decomposition.
(2) There exists unique $x \in \mathcal{A}$ such that

$$x = awx^2, (awx)^* = awx, x(aw)^2x = awx, \lim_{n\to\infty}||(aw)^n-awx(aw)^n||^{\frac{1}{n}}=0.$$

Proof. (2) $\Rightarrow$ (1) This is obvious by Theorem 2.1.

(1) $\Rightarrow$ (2) By hypothesis, there exists unique $x \in \mathcal{A}$ such that

$$x = awx^2, (awx)^* = awx, x(aw)^2x = awx, \lim_{n\to\infty}||(aw)^n-awx(aw)^n||^{\frac{1}{n}}=0.$$

Assume that there exists $y \in \mathcal{A}$ such that

$$y = awy^2, (awy)^* = awy, y(aw)^2y = awy, \lim_{n\to\infty}||(aw)^n-awy(aw)^n||^{\frac{1}{n}}=0.$$

Set $a_1 = axa, a_2 = a-a_1$ and $b_1 = aya, b_2 = a-b_1$. As in the proof of Theorem 2.1, we prove that

$$\begin{array}{c} a_1^*a_2 = a_2wa_1 = 0, a_1 \in \mathcal{A}^{\oplus}, a_2 \in \mathcal{A}_w^{qnil}, \\ b_1^*b_2 = b_2wb_1 = 0, b_1 \in \mathcal{A}^{\oplus}, b_2 \in \mathcal{A}_w^{qnil}. \end{array}$$

As in the proof of Corollary 2.2 , we verify that $axa = a_1 = b_1 = aya$. Therefore

$$x = (axa)_w^{\oplus} = (a_1)_w^{\oplus} = (b_1)_w^{\oplus} = (aza)_w^{\oplus} = z.$$

Accordingly, $x = z$, the result follows. □

We denote x in Theorem 2.3 by $a_w^{\textcircled{d}}$, and call it the generalized w-core inverse of a.

Corollary 2.4. *Let $a = x + y$ be the generalized w-core decomposition of $a \in \mathcal{A}$. Then $a_w^{\textcircled{d}} = x_w^{\textcircled{\#}}$.*

Proof. Let $a = x + y$ be the generalized w-core decomposition of $a \in \mathcal{A}$. Similarly to the proof of Theorem 2.1, $x_w^{\textcircled{\#}}$ is the generalized w-core inverse of a. So the theorem is true. □

Theorem 2.5. *Let $a, w \in \mathcal{A}$. Then $a \in \mathcal{A}_w^{\textcircled{d}}$ if and only if there exists a projection $p \in \mathcal{A}$ such that*

(1) $(1-p)a \in (1-p)aw\mathcal{A}$ and $(1-p)aw \in \mathcal{A}^{\#}$;
(2) $a + p \in \mathcal{A}$ is right invertible and $paw = pawp \in \mathcal{A}^{qnil}$.

Proof. (1) ⇒ (2) Since $a \in \mathcal{A}_w^{\textcircled{d}}$, by using Theorem 2.3, there exist $x, y \in \mathcal{A}$ such that

$$a = x + y, x^*y = ywx = 0, x \in \mathcal{A}_w^{\textcircled{\#}}, y \in \mathcal{A}_w^{qnil}.$$

By virtue of [**?**, Theorem 2.1], we have

$$x_w^{\textcircled{\#}} = xw(x_w^{\textcircled{\#}})^2, (xwx_w^{\textcircled{\#}})^* = xwx_w^{\textcircled{\#}}, x_w^{\textcircled{\#}}xwx_w^{\textcircled{\#}} = x_w^{\textcircled{\#}}, x_w^{\textcircled{\#}}xwx = x, x = xwx_w^{\textcircled{\#}}x.$$

Let $p = 1 - xwx_w^{\textcircled{\#}}$. Then $p^2 = p = p^*$ and $px = 0$. We directly check that

$$(x + 1 - xwx_w^{\textcircled{\#}})(wx_w^{\textcircled{\#}} + 1 - xwx_w^{\textcircled{\#}}) = 1 + x(1 - xwx_w^{\textcircled{\#}}).$$

Hence, Let $q = [wx_w^{\textcircled{\#}} + 1 - xwx_w^{\textcircled{\#}}][1 + x(1 - xwx_w^{\textcircled{\#}})]^{-1}$. Then $(x + p)q = 1$.

$$\begin{array}{rcl} 1 + yq & = & 1 + (ywx_w^{\textcircled{\#}} + y - yxwx_w^{\textcircled{\#}})[1 + x(1 - xwx_w^{\textcircled{\#}})]^{-1} \\ & = & 1 + [y - yxwx_w^{\textcircled{\#}}][1 + x(1 - xwx_w^{\textcircled{\#}})]. \end{array}$$

We check that

$$\begin{array}{rl} & 1 + [1 - xwx_w^{\textcircled{\#}}][1 + x(1 - xwx_w^{\textcircled{\#}})]y \\ = & 1 + [1 - xwx_w^{\textcircled{\#}}][y + xy] \\ = & 1 + y + [1 - xwx_w^{\textcircled{\#}}]xy \\ = & 1 + y \in \mathcal{A}^{-1}. \end{array}$$

Hence, $1 + qy \in \mathcal{A}^{-1}$. Therefore we check that

$$\begin{array}{rcl} pa & = & p(x + y) = py = (1 - xwx_w^{\textcircled{\#}})y = y \in \mathcal{A}^{qnil}, \\ paw(1-p) & = & ywxwx_w^{\textcircled{\#}} = 0, \\ a + p & = & x + y + p = (x + p)[1 + qy] \in \mathcal{A} \text{ is right invertible.} \end{array}$$

Moreover, we see that Since $(1-p)a = xwx_w^{\oplus}(x+y) = xwx_w^{\oplus}x = x \in \mathcal{A}_w^{\oplus}$, it follows by [30, Theorem 2.10] that $(1-p)aw \in \mathcal{A}^{\#}$ and $(1-p)a \in (1-p)aw\mathcal{A}$, as required.

(2) $\Rightarrow$ (1) By hypothesis, there exists a projection $p \in \mathcal{A}$ such that $(1-p)a \in \mathcal{A}_w^{\oplus}$;

$$a + p \in \mathcal{A} \text{ right invertible}, paw(1-p) = 0, pa \in \mathcal{A}_w^{qnil}.$$

Set $x = (1-p)a$ and $y = pa$. Then

$$\begin{array}{rcl} x^*y & = & [a^*(1-p)^*]pa = 0, \\ ywx & = & paw(1-p)a = 0, \\ y & = & pa \in \mathcal{A}_w^{qnil}. \end{array}$$

Write $(a+p)q = 1$ for some $q \in \mathcal{A}$. Then $(1-p)aq = (1-p)(a+p)q = 1-p$, and so $(1-p)aq(1-p)a = (1-p)a$ and $[(1-p)aq]^* = (1-p)aq$. Hence, $(1-p)a \in \mathcal{A}^{(1,3)}$.

By hypothesis, $(1-p)a \in (1-p)aw\mathcal{A}$ and $(1-p)aw \in \mathcal{A}^{\#}$. In light of [30, Lemma 2.8], $w \in \mathcal{A}^{||(1-p)a}$. According to [30, Theorem 2.6], $(1-p)a \in \mathcal{A}_w^{\oplus}$. That is, $x \in \mathcal{A}_w^{\oplus}$. Therefore $a \in \mathcal{A}_w^{\textcircled{d}}$. □

Corollary 2.6. *Let $a, w \in \mathcal{A}$. Then $a \in \mathcal{A}^{\textcircled{d}}$ if and only if there exists a projection $p \in \mathcal{A}$ such that*

(1) $(1-p)a \in \mathcal{A}^{\#}$;
(2) $a + p \in \mathcal{A}$ is right invertible and $pa = pap \in \mathcal{A}^{qnil}$.

