Causal effect vector and multiple correlation

Ait-taleb Nabil [*]

Abstract

In this article, we will describe the mechanism that links the notion of causality to correlations. This article answers yes to the following question: Can we deduce a causal relationship from correlations?

[*]Corresponding author: nabiltravail1982@gmail.com

1 Introduction

In this paper, we will understand from a proof how to relate the notion of causality to the correlation. For this, we will have to introduce the causal effect vector $\vec{\Delta}E$. This vector $\vec{\Delta}E$ will be related to the square root of a quadratic form containing the correlations $\sqrt{K_{X\Omega}.K_{\Omega^2}^{-1}.K_{\Omega X}}$.

2 Multiple correlation and causal effect vector

We will describe below the relationship relating the causal effect vector $\vec{\Delta}E$ and the correlations $\sqrt{K_{X\Omega}.K_{\Omega^2}^{-1}.K_{\Omega X}}$:

$$\boxed{\frac{\sqrt{Var(\frac{\vec{X}+\vec{\Delta}E}{2})}}{\sqrt{Var(\vec{X})}} = \sqrt{K_{X\Omega}.K_{\Omega^2}^{-1}.K_{\Omega X}}}$$

Where:

1. $Var(.)$ is the variance.
2. $\vec{X}$ is the signal obtained when Ω **does not impact** the signal $\vec{X}$.
3. $\frac{\vec{X}+\vec{\Delta}E}{2}$ is the signal obtained when Ω **impacts** the signal $\vec{X}$.
4. $\vec{\Delta}E = \vec{E}(X|\Omega) - \vec{E}(X|\tilde{\Omega})$ is the causal effect vector of the causes Ω acting on the vector $\vec{X}$.
5. $\sqrt{K_{X\Omega}.K_{\Omega^2}^{-1}.K_{\Omega X}}$ is the multiple correlation and $0 \leq \sqrt{K_{X\Omega}.K_{\Omega^2}^{-1}.K_{\Omega X}} \leq 1$.

Proof:

From the conditional average $\vec{E}(.|.)$, we will define the following signals:

1. $\vec{E}(X|\Omega)$ is the causal signal obtained when the causes Ω **act** on the variable X. This causal signal is produced by the intervention of causes Ω in the signal X.
2. $\vec{E}(X|\tilde{\Omega})$ is the signal obtained when the causes Ω **do not act** on the variable X. This signal is produced without the intervention of causes Ω in the signal X.

The total signal $\vec{X}$ is obtained by summing the two previous signals:

$$\vec{X} = \vec{E}(X|\Omega) + \vec{E}(X|\tilde{\Omega}) \qquad (1)$$

Note that it is easy to show that the two signals $\vec{E}(X|\Omega)$ and $\vec{E}(X|\tilde{\Omega})$ are uncorrelated: $cor(\vec{E}(X|\Omega), \vec{E}(X|\tilde{\Omega})) = 0$, where $cor()$ is the correlation.

We will now describe the causal effect vector $\vec{\Delta}E$:

$$\vec{\Delta}E = \vec{E}(X|\Omega) - \vec{E}(X|\tilde{\Omega}) \qquad (2)$$

By combining relations (1) and (2):

$$\vec{E}(X|\Omega) = \frac{1}{2}.(\vec{\Delta}E + \vec{X}) \qquad (3)$$

As we know that (see appendix):

$$\frac{\sqrt{Var(\vec{E}(X|\Omega))}}{\sqrt{Var(\vec{X})}} = \sqrt{K_{X\Omega}.K_{\Omega^2}^{-1}.K_{\Omega X}} \qquad (4)$$

We obtain by combining the relations (3) and (4):

$$\frac{1}{2}.\frac{\sqrt{Var(\vec{\Delta}E + \vec{X})}}{\sqrt{Var(\vec{X})}} = \sqrt{K_{X\Omega}.K_{\Omega^2}^{-1}.K_{\Omega X}}$$

$$\frac{\sqrt{Var(\frac{\vec{X}+\vec{\Delta}E}{2})}}{\sqrt{Var(\vec{X})}} = \sqrt{K_{X\Omega}.K_{\Omega^2}^{-1}.K_{\Omega X}}$$

This relationship therefore links the causal effect vector $\vec{\Delta}E$ of the causes Ω acting on the variable X to the correlations $\sqrt{K_{X\Omega}.K_{\Omega^2}^{-1}.K_{\Omega X}}$.

