# A unified theory of gravity and inertia

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Abstract. In this paper, we show how the phenomenon of inertia can be explained in non-relativistic classical mechanics using a unified theory of gravity and inertia. As a basis, we used the inertia-free mechanics of H.J. Treder. It can implement both Mach's principle and the idea of inertia having a gravitational origin without the shortcomings of an anisotropic inertial mass. Inertia arises from a velocity-dependent part of the gravitational potential. Thus, it will be possible to formulate classical mechanics with postulating neither the weak equivalence principle, nor a gravitational constant, nor any concept of inertial mass or inertial forces a priori. We will show that all four can be derived from the theory. The theory is valid in arbitrary accelerated frames of reference and the inertial frames are determined by all other particles in the universe, as demanded by Mach's principle. The exact Newtonian inertial forces will appear in any non-inertial frame, for translational and rotational acceleration, showing that they are not fictitious, but real parts of the gravitational force. In the lowest order v/c of the theory, Newtonian mechanics is obtained. The corrections that appear are shown to be just the terms present in Gravitoelectromagnetism. Ultimately, explaining inertia as a gravitational effect will allow us to derive an expression for the gravitational constant, enabling us to explain the apparent weakness of gravity.

Such a unified theory of gravity and inertia has profound implications for the nature of mass and structure of elementary particles, as well as the origin of relativistic and quantum effects. This suggests a very different path towards a combined theory of relativity, gravity, and quantum mechanics, as well as elementary particles. This will be discussed in a subsequent paper.

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*Date*: Draft of October 14, 2024.

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## 1. Introduction

The origin of inertia remains unknown. Neither classical mechanics nor general relativity provides a satisfactory explanation for the inertial properties of matter. As this problem is already manifestly present in classical mechanics, it is necessary to first look for a non-relativistic theory that correctly explains the origin of inertia. Subsequently, one can deal with the relativistic case, using this theory as a starting point, to not carry over this shortcoming of Newtonian mechanics. It turns out that the solution of the non-relativistic problem of the origin of inertia indeed has profound and far-reaching consequences for the nature of mass and structure of the elementary particles, suggesting a very different origin of relativistic and quantum effects. In this paper, we therefore only deal with the classical, non-relativistic problem, and discuss the quantum-relativistic in a subsequent paper.

The question about the origin of inertia can be divided into two sub-questions. The first is "What is the origin of the inertial forces that arise when a body is accelerated?" and the second is "An acceleration relative to *what* causes inertial forces to appear?". The first question is necessarily closely related to the origin of inertial mass.

Although the first question remains unanswered to this day, the second one has already been addressed by Newton. He postulated that there exists an absolute space, and that a body experiences inertial forces when it accelerates relative to it. He tried to prove this in his famous bucket experiment. A bucket hung by a long cord was filled with water and the cord was twisted. Then, an external force holds the bucket in place. The water surface remains flat. Afterwards, the bucket is released and begins to spin. At first, the water remains at rest, but after a while, it starts to more and more co-rotate with the bucket. Its surface is then pushed up the walls of the bucket and forms a parabolic shape. Newton argued that this is proof of acceleration being absolute, that is, the inertial forces arise when a body is accelerating (in this case rotating) relative to absolute space. One could always tell whether a body is accelerating, since then the water would be pushed up against the walls. On the other hand, the water surface is flat when the bucket is rotating relative to the water at the beginning, but curved, when the water is co-rotating to the bucket. Thus, the rotation of the water relative to the bucket appears to play no role in whether inertial forces arise or not. Criticizing Newton in his bucket experiment, Mach [1] objected that 'it (the bucket experiment) only informs us, that the motion of the water relative to the sides of the vessel produces no noticeable centrifugal forces, but that such forces are produced by its rotation relative to the mass of the Earth and the other celestial bodies.' and that 'no one is competent to say how the experiment would turn out if the sides of the vessel increased in thickness and mass till they were ultimately several leagues thick' [1, p. 216 f]. This last statement already hints at the influence of the motion of the water relative to other particles in the universe on the question of when a body experiences inertial forces. And further, that the strength of this influence is weighted by the masses of those bodies. Indeed, Mach argued, that only relative quantities are determined by the dynamical laws of the universe, and in turn, only these relative quantities must enter the dynamical laws of the universe. He wrote that '[...] The universe is not twice given, with an earth at rest and an earth in motion; but only once, with its relative motions, alone determinable.' Thus, as was also first demanded for such purely relative mechanics by Huygens [2] and later also Poincaré [3,4] (see also Treder [5]), no absolute quantities like the positions $\mathbf{r}_k$ or velocities $\mathbf{v}_k$ must enter the Lagrange function of the universe. Instead, it must purely depend on relative quantities $r_{ij} = |\mathbf{r}_i - \mathbf{r}_j|$, $\dot{r}_{ij}$ , $\mathbf{v}_{ij} = \mathbf{v}_i - \mathbf{v}_j$. Therefore, the kinetic energy must have the general form

$$T = \sum_{i>j} m_i m_j f(r_{ij}, \dot{r}_{ij}, \mathbf{v}_{ij}). \tag{1.1}$$

According to Mach, also the inertial frames of reference, the frames in which Newton's laws of motion hold, should be completely determined by the relative motion of all particles in the universe, which is a direct consequence of a kinetic energy of the form (1.1). And not like in Newtonian theory, by a postulated absolute space, which is unobservable. Consequently, inertial forces should arise when a body is accelerated relative to the other masses in the universe, instead of absolute space.