Proof. This is obvious by choosing $w = 1$ in Theorem 2.5. □

3. Characterizations by weighted g-Drazin inverses

This aim of this section is to characterize the generalized w-core inverse by using involved image of the weighted g-Drazin inverse. Evidently, $a_w^{\oplus} = (aw)^{\oplus}$. But $aw \in \mathcal{A}^{\oplus}$ do not imply $a \in \mathcal{A}_w^{\oplus}$. For instance, letting $a = \begin{pmatrix} -i & 1 \\ 0 & i \end{pmatrix}$, $w = \begin{pmatrix} i & -1 \\ 0 & 0 \end{pmatrix}$. Then $aw \in \mathbb{C}^{2\times 2}$ has the core-inverse $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, while a has not w-core inverse. Contract to this observation, we now derive the following result which enable us to investigate the generalized w-core inverse by using the weighted g-Drazin inverse.

Lemma 3.1. *Let $a, w \in \mathcal{A}$. Then the following are equivalent:*

(1) $a \in \mathcal{A}$ has generalized w-core decomposition.
(2) $aw \in \mathcal{A}^{\textcircled{d}}$.

(3) $a \in \mathcal{A}^{d,w}$ and there exists unique $x \in \mathcal{A}$ such that

$$x = awx^2, (awx)^* = awx, \lim_{n\to\infty} ||(aw)^n - awx(aw)^n||^{\frac{1}{n}} = 0.$$

In this case, $a_w^{\textcircled{d}} = x = (aw)^{\textcircled{d}}$.

Proof. (1) ⇒ (2) Since $a \in \mathcal{A}_w^{\textcircled{d}}$, there exist $x, y \in \mathcal{A}$ such that

$$a = x + y, x^*y = ywx = 0, x \in \mathcal{A}_w^{\textcircled{\#}}, y \in \mathcal{A}_w^{qnil}.$$

Hence, $aw = xw + yw$. Since $x \in \mathcal{A}_w^{\textcircled{\#}}$, we have

$$xw(x_w^{\textcircled{\#}})^2 = x_w^{\textcircled{\#}}, (xwx_w^{\textcircled{\#}})^* = xwx_w^{\textcircled{\#}}, xwx_w^{\textcircled{\#}}x = x.$$

Then

$$xw(x_w^{\textcircled{\#}})^2 = x_w^{\textcircled{\#}}, (xwx_w^{\textcircled{\#}})^* = xwx_w^{\textcircled{\#}}, x(wx_w^{\textcircled{\#}})^2 = wx_w^{\textcircled{\#}}.$$

This shows that $xw \in \mathcal{A}^{\textcircled{\#}}$. Obviously, $yw \in \mathcal{A}^{qnil}$. Moreover, we check that

$$(xw)^*(yw) = w^*(x^*y)w = 0, (yw)(xw) = (ywx)w = 0.$$

In light of [4, Corollary 2.2], $aw \in \mathcal{A}^{\textcircled{d}}$. Moreover, we have $a_w^{\textcircled{d}} = x_w^{\textcircled{\#}} = (xw)^{\textcircled{\#}} = (aw)^{\textcircled{d}}$.

(2) ⇒ (1) Let $x = (aw)^{\textcircled{d}}$. Then $aw \in \mathcal{A}^d$. In view of [4, Theorem 2.5], we have

$$x = awx^2, (awx)^* = awx, \lim_{n\to\infty} ||(aw)^n - awx(aw)^n||^{\frac{1}{n}} = 0.$$

We easily check that

$$\begin{array}{rcl} ||awx - x(aw)^2x||^{\frac{1}{n}} & = & ||(aw)^n x^{n+1} - x(aw)^{n+1}x^{n+1}||^{\frac{1}{n}} \\ & \leq & ||(aw)^n - x(aw)^{n+1}||^{\frac{1}{n}}||x^{n+1}||^{\frac{1}{n}}. \end{array}$$

Hence,

$$\lim_{n\to\infty} ||awx - x(aw)^2x||^{\frac{1}{n}} = 0.$$

Therefore $x(aw)^2x = awx$, and then $a \in \mathcal{A}_w^{\textcircled{d}}$.

(2) ⇒ (3) In view of [4, Theorem 2.5], $a \in \mathcal{A}^{d,w}$ and there exists $x \in \mathcal{A}$ such that

$$x = awx^2, (awx)^* = awx, \lim_{n\to\infty} ||(aw)^n - awx(aw)^n||^{\frac{1}{n}} = 0.$$

Moreover, we have $(aw)(aw)^dx = (aw)^{\textcircled{d}}$. Since $x = awx^2$, by induction, we have $x = (aw)^n x^{n+1}$ for any $n \in \mathbb{N}$. Then

$$\begin{array}{rcl} ||x - (aw)(aw)^dx||^{\frac{1}{n}} & = & ||x - (aw)(aw)^dx||^{\frac{1}{n}} \\ & = & ||[1 - (aw)(aw)^d](aw)^n x^{n+1}||^{\frac{1}{n}} \\ & = & ||(aw)^n - (aw)^d(aw)^{n+1}||^{\frac{1}{n}}||x||^{1+\frac{1}{n}}. \end{array}$$

As $\lim\limits_{n\to\infty} ||(aw)^n - (aw)^d(aw)^{n+1}||^{\frac{1}{n}} = 0$, we see that

$$\lim_{n\to\infty} ||x - (aw)(aw)^d x||^{\frac{1}{n}} = 0,$$

and therefore $x = (aw)(aw)^d x = (aw)^{ⓓ}$, as required.

(3) ⇒ (2) Since $a \in \mathcal{A}^{d,w}$, we have $aw \in \mathcal{A}^d$. Therefore $aw \in \mathcal{A}^{ⓓ}$ by Lemma 3.1. □

We are ready to prove:

Theorem 3.2. *Let $a, w \in \mathcal{A}$. Then $a \in \mathcal{A}_w^{ⓓ}$ if and only if*

(1) $a \in \mathcal{A}^{d,w}$;
(2) There exists $x \in \mathcal{A}$ such that

$$xawx = x, x\mathcal{A} = a^{d,w}\mathcal{A}, \mathcal{A}x = \mathcal{A}(a^{d,w})^*.$$

In this case, $a_w^{ⓓ} = x$.

Proof. $\Longrightarrow$ Choose $x = a_w^{ⓓ}$. Then $aw \in \mathcal{A}^{ⓓ}$ and $x = (aw)^{ⓓ}$. In view of [4, Theorem 3.3], $aw \in \mathcal{A}^d$ and

$$x(aw)x = x, x\mathcal{A} = (aw)^d\mathcal{A}, \mathcal{A}x = \mathcal{A}((aw)^d)^*.$$

Since $a^{d,w} = [(aw)^d]^2 a = a[(wa)^d]^2 = (aw)^d a(wa)^d$, we easily check that $(aw)^d = [(aw)^d]^2 aw = a^{d,w}a$, and then

$$(aw)^d\mathcal{A} = a^{d,w}\mathcal{A}.$$

On the other hand, we have $(a^{d,w})^* = [(aw)^d aw]^*[(aw)^d]^*$ and $[(aw)^d]^* = [((aw)^d)^2 aw]^* = w^*(a^{d,w})^*$. Thus, $\mathcal{A}[(aw)^d]^* = \mathcal{A}(a^{d,w})^*$. Therefore

$$x\mathcal{A} = a^{d,w}\mathcal{A}, \mathcal{A}x = \mathcal{A}(a^{d,w})^*$$

.