3 Appendix

3.1 Conditional average vector and correlations

The relationship which links the conditional average vector to the correlations can be written as follows:

$$\boxed{\sqrt{K_{X,\Omega}.K_{\Omega^2}^{-1}.K_{\Omega,X}} = \sqrt{\frac{Var(E(X|\Omega))}{Var(X)}}}$$

where $0 \leq \sqrt{K_{X,\Omega}.K_{\Omega^2}^{-1}.K_{\Omega,X}} \leq 1$ is the multiple correlation, $E(.|.)$ is the conditional average and $Var(.)$ is the variance.

Proof:

In what follows, we will factorize the variance Σ_{X^2} of the conditional variance $\Sigma_{X^2|\Omega}$ to show the correlations K:

$\Sigma_{X^2|\Omega} = \Sigma_{X^2} - \Sigma_{X,\Omega}.\Sigma_{\Omega^2}^{-1}.\Sigma_{X,\Omega}$

$\Sigma_{X^2|\Omega} = \Sigma_{X^2} - \Sigma_{X,\Omega}.(diag^{-1}(\Sigma_{\Omega^2}))^{\frac{1}{2}}.K_{\Omega^2}^{-1}.(diag^{-1}(\Sigma_{\Omega^2}))^{\frac{1}{2}}.\Sigma_{\Omega,X}$

$\Sigma_{X^2|\Omega} = \Sigma_{X^2} - \Sigma_{X^2}^{\frac{1}{2}}.K_{X,\Omega}.K_{\Omega^2}^{-1}.\Sigma_{X^2}^{\frac{1}{2}}.K_{\Omega,X}$

$\Sigma_{X^2|\Omega} = \Sigma_{X^2}.(1 - K_{X,\Omega}.K_{\Omega^2}^{-1}.K_{\Omega,X})$

The relationship can also be written:

$$K_{X,\Omega}.K_{\Omega^2}^{-1}.K_{\Omega,X} = 1 - \frac{\Sigma_{X^2|\Omega}}{\Sigma_{X^2}} = 1 - \frac{||X - E(X|\Omega)||^2}{||X - E(X)||^2} = \frac{||X - E(X)||^2 - ||X - E(X|\Omega)||^2}{||X - E(X)||^2}$$

Using the Pythagorean Theorem:

$$||X - E(X)||^2 = ||E(X|\Omega) - E(X)||^2 + ||X - E(X|\Omega)||^2$$

$$K_{X,\Omega}.K_{\Omega^2}^{-1}.K_{\Omega,X} = \frac{||E(X|\Omega) - E(X)||^2}{||X - E(X)||^2} = \frac{\frac{||E(X|\Omega)-E(X)||^2}{N}}{\frac{||X-E(X)||^2}{N}}$$

As we have:$E_\Omega(E(X|\Omega)) = \frac{1}{N}\sum_{\Omega} E(X|\Omega) = E(X)$, we obtain:

$$K_{X,\Omega}.K_{\Omega^2}^{-1}.K_{\Omega,X} = \frac{Var(E(X|\Omega))}{Var(X)}$$

By taking the square root we obtain the relationship.

4 Conclusion

In this paper, we have shown mathematically the steps to follow to obtain a relationship relating the notion of causality and correlation.

[1]Optimal stastical decisions. Author: Morris H.DeGroot. Copyright 1970-2004 John Wiley and sons.

[2]Matrix Analysis. Author: Roger A.Horn and Charles R.Johnson. Copyright 2012, Cambridge university press.