$\Longleftarrow$ By hypothesis, There exists $x \in \mathcal{A}$ such that

$$xawx = x, x\mathcal{A} = a^{d,w}\mathcal{A}, \mathcal{A}x = \mathcal{A}(a^{d,w})^*.$$

As the argument above, we have

$$(aw)^d\mathcal{A} = a^{d,w}\mathcal{A}, \mathcal{A}[(aw)^d]^* = \mathcal{A}(a^{d,w})^*.$$

Therefore we have

$$xawx = x, x\mathcal{A} = (aw)^d\mathcal{A}, \mathcal{A}x = \mathcal{A}((aw)^d)^*.$$

In light of [4, Theorem 3.3], $aw \in \mathcal{A}^{ⓓ}$. According to Lemma 3.1, $a_w^{ⓓ} = x$. □

An element $a \in \mathcal{A}$ has pseudo w-core decomposition if there exist $x, y \in \mathcal{A}$ such that

$$a = x + y, x^*y = ywx = 0, x \in \mathcal{A}_w^{\textcircled{\#}}, y \in \mathcal{A}_w^{nil}.$$

The preceding x is unique if it exists, and we denote its w-core inverse by the pseudo w-core inverse of a, i.e., $a_w^{\textcircled{D}} = x_w^{\textcircled{\#}}$. Evidently, $a_w^{\textcircled{D}} = z$ if and only if there exists $n \in \mathbb{N}$ such that

$$z = awz^2, (awz)^* = awz, z(aw)^2 z = awz \text{ and } (aw)^n = awz(aw)^n.$$

In this case, $a_w^{\textcircled{D}} = z$.

Corollary 3.3. *Let $a, w \in \mathcal{A}$. Then a has pseudo w-core inverse if and only if*

(1) $a \in \mathcal{A}_w^{\textcircled{d}}$.
(2) aw has Drazin inverse.

Proof. $\Longrightarrow$ Obviously, $a \in \mathcal{A}_w^{\textcircled{d}}$ and $aw \in \mathcal{A}^{\textcircled{D}}$. In view of [8, Theorem 2.3], aw has Drazin inverse, as desired.

$\Longrightarrow$ In view of Theorem 3.2, there exists $x \in \mathcal{A}$ such that

$$xawx = x, x\mathcal{A} = a^{d,w}\mathcal{A}, \mathcal{A}x = \mathcal{A}(a^{d,w})^*.$$

Since aw has Drazin inverse, we have $a^{d,w} = [(aw)^d]^2 a = [(aw)^D]^2 a = a^{D,w}$. Hence,

$$x(aw)x = x, x\mathcal{A} = (aw)^D\mathcal{A}, \mathcal{A}x = \mathcal{A}((aw)^D)^*.$$

Analogously to Theorem 3.2, we prove that $aw \in \mathcal{A}^{\textcircled{D}}$. Similarly to Lemma 3.1, a has pseudo w-core inverse. □

Let $\mathcal{R}(X)$ represent the range space of a complex matrix X and X^* be the conjugate transpose of X. We improve [2, Theorem 3.5] and provide a new characterizations of pseudo W-core inverse for any complex matrix.

Corollary 3.4. *Let $A, X, W \in \mathbb{C}^{n\times n}$. Then the following are equivalent:*

(1) X is the pseudo W-core inverse of A.
(2) $XAWX = X, \mathcal{R}(X) = \mathcal{R}(X^*) = \mathcal{R}(AW)^D$.

Proof. Obviously, $AW \in \mathbb{C}^{n\times n}$ has Drazin inverse. Since $A^{D,W} = [(AW)^D]^2 A$ and $(AW)^D = [(AW)^D]^2 AW = A^{D,W}W$, we see that $\mathcal{R}(A^{D,W}) = \mathcal{R}(AW)^D$. Therefore we obtain the result by Theorem 3.2 and Corollary 3.3. □

If a and x satisfy the equations $a = axa$ and $(ax)^* = ax$, then x is called $(1,3)$-inverse of a and is denoted by $a^{(1,3)}$. We use $\mathcal{A}^{(1,3)}$ to stand for sets of all $(1,3)$-invertible elements in $\mathcal{A}$. We now derive

Theorem 3.5. *Let $a, w \in \mathcal{A}$. Then the following are equivalent:*

(1) $a \in \mathcal{A}_w^{\textcircled{d}}$.
(2) $a \in \mathcal{A}^{d,w}$ and $a^{d,w} \in \mathcal{A}^{(1,3)}$.
(3) $a \in \mathcal{A}^{d,w}$ and there exists a projection $q \in \mathcal{A}$ such that $a^{d,w}\mathcal{A} = q\mathcal{A}$.

In this case, $a_w^{\textcircled{d}} = a^{d,w}wa^{d,w}(a^{d,w})^{(1,3)} = a^{d,w}wq$.

Proof. (1) $\Rightarrow$ (2) In view of Theorem 3.2, $a \in \mathcal{A}^{d,w}$. Let $x = a_w^{\textcircled{d}}$. By virtue of Theorem 2.1, there exists $x \in \mathcal{A}$ such that

$$x = awx^2, (awx)^* = awx, x(aw)^2x = awx, \lim_{n\to\infty} ||(aw)^n - awx(aw)^n||^{\frac{1}{n}} = 0.$$

Let $z = (wa)^2wx$. Then

$$\begin{array}{rcl} a^{d,w}z & = & [(aw)^d]^2a[(wa)^2wx] \\ & = & (aw)^2(aw)^dx \\ & = & (aw)^2(aw)^d(aw)^{\textcircled{d}} \\ & = & (aw)(aw)^{\textcircled{d}}. \end{array}$$

Therefore $(a^{d,w}z)^* = [(aw)(aw)^{\textcircled{d}}]^* = (aw)(aw)^{\textcircled{d}} = a^{d,w}z$. Moreover, we verify that

$$\begin{array}{rcl} a^{d,w}za^{d,w} & = & (aw)(aw)^{\textcircled{d}}a^{d,w} \\ & = & (aw)(aw)^{\textcircled{d}}[(aw)^d]^2a \\ & = & (aw)(aw)^{\textcircled{d}}(aw)^2[(aw)^d]^4a \\ & = & (aw)^2[(aw)^d]^4a \\ & = & [(aw)^d]^2a \\ & = & a^{d,w}. \end{array}$$

Accordingly, $a^{d,w} \in \mathcal{A}^{(1,3)}$, as required.

(2) $\Rightarrow$ (1) Let $x = a^{d,w}wa^{d,w}(a^{d,w})^{(1,3)}$. Then we check that

$$\begin{array}{rcl} xawx & = & a^{d,w}wa^{d,w}(a^{d,w})^{(1,3)}awa^{d,w}wa^{d,w}(a^{d,w})^{(1,3)} \\ & = & a^{d,w}wa^{d,w}(a^{d,w})^{(1,3)}(aw)a^{d,w}wa^{d,w}(a^{d,w})^{(1,3)} \\ & = & a^{d,w}wa^{d,w}(a^{d,w})^{(1,3)}a^{d,w}(wa)wa^{d,w}(a^{d,w})^{(1,3)} \\ & = & a^{d,w}wa^{d,w}w(awa^{d,w})(a^{d,w})^{(1,3)} \\ & = & a^{d,w}w[a^{d,w}wa]wa^{d,w}(a^{d,w})^{(1,3)} \\ & = & [a^{d,w}wawa^{d,w}]wa^{d,w}(a^{d,w})^{(1,3)} \\ & = & [a^{d,w}wawa^{d,w}]wa^{d,w}(a^{d,w})^{(1,3)} \\ & = & a^{d,w}wa^{d,w}(a^{d,w})^{(1,3)} \\ & = & x. \end{array}$$

Clearly, $x\mathcal{A} \subseteq a^{d,w}\mathcal{A}$. Also we see that

$$a^{d,w} = (a^{d,w}w)^2a = a^{d,w}wa^{d,w}(a^{d,w})^{(1,3)}a^{d,w}wa = xa^{d,w}wa;$$

hence, $a^{d,w}\mathcal{A}\subseteq x\mathcal{A}$. Thus $x\mathcal{A}=a^{d,w}\mathcal{A}$.

We easily verify that

$$\begin{array}{rcl} x &=& a^{d,w}wa^{d,w}(a^{d,w})^{(1,3)} = a^{d,w}w[a^{d,w}(a^{d,w})^{(1,3)}]\\ &=& a^{d,w}w[a^{d,w}(a^{d,w})^{(1,3)}]^* = a^{d,w}w[(a^{d,w})^{(1,3)}]^*[a^{d,w}]^*; \end{array}$$

and then, $\mathcal{A}x\subseteq \mathcal{A}(a^{d,w})^*$. Also we check that

$$\begin{array}{rcl} [a^{d,w}]^* &=& [a^{d,w}(a^{d,w})^{(1,3)}a^{d,w}]^* = [(a^{d,w}(a^{d,w})^{(1,3)})^*a^{d,w}]^*\\ &=& [a^{d,w}]^*a^{d,w}(a^{d,w})^{(1,3)}\\ &=& [a^{d,w}]^*[a^{d,w}wawa^{d,w}](a^{d,w})^{(1,3)}\\ &=& [a^{d,w}]^*[a^{d,w}wa]wa^{d,w}(a^{d,w})^{(1,3)}\\ &=& [a^{d,w}]^*[awa^{d,w}]wa^{d,w}(a^{d,w})^{(1,3)}\\ &=& [a^{d,w}]^*aw[a^{d,w}wa^{d,w}(a^{d,w})^{(1,3)}]\\ &=& [a^{d,w}]^*awx, \end{array}$$

and then $\mathcal{A}(a^{d,w})^*\subseteq \mathcal{A}x$. Hence $\mathcal{A}x=\mathcal{A}(a^{d,w})^*$. Accordingly, $a\in \mathcal{A}^{\textcircled{d}}_w$ by Theorem 3.2.

(2) $\Rightarrow$ (3) By hypothesis, $a^{d,w}\in \mathcal{A}^{(1,3)}$, and so $a^{d,w}=a^{d,w}(a^{d,w})^{(1,3)}a^{d,w}$ and $[a^{d,w}(a^{d,w})^{(1,3)}]^*=a^{d,w}(a^{d,w})^{(1,3)}$. Let $q=a^{d,w}(a^{d,w})^{(1,3)}$. Then $a^{d,w}\mathcal{A}=q\mathcal{A}, q^2=q=q^*$, as required.

(3) $\Rightarrow$ (2) Let $x=a^{d,w}wq$. Then $awx=awa^{d,w}wq=aw[(aw)^d]^2awq=aw(aw)^dq=q$, and so $(awx)^*=q^*=q=awx$. Moreover, we have

$$awx^2=(awx)x=qa^{d,w}wq=a^{d,w}wq=x.$$

Obviously, $a^{d,w}w(aw)=(aw)a^{d,w}w$, and then we verify that

$$\begin{array}{rl} & ||(aw)^n-x(aw)^{n+1}||\\ = & ||[(aw)^n-(a^{d,w}wq)a^{d,w}w(aw)^{n+2}]-[x((aw)^{n+1}-a^{d,w}w(aw)^{n+2})]||\\ \leq & ||(aw)^n-a^{d,w}w(aw)^{n+1}||+||x||||(aw)^{n+1}-a^{d,w}w(aw)^{n+2}||\\ \leq & \big(1+||x||||a||\big)||(aw)^n-a^{d,w}w(aw)^{n+1}||\\ = & \big(1+||x||||a||\big)||(aw)^n(1-a^{d,w}waw)^n||\\ = & \big(1+||x||||a||\big)||(aw-a^{d,w}w(aw)^2)^n||. \end{array}$$

Since $aw-a^{d,w}w(aw)^2=aw-(aw)^d(aw)^2\in \mathcal{A}^{qnil}$, we have

$$\lim_{n\to\infty}||(aw-a^{d,w}w(aw)^2)^n||^{\frac{1}{n}}=0.$$

Therefore

$$\lim_{n\to\infty}||(aw)^n-x(aw)^{n+1}||^{\frac{1}{n}}=0.$$

Then $x=(aw)^{\textcircled{d}}$. In view of Theorem 2.1, $a\in \mathcal{A}^{\textcircled{d}}_w$. In this case, $a^{\textcircled{d}}_w=x=a^{d,w}wq=a^{d,w}wa^{d,w}(a^{d,w})^{(1,3)}$. □

Corollary 3.6. *Let $a, w \in \mathcal{A}$. Then the following are equivalent:*

(1) a has pseudo w-core inverse.
(2) $a \in \mathcal{A}^{D,w}$ and $a^{D,w} \in \mathcal{A}^{(1,3)}$.
(3) $a \in \mathcal{A}^{D,w}$ and there exists a projection $q \in \mathcal{A}$ such that $a^{D,w}\mathcal{A} = q\mathcal{A}$.

In this case, $a_w^{\textcircled{D}} = a^{D,w}wa^{D,w}(a^{D,w})^{(1,3)} = a^{D,w}wq$.

Proof. As $a \in \mathcal{A}^D$, we have $a^d = a^D$. Therefore we complete the proof by Theorem 3.5. □

4. Relations with weighted core inverses

The aim of this section is to establish the relations between generalized weighted core inverse and other weighted generalized inverses. We come now to the demonstration for which this section has been developed.

Theorem 4.1. *Let $a, w \in \mathcal{A}$. Then the following are equivalent:*

(1) $a \in \mathcal{A}_w^{\textcircled{d}}$.
(2) $a \in \mathcal{A}^{d,w}$ and $a^{d,w} \in \mathcal{A}_w^{\textcircled{\#}}$.

In this case,

$$a_w^{\textcircled{d}} = [a^{d,w}w]^2(a^{d,w})_w^{\textcircled{\#}}.$$

Proof. (1) $\Rightarrow$ (2) In view of Theorem 3.2, $a \in \mathcal{A}^{d,w}$. Let $x = a_w^{\textcircled{d}}$. Then we have

$$x = awx^2, (awx)^* = awx, x(aw)^2x = awx, \lim_{n\to\infty} ||(aw)^n - awx(aw)^n||^{\frac{1}{n}} = 0.$$

We verify that

$$\begin{array}{rcl} ||aw(aw)^d - awx(aw)(aw)^d|| & = & ||(aw)^n[(aw)^d]^n - awx(aw)^n[(aw)^d]^n|| \\ & \leq & ||(aw)^n - awx(aw)^n||||[(aw)^d]^n||. \end{array}$$

Since $\lim\limits_{n\to\infty} ||(aw)^n - awx(aw)^n||^{\frac{1}{n}} = 0$, we deduce that

$$\lim_{n\to\infty} ||aw(aw)^d - awx(aw)(aw)^d||^{\frac{1}{n}} = 0.$$

Hence, $awx(aw)(aw)^d = aw(aw)^d$. Let $z = (aw)^2x$. Then

$$\begin{array}{rcl} a^{d,w}wz & = & a^{d,w}w(aw)^2x = [(aw)^d]^2aw(aw)^2x = awx, \\ a^{d,w}wz^2 & = & (awx)z = (awx)(aw)^2x = aw[x(aw)^2x] = (aw)^2x = z, \\ (a^{d,w}wz)^* & = & (awx)^* = awx = a^{d,w}wz, \\ za^{d,w}wa^{d,w} & = & (aw)^2xa^{d,w}wa^{d,w} = aw[awx(aw)(aw)^d]a^{d,w}wa^{d,w} \\ & = & aw[(aw)(aw)^d]a^{d,w}wa^{d,w} = aw[(aw)(aw)^d][(aw)^d]^2awa^{d,w} \\ & = & aw(aw)^da^{d,w} = a^{d,w}. \end{array}$$

Accordingly,

$$a^{d,w}wz^2 = z, (a^{d,w}wz)^* = a^{d,w}wz, za^{d,w}wa^{d,w} = a^{d,w}.$$

Then $a^{d,w} \in \mathcal{A}_w^{\#}$ and $(a^{d,w})_w^{\#} = z = (aw)^2 a_w^{ⓓ}$, as desired.

(2) ⇒ (1) Set $x = (a^{d,w})_w^{\#}$. Then we have

$$a^{d,w}wxa^{d,w} = a^{d,w}, [a^{d,w}wx]^* = a^{d,w}wx.$$

Hence, $a^{d,w} \in \mathcal{A}^{(1,3)}$. According to Theorem 3.5, $a \in \mathcal{A}_w^{ⓓ}$. Moreover, we have

$$a_w^{ⓓ} = a^{d,w}wa^{d,w}(wx) = a^{d,w}wa^{d,w}w(a^{d,w})_w^{\#} = [a^{d,w}w]^2(a^{d,w})_w^{\#}.$$

□

As an immediate consequence, we provide formulas of the pseudo weighted core inverse of a complex matrix.

Corollary 4.2. *Let $A, W \in \mathbb{C}^{n\times n}$. Then*

$$\begin{array}{rcl} A_W^{ⓓ} & = & [A^{D,W}W]^2(A^{D,W})_W^{\#} \\ & = & (AW)^k A[(AW)^{k+1}A]^\dagger, \end{array}$$

where $k = max\{ind(AW), ind(WA)\}$.

Proof. By virtue of Theorem 4.1, $A_W^{ⓓ} = [A^{D,W}W]^2(A^{D,W})_W^{\#}$. In view of [9, Theorem 2.10],

$$W^{ⓓ,A} = (WA)^k[A(WA)^{k+1}]^\dagger.$$

According to Lemma 3.1, we get

$$\begin{array}{rcl} A_W^{ⓓ} & = & (AW)^{ⓓ} = AW[(AW)^{ⓓ}]^2 \\ & = & A[W\big((AW)^{ⓓ}\big)^2] = A[W^{ⓓ,A}] \\ & = & A[(WA)^k[A(WA)^{k+1}]^\dagger] = (AW)^k A[(AW)^{k+1}A]^\dagger, \end{array}$$

as asserted. □

Example 4.3.

Let $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & -1 & -1 \end{pmatrix}, W = \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \in \mathbb{C}^{3\times 3}$. We take the involution on $\mathbb{C}^{4\times 4}$ as the conjugate transpose. Then

$$AW = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, WA = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}.$$

Hence, $max\{ind(AW), ind(WA)\} = 1$. Moreover, we have

$$\begin{aligned} A_W^{\textcircled{D}} &= AWA[(AW)^2A]^{\dagger} \\ &= \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}^{\dagger} \\ &= \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

Evidently, we check that

$$AW(A_W^{\textcircled{D}})^2 = A_W^{\textcircled{D}}, (AWA_W^{\textcircled{D}})^* = AWA_W^{\textcircled{D}}, AW = AWA_W^{\textcircled{D}}AW.$$

Let $a, x \in \mathcal{A}$. x is called $(1,4)$-inverse of a and is denoted by $a^{(1,4)}$ provided that $axa = a$ and $(xa)^* = xa$. We use $\mathcal{A}^{(1,4)}$ to stand for sets of all $(1,4)$ invertible elements in $\mathcal{A}$.

Lemma 4.4. *Let $f \in \mathcal{A}$ be an idempotent. Then the following are equivalent:*

(1) $f \in \mathcal{A}^{(1,3)}$.
(2) $f^{\pi} \in \mathcal{A}^{(1,4)}$.

Proof. See [20, Lemma 3.2]. □

Theorem 4.5. *Let $a, w \in \mathcal{A}$. Then the following are equivalent:*

(1) $a \in \mathcal{A}_w^{\textcircled{d}}$.
(2) $a \in \mathcal{A}^{d,w}$ and $awa^{d,w}w \in \mathcal{A}^{(1,3)}$.
(3) $a \in \mathcal{A}^{d,w}$ and $(aw)^{\pi} \in \mathcal{A}^{(1,4)}$.

In this case, $a_w^{\textcircled{d}} = a^{d,w}w(awa^{d,w}w)^{(1,3)} = a^{d,w}w[1 - (a^{\pi})^{(1,4)}a^{\pi}]$.

Proof. (1) $\Rightarrow$ (2) In view of Theorem 3.2, $a \in \mathcal{A}^{d,w}$. For any $m \in \mathbb{N}$, we check that

$$\begin{aligned} ||awa^{d,w}w - awa_w^{\textcircled{d}}awa^{d,w}w|| &= ||awa^{d,w}w - awa_w^{\textcircled{d}}awa^{d,w}w|| \\ &\leq ||(aw)^m - awa_w^{\textcircled{d}}(aw)^m||||((aw)^d)^m a^{d,w}w||. \end{aligned}$$

Since

$$\lim_{m\to\infty} ||(aw)^m - awa_w^{\textcircled{d}}(aw)^m||^{\frac{1}{m}} = 0,$$

we have

$$\lim_{m\to\infty} ||awa^{d,w}w - awa_w^{\textcircled{d}}awa^{d,w}w||^{\frac{1}{m}} = 0.$$

Hence $awa_w^{\textcircled{d}}awa^{d,w}w = awa^{d,w}w$, and then

$$\begin{array}{rcl} [awa^{d,w}w][awa_w^{\textcircled{d}}] & = & aw[(aw)^d]^2awawa_w^{\textcircled{d}} = awa_w^{\textcircled{d}}, \\ ((awa^{d,w}w)(awa_w^{\textcircled{d}}))^* & = & (awa_w^{\textcircled{d}})^* = awa_w^{\textcircled{d}} = (awa^{d,w}w)(awa_w^{\textcircled{d}}), \\ (awa^{d,w}w)(awa_w^{\textcircled{d}})(awa^{d,w}w) & = & awa_w^{\textcircled{d}}(awa^{d,w}w) = awa_w^{\textcircled{d}}awa^{d,w}w \\ & = & awa^{d,w}w. \end{array}$$

Accordingly, $awa^{d,w}w \in \mathcal{A}^{(1,3)}$, as desired.

(2) $\Rightarrow$ (1) Let $x = a^{d,w}w(awa^{d,w}w)^{(1,3)}$. Then we verify that

$$\begin{array}{rcl} awx & = & awa^{d,w}w(awa^{d,w}w)^{(1,3)} = aw(aw)^d(aw(aw)^d)^{(1,3)}, \\ (awx)^* & = & awx, \\ awx^2 & = & aw(aw)^d(aw(aw)^d)^{(1,3)}a^{d,w}w(awa^{d,w}w)^{(1,3)} \\ & = & aw(aw)^d(aw(aw)^d)^{(1,3)}aw[(aw)^d]^2(awa^{d,w}w)^{(1,3)} \\ & = & (aw)^d(awa^{d,w}w)^{(1,3)} = x, \end{array}$$

$$\begin{array}{rl} & ||(aw)^n - awx(aw)^n|| \\ = & ||(aw)^n - aw(aw)^d(aw(aw)^d)^{(1,3)}(aw)^n|| \\ \leq & ||(aw)^n - (aw)^d(aw)^{n+1}|| + ||(aw)(aw)^d(aw)^n - aw(aw)^d(aw(aw)^d)^{(1,3)}(aw)^n|| \\ \leq & ||(aw)^n - (aw)^d(aw)^{n+1}|| + ||(aw)(aw)^d(aw(aw)^d)^{(1,3)}(aw)(aw)^d(aw)^n \\ - & aw(aw)^d(aw(aw)^d)^{(1,3)}(aw)^n|| \\ \leq & ||(aw)^n - (aw)^d(aw)^{n+1}|| + ||(aw)(aw)^d(aw(aw)^d)^{(1,3)}||||(aw)^d(aw)^{n+1} - (aw)^n|| \\ = & ||(aw)^n - (aw)^d(aw)^{n+1}||[1 + ||(aw)(aw)^d(aw(aw)^d)^{(1,3)}||]. \end{array}$$

Hence,

$$\lim_{n\to\infty} ||(aw)^n - awx(aw)^n||^{\frac{1}{n}} = 0.$$

Therefore $a \in \mathcal{A}_w^{\textcircled{d}}$. In this case, $a_w^{\textcircled{d}} = a^{d,w}w(awa^{d,w}w)^{(1,3)}$.

(2) $\Leftrightarrow$ (3) In view of Lemma 4.4, $aw(aw)^d \in \mathcal{A}^{(1,3)}$ if and only if $(aw)^\pi = 1 - (aw)(aw)^d \in \mathcal{A}^{(1,4)}$, as desired. □

Corollary 4.6. *Let $a, w \in \mathcal{A}$. Then the following are equivalent:*

(1) $a \in \mathcal{A}^{\textcircled{d}}$.
(2) $a \in \mathcal{A}^d$ and $aa^d \in \mathcal{A}^{(1,3)}$.
(3) $a \in \mathcal{A}^d$ and $a^\pi \in \mathcal{A}^{(1,4)}$.

In this case, $a^{\textcircled{d}} = a^d(aa^d)^{(1,3)} = a^d[1 - (a^\pi)^{(1,4)}a^\pi]$.

Proof. This is obvious by choosing $w = 1$ in Theorem 4.6. □

5. GENERALIZED w-CORE ORDERS

This section is devoted to the generalized weighted core order for two elements is a Banach *-algebra. The following result will frequently be applied in investigating properties of generalized weighted core orders.

Lemma 5.1. *Let $a, b \in \mathcal{A}_w^{\circledS}$. Then the following are equivalent:*

(1) $a \leq_w^{\circledS} b$.
(2) $awa^{d,w} = bwa^{d,w}, a^*a^{d,w} = b^*a^{d,w}$.

Proof. (1) $\Rightarrow$ (2) Since $a \leq_w^{\circledS} b$, we have

$$awa_w^{\circledS} = bwa_w^{\circledS}, a_w^{\circledS}a = a_w^{\circledS}b.$$

By virtue of 4.5, $a_w^{\circledS} = a^{d,w}wa^{d,w}(a^{d,w})^{(1,3)}$. Then

$$awa^{d,w}wa^{d,w}(a^{d,w})^{(1,3)} = bwa^{d,w}wa^{d,w}(a^{d,w})^{(1,3)},$$

and so

$$awa^{d,w}wa^{d,w} = bwa^{d,w}wa^{d,w}.$$

Since $a^{d,w} = (a^{d,w}w)^2a$, we have $awa^{d,w} = [awa^{d,w}wa^{d,w}]wa = [bwa^{d,w}wa^{d,w}]wa = bwa^{d,w}$.

Since $a_w^{\circledS}a = a_w^{\circledS}b$, we have

$$a^{d,w}wa^{d,w}(a^{d,w})^{(1,3)}a = a^{d,w}wa^{d,w}(a^{d,w})^{(1,3)}b.$$

As $a^{d,w} = a(wa^{d,w})^2$, we deduce that

$$a^{d,w}(a^{d,w})^{(1,3)}a = a^{d,w}(a^{d,w})^{(1,3)}b.$$

This implies that

$$a^*a^{d,w}(a^{d,w})^{(1,3)} = b^*a^{d,w}(a^{d,w})^{(1,3)}.$$

Therefore $a^*a^{d,w} = b^*a^{d,w}$, as required.

(2) $\Rightarrow$ (1) Since $awa^{d,w} = bwa^{d,w}$, by virtue of Theorem 3.5, we have

$$\begin{aligned} awa_w^{\circledS} &= awa^{d,w}wa^{d,w}(a^{d,w})^{(1,3)} \\ &= bwa^{d,w}wa^{d,w}(a^{d,w})^{(1,3)} \\ &= bwa_w^{\circledS}. \end{aligned}$$

Since $a^*a^{d,w}=b^*a^{d,w}$, we have $a^*a^{d,w}(a^{d,w})^{(1,3)}=b^*a^{d,w}(a^{d,w})^{(1,3)}$, and then $a^{d,w}(a^{d,w})^{(1,3)}a=a^{d,w}(a^{d,w})^{(1,3)}b$. Therefore we derive

$$\begin{array}{rcl} a_w^{\textcircled{d}}a & = & a^{d,w}wa^{d,w}(a^{d,w})^{(1,3)}a \\ & = & a^{d,w}w[a^{d,w}(a^{d,w})^{(1,3)}a] \\ & = & a^{d,w}w[a^{d,w}(a^{d,w})^{(1,3)}b] \\ & = & a_w^{\textcircled{d}}b, \end{array}$$

thus yielding the result. □

Let $a\in\mathcal{A}_w^{\textcircled{\#}}, b\in\mathcal{A}$. Recall that $a\leq_w^{\textcircled{\#}} b$ if $awa_w^{\textcircled{\#}}=bwa_w^{\textcircled{\#}}$ and $a_w^{\textcircled{\#}}a=a_w^{\textcircled{\#}}b$ (see [32]). We are now ready to prove:

Theorem 5.2. *Let $a,b\in\mathcal{A}_w^{\textcircled{d}}$. If $a=a_1+a_2, b=b_1+b_2$ are generalized w-core decompositions of a and b. Then the following are equivalent:*

(1) $a\leq_w^{\textcircled{d}} b$.
(2) $a_1\leq_w^{\textcircled{\#}} b_1$.

Proof. (1) ⇒ (2) Since $a\leq_w^{\textcircled{d}} b$, we have $awa_w^{\textcircled{d}}=bwa_w^{\textcircled{d}}$ and $a_w^{\textcircled{d}}a=a_w^{\textcircled{d}}b$. For any $m\in\mathbb{N}$, we derive

$$\begin{array}{rcl} a_1w(a_1)_w^{\textcircled{\#}} & = & (a_1+a_2)w(a_1)_w^{\textcircled{\#}}=awa_w^{\textcircled{d}}=bwa_w^{\textcircled{d}} \\ & = & bwaw(a_w^{\textcircled{d}})^2=bw[awa_w^{\textcircled{d}}]a_w^{\textcircled{d}}=bw[bwa_w^{\textcircled{d}}]a_w^{\textcircled{d}} \\ & = & (bw)^2(a_w^{\textcircled{d}})^2=\cdots=(bw)^m(a_w^{\textcircled{d}})^m, \\ b_1w(a_1)_w^{\textcircled{\#}} & = & bwb_w^{\textcircled{d}}bwa_w^{\textcircled{d}}=bwb_w^{\textcircled{d}}bwaw(a_w^{\textcircled{d}})^2=bwb_w^{\textcircled{d}}(bw)^2(a_w^{\textcircled{d}})^2 \\ & = & \cdots=bwb_w^{\textcircled{d}}(bw)^m(a_w^{\textcircled{d}})^m. \end{array}$$

Thus, we have

$$\begin{array}{rl} & ||a_1w(a_1)_w^{\textcircled{\#}}-b_1w(a_1)_w^{\textcircled{\#}}|| \\ = & ||(bw)^m(a_w^{\textcircled{d}})^m-bwb_w^{\textcircled{d}}(bw)^m(a_w^{\textcircled{d}})^m|| \\ \leq & ||(bw)^m-bwb_w^{\textcircled{d}}(bw)^m||||(a_w^{\textcircled{d}})^m||. \end{array}$$

In view of Theorem 2.1,

$$\lim_{m\to\infty}||(bw)^m-bwb_w^{\textcircled{d}}(bw)^m||^{\frac{1}{m}}=0.$$

Hence,

$$\lim_{m\to\infty}||a_1w(a_1)_w^{\textcircled{\#}}-b_1w(a_1)_w^{\textcircled{\#}}||^{\frac{1}{m}}=0.$$

Therefore $a_1w(a_1)_w^{\textcircled{\#}}=b_1w(a_1)_w^{\textcircled{\#}}$.

Since $b_1=bwb_w^{\textcircled{d}}b$, we verify that

$$awa_w^{\textcircled{d}}=a_1w(a_1)_w^{\textcircled{\#}}=b_1w(a_1)_w^{\textcircled{\#}}=bwb_w^{\textcircled{d}}bwa_w^{\textcircled{d}}=bwb_w^{\textcircled{d}}awa_w^{\textcircled{d}}.$$

Thus,

$$[awa^{\textcircled{d}}_w]^* = [bwb^{\textcircled{d}}_w awa^{\textcircled{d}}_w]^*,$$

and so

$$awa^d_w = awa^{\textcircled{d}}_w bwb^{\textcircled{d}}_w.$$

Then we see that

$$\begin{array}{lll} (a_1)^{\textcircled{\#}}_w a_1 & = & a^{\textcircled{d}}_w (awa^{\textcircled{d}}_w a) = a^{\textcircled{d}}_w (awa^{\textcircled{d}}_w) a \\ & = & a^{\textcircled{d}}_w (awa^{\textcircled{d}}_w) b \\ & = & a^{\textcircled{d}}_w (awa^{\textcircled{d}}_w bwb^{\textcircled{d}}_w) b \\ & = & (a^{\textcircled{d}}_w awa^{\textcircled{d}}_w) bwb^{\textcircled{d}}_w) b \\ & = & a^{\textcircled{d}}_w (bwb^{\textcircled{d}}_w b) = (a_1)^{\textcircled{\#}}_w b_1. \end{array}$$

Therefore $a_1 \leq^{\textcircled{\#}}_w b_1$.

(2) $\Rightarrow$ (1) Obviously, we have

$$awa^{\textcircled{d}}_w = (a_1 + a_2) wa^{\textcircled{\#}}_1 = a_1 wa^{\textcircled{\#}}_1 = b_1 wa^{\textcircled{\#}}_1 = bwb^{\textcircled{d}}_w bwa^{\textcircled{d}}_w.$$

Then

$$a^{\textcircled{d}}_w = aw(a^{\textcircled{d}}_w)^2 = bwb^{\textcircled{d}}_w bw(a^{\textcircled{d}}_w)^2.$$

Since $\lim\limits_{n\to\infty} ||(bw)^n - bwb^{\textcircled{d}}_w (bw)^n||^{\frac{1}{n}} = 0$, we deduce that

$$bwb^{\textcircled{d}}_w bwa^{\textcircled{d}}_w = bwa^{\textcircled{d}}_w.$$

This implies that

$$awa^{\textcircled{d}}_w = bwa^{\textcircled{d}}_w.$$

Clearly, $a^{\textcircled{d}}_w a_2 = (a_1)^{\textcircled{\#}}_w a_2 = (a_1)^{\textcircled{\#}}_w a_1 w (a_1)^{\textcircled{\#}}_w a_2 = (a_1)^{\textcircled{\#}}_w (a_1 w (a_1)^{\textcircled{\#}}_w)^* a_2 = (a_1)^{\textcircled{\#}}_w [w(a_1)^{\textcircled{\#}}_w]^* (a_1)^* a_2 = 0$.

Moreover, we have

$$\begin{array}{lll} awa^{\textcircled{d}}_w & = & bwb^{\textcircled{d}}_w bwa^{\textcircled{d}}_w \\ & = & (bwb^{\textcircled{d}}_w)(awa^{\textcircled{d}}_w). \end{array}$$

Then

$$\begin{array}{lll} awa^{\textcircled{d}}_w & = & (awa^{\textcircled{d}}_w)^* \\ & = & (awa^{\textcircled{d}}_w)^* (bwb^{\textcircled{d}}_w)^* \\ & = & awa^{\textcircled{d}}_w bwb^{\textcircled{d}}_w. \end{array}$$

Hence, $a^{\textcircled{d}}_w = a^{\textcircled{d}}_w awa^{\textcircled{d}}_w = a^{\textcircled{d}}_w awa^{\textcircled{d}}_w bwb^{\textcircled{d}}_w = a^{\textcircled{d}}_w bwb^{\textcircled{d}}_w$. Accordingly, $a^{\textcircled{d}}_w b = a^{\textcircled{d}}_w bwb^{\textcircled{d}}_w b = (a_1)^{\textcircled{d}}_w b_1 = (a_1)^{\textcircled{d}}_w a_1 = a^{\textcircled{d}}_w (a_1 + a_2) = a^{\textcircled{d}}_w a$, thus yielding the result. □

Corollary 5.3. *The relation $\leq^{\textcircled{d}}_w$ for generalized w-core invertible elements is a pre-order on $\mathcal{A}$.*

Proof. Step 1. $a \leq_w^{\textcircled{d}} a$. Let $a = a_1 + a_2$ be the generalized w-core decomposition. In view of [32, Theorem 2.3], $a_1 \leq_w^{\textcircled{\#}} a_1$. By using Theorem 5.1, $a \leq_w^{\textcircled{d}} a$.

Step 2. Assume that $a \leq_w^{\textcircled{d}} b$ and $b \leq_w^{\textcircled{d}} c$. We claim that $a \leq_w^{\textcircled{d}} c$. Let $a = a_1 + a_2, b = b_1 + b_2$ and $c = c_1 + c_2$ be the generalized w-core decompositions of a, b and c, respectively. By virtue of Lemma 5.1, we have $a_1 \leq_w^{\textcircled{\#}} b_1$ and $b_1 \leq_w^{\textcircled{\#}} c_1$. In view of [32, Theorem 2.3], we have $a_1 \leq_w^{\textcircled{\#}} c_1$. By using Lemma 5.1 again, $a \leq_w^{\textcircled{d}} c$.

Therefore the relation $\leq_w^{\textcircled{d}}$ for generalized w-core invertible elements is a pre-order. $\square$

The relation $\leq_w^{\textcircled{d}}$ for generalized w-core invertible elements is a pre-order, while it is not partial order as the following shows.

Example 5.4.

Let $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, W = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathbb{C}^{3\times 3}$.

We take the involution on $\mathbb{C}^{3\times 3}$ as the conjugate transpose. Then $A^{D,W} = [(AW)^D]^2 A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = [(BW)^D]^2 B = B^{D,W}$. By using Lemma 5.1, we directly verify that $A \leq_W^{\textcircled{d}} B$ and $B \leq_W^{\textcircled{d}} A$. But $A \neq B$.

Theorem 5.5. *Let $a, b \in \mathcal{A}_w^{\textcircled{d}}$. Then the following are equivalent:*

(1) $a \leq_w^{\textcircled{d}} b$.
(2) a and b are represented as

$$a = \begin{pmatrix} a_1 & a_{12} \\ 0 & a_2 \end{pmatrix}_{(p,q)}, b = \begin{pmatrix} a_1 & a_{12} \\ 0 & a_2 + (b-a) \end{pmatrix}_{(p,q)},$$

where $p = awa_w^{\textcircled{d}}$ and $q = (wa)(wa)^d$.

Proof. (1) $\Rightarrow$ (2) Let $p = awa_w^{\textcircled{d}}$ and $q = (wa)(wa)^d$. Then $p^2 = p = p^* \in \mathcal{A}$ and $q^2 = q \in \mathcal{A}$. We check that

$$\begin{aligned} (1-p)aq &= [1 - awa_w^{\textcircled{d}}]a(wa)(wa)^d \\ &= [1 - awa_w^{\textcircled{d}}]awa(wa)^n[(wa)^d]^n \\ &= [1 - aw(aw)^{\textcircled{d}}](aw)^{n+1}a[(wa)^d]^n \\ &= [(aw)^{n+1} - (aw)a_w^{\textcircled{d}}(aw)^{n+1}]a[(wa)^d]^n \\ &= 0; \end{aligned}$$

then $(1-p)ap = 0$. Moreover, we verify that

$$\begin{array}{rcl}(1-p)bq & = & [1-awa_w^{ⓓ}]b(wa)(wa)^d\\ & = & [1-awa_w^{ⓓ}]bwa^{d,w}wa\\ & = & [1-awa_w^{ⓓ}]awa^{d,w}wa\\ & = & 0.\end{array}$$

Write $b = \left(\begin{array}{cc} b_1 & b_{12}\\ 0 & b_2\end{array}\right)_p$. Clearly, we have

$$\begin{array}{rcl} pbq & = & awa_w^{ⓓ}b(wa)(wa)^d\\ & = & awa_w^{ⓓ}bwa^{d,w}wa\\ & = & awa_w^{ⓓ}awa^{d,w}wa\\ & = & awa_2^{ⓓ}a(wa)(wa)^d\\ & = & paq,\end{array}$$

and so $a_1 = b_1$.

Also we have

$$\begin{array}{rcl} pb(1-q) & = & awa_w^{ⓓ}b[1-(wa)(wa)^d]\\ & = & aw(a_w^{ⓓ}b)[1-(wa)(wa)^d]\\ & = & aw(a_w^{ⓓ}a)[1-(wa)(wa)^d]\\ & = & pa(1-p).\end{array}$$

Moreover, $(1-p)b(1-q) = (1-p)b = b - pb = b - a(a_w^{ⓓ}b) = b - aa_w^{ⓓ}a = b - pa = (1-p)a + (b-a) = a_2 + (b-a)$, as desired.

(2) $\Rightarrow$ (1) By hypothesis, $paq = pbq$ and $pa(1-q) = pb(1-q)$. Then $pa = pb$. Hence, $awa_w^{ⓓ}a = awa_w^{ⓓ}b$. This implies that $a_w^{ⓓ}a = a_w^{ⓓ}b$.

Moreover, we have $(1-p)aq = 0 = (1-p)bq$. As $paq = pbq$, we have $aq = bq$, and so $a(wa)(wa)^d = b(wa)(wa)^d$. Then $awa_w^{ⓓ} = bwa_w^{ⓓ}$. In light of Lemma 5.1, $a \leq_w^{ⓓ} b$, as asserted. □

The generalized core-EP inverse for a Banach algebra element was introduced in [5]. $a \leq^{ⓓ} b$ if and only if $aa^{ⓓ} = ba^{ⓓ}$ and $a^{ⓓ}a = a^{ⓓ}b$. As an immediate consequence of Theorem 5.5, we derive

Corollary 5.6. *Let $a, b \in \mathcal{A}^{ⓓ}$. Then the following are equivalent:*

(1) $a \leq^{ⓓ} b$.

(2) a *and* b *are represented as*

$$a = \left(\begin{array}{cc} a_1 & a_{12}\\ 0 & a_2\end{array}\right)_{(p,q)}, b = \left(\begin{array}{cc} a_1 & a_{12}\\ 0 & a_2+(b-a)\end{array}\right)_{(p,q)},$$

where $p = aa^{ⓓ}$ *and* $q = aa^d$.

Conflict of interest

The authors declare there is no conflicts of interest.

Data Availability Statement

The data used to support the findings of this study are included within the article.

References

[1] O.M. Baksalary and G. Trenkler, Core inverse of matrices, *Linear Multilinear Algebra*, **58**(2010), 681–697.

[2] R. Behera; G. Maharana and J.K. Sahoo, Further results on weighted core-EP inverse of matrices, *Result. Math.*, **75**(2020), Paper No. 174, 20 p.

[3] H. Chen and M. Sheibani, *Theory of Clean Rings and Matrices*, World Scientific, Hackensack, NJ, 2023. https://doi.org/10.1142/12959.

[4] H. Chen and M. Sheibani, Generalized weighted core inverse in Banach *-algebras, *Filomat*, **38**(2024), 3691–3706.

[5] H. Chen and M. Sheibani, Some new results in generalized weighted core inverse in Banach *-algebras, preprint, 2023. https://doi.org/10.21203/rs.3.rs-3329741/v1.

[6] J. Chen; H. Zhu; P. Patricio and Y. Zhang, Characterizations and representations of core and dual core inverses, *Canad. Math. Bull.*, 2016, http://dx.doi.org/10.4153/CMB-2016-045-7.

[7] S. Das; J.K. Sahoo; K. Jajati and R. Behera, Further results on weighted core inverse in a ring, *Linear Multilinear Algebra*, **71**(2023), 2915–2934.

[8] Y. Gao and J. Chen, Pseudo core inverses in rings with involution, *Commun. Algebra*, **46**(2018), 38–50.

[9] Y. Gao; J. Chen and P. Patrício, Representations and properties of the W-weighted core-EP inverse, *Linear Multilinear Algebra*, **68**(2020), 1160–1174.

[10] T. Jin; H. Zhu and L. Wu, New characterizations for w-core inverses in rings with involution, *Filomat*, 37(2023), 2131–2139.

[11] T. Li, Characterizations of weighted core inverse in rings with involution, *J. Algebra Appl.*, 2022, https://doi.org/10.1142/S021949882350216X.

[12] T. Li and M. Zhou, The absorption laws for the weighted core inverse in rings, *Linear Multilinear Algebra*, **71**(2023), 480–495 .

[13] D. Mosić, Weighted generalized Drazin inverse in rings, *Georgian Math. J.*, **23**(2016), 587–594.

[14] D. Mosić, Core-EP pre-order of Hilbert space operators, *Quaest. Math.*, **41**(2018), 585–600.

[15] D. Mosić, Core-EP inverse in ring with involution, *Publ. Math. Debrecen*, **96**(2020), 427–443.

[16] D. Mosić, Core-EP inverses in Banach algebras, *Linear Multilinear Algebra*, **69**(2021), 2976–2989.

[17] D. Mosić, Weighted core-EP inverse and weighted core-EP pre-orders in a C^*-algebra, *J. Aust. Math. Soc.*, **111**(2021), 76–110.

[18] D. Mosić; G. Dolinar and J. Marovt, EP-quasinilpotent decomposition and its applications, *Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat.,*, **115**(2021), No. 4, Paper No. 188, 25 p.

[19] K.M. Prasad and K.S. Mohana, Core-EP inverse, *Linear Multilinear Algebra*, **62**(2014), 792–802.

[20] G. Shi; J. Chen; T. Li and M. Zhou, Jacobson's lemma and Cline's formula for generalized inverses in a ring with involution, *Commun. Algebra*, **48**(2020), 3948–3961.

[21] D.S Rakic; N.C. Dincic and D.S. Djordjevic, Group, Moore-Penrose, core and dual core inverse in rings with involution, *Linear Algebra Appl.*, **463**(2014), 115–133.

[22] H. Wang, Core-EP Decomposition and its applications, *Linear Algebra Appl.*, **508** (2016), 289–300.

[23] S. Xu; J. Chen and X. Zhang, New characterizations for core inverses in rings with involution, *Front. Math. China*, 2017, DOI: 10.1007/s11464-016-0591-2.

[24] Y. Yang and H. Zhu, The w-core inverse of a product and its applications, *Filomat*, 37(2023), 4587–4601.

[25] Q. Zhang; C. Wang and H. Zhu, Characterizations and representations of w-core inverses in rings, *Filomat*, 37(2023), 3183–3190.

[26] Q. Zhang; H. Zhu and H. Zou, Left and right w-core partial orders in rings, *Acta Math. Hung.*, **169**(2023), 432–446.

[27] M. Zhou and J. Chen, Characterizations and maximal classes of elements related to pesudo core inverses, *Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat.*, **114**(2020), https://doi.org/ 10.1007/s13398-020-00832-4.

[28] H. Zhu and P. Patricio, Characterizations for pseudo core inverses in a ring with involution, *Linear Multilinear Algebra*, **67**(2019), 1109–1120.

[29] H. Zhu and Q. Wang, Weighted Moore-Penrose inverses and weighted core inverses in rings with involution, *Chin. Ann. Math., Ser. B*, **42**(2021), 613–624.

[30] H. Zhu; L. Wu and J. Chen, A new class of generalized inverses in semigroups and rings with involution, *Commun. Algebra*, **51**(2023), 2098–2113.

[31] H. Zhu; L. Wu and D. Mosić, One-sided w-core inversesin rings with an involution, *Linear Multilinear Algebra*, **71**(2023), 528–544.

[32] H. Zhu and L. Wu, A new class of partial orders, arXiv:2309.13588v1 [math.RA] 24 Sep 2023.

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