For the first question, there was no answer to it given by Newton at all. Newton's second law

$$\mathbf{F} = m\mathbf{a} \tag{1.2}$$

only holds in an inertial frame of reference and when you go into a non-inertial frame, the inertial forces have to be *postulated.* The force on the right side of (1.2) doesn't automatically obey the transformation law

$$\mathbf{F} \to \mathbf{F} + m\mathbf{a} \tag{1.3}$$

when transformed into an accelerated frame moving with a velocity V(t). This one can easily see, especially for the Newtonian gravitational force

$$\mathbf{F}_k = \sum_{j \neq k} \frac{Gm_k m_j}{r_{kj}^3} \mathbf{r}_{kj}. \tag{1.4}$$

There was also no answer given by Mach. He also did not attach any particular importance to the explanation of inertial mass. In his opinion, it is just empirically defined by Newton's third law: If two bodies act on each other, they experience accelerations in opposite directions and of magnitude $a_1/a_2 = m_2/m_1$ . Inertial mass is then just empirically defined as the inverse of the ratio of accelerations. Mach held the opinion that 'every venture beyond this will only be productive in obscurity.' However, Mach's demand that only relative quantities enter the dynamical laws of the universe, already implies that inertial mass is not an intrinsic property of matter, but results from an interaction with all other particles in the universe. This is an immediate mathematical consequence of a kinetic energy satisfying Mach's principle having the form (1.1). A hint on the nature of this interaction lies in the empirical equality of gravitational and inertial acceleration, as well as the proportionality between inertial and gravitational mass (today termed the equivalence principle). This suggests that inertial mass and forces are of gravitational origin, that is, a part of the gravitational force itself. It was first proposed by Friedlaender [6, p. 17], that '[...] the correct form of the law of inertia will only then have been found when relative inertia as an effect of masses on each other and gravitation, which is also an effect of masses on each other, have been derived on the basis of a unified law.' This was later picked up by Einstein. In [7] he argued that 'the G-field (the metric tensor) is completely defined by the masses of the bodies (of the universe)'. Since the metric tensor determines the inertial mass of a body in special and general relativity, his definition implied that inertial mass is of gravitational origin. His general theory of relativity was intended to incorporate this idea, but, according to his own words, failed to do so: A particle in an empty universe, which corresponds to flat Minkowski space, does have a non-vanishing inertial mass. If it were indeed, according to his definition, completely determined by the gravitational interaction with other masses, this could not be the case [8].

However, it is clear, that both answers to each question together and correctly implemented will result in a theory correctly accounting for the inertial properties of matter. Historically, there were many attempts to build theories at least partially incorporating both ideas. As was proposed by Barbour & Bertotti [9], a non-relativistic theory realizing Mach's principle should be invariant under transformations of the form

$$\mathbf{r} \rightarrow A(t)\mathbf{r} + \mathbf{g}(t), \tag{1.5}$$

with A an orthogonal matrix and g a displacement vector. This invariance ensures the dependence of the theory on purely relative quantities, for translational and rotational motion, as demanded by Mach's principle. First, Barbour & Bertotti [10] and later also Lynden-Bell & Katz [11] developed a mathematically equivalent non-relativistic theory invariant under (1.5). However, they chose a too restricted set of solutions and thus were unable to explain the inertial forces, although their theory is in principle capable of doing so. Further, they didn't incorporate the idea of inertia being of gravitational origin. It is for this reason they were not able to derive the concepts of inertial mass, the weak equivalence principle as well as obtain an explanation for the gravitational constant. We will discuss this at the end of section V.

Many, especially earlier, attempts to incorporate both ideas resulted in theories based on velocity-dependent gravitational potentials, only depending on relative distances $r_{ij} = \mid \mathbf{r}_i - \mathbf{r}_j \mid$ between particles, and their rates of changes $\dot{r}_{ij}$ [9, 12-15]. Most were for example built on the velocity-dependent Weber potential [13-15]

$$V_{Weber} = -\frac{Gm_i m_j}{r_{ij}}(1 - \frac{\dot{r}_{ij}^2}{2c^2}), \tag{1.6}$$

with G the gravitational constant and c the speed of light. This potential then takes the role of both kinetic and potential energy. Those theories indeed explain inertia as of gravitational origin, since the kinetic energy is part of the gravitational potential. At the same time, they are invariant under (1.5). However, they lead to an anisotropic inertial mass, which is ruled out experimentally[1]). In addition, they also yield the wrong inertial forces. This has led to a general refutation of theories built on such velocity-dependent potentials, which is unjustified, as we shall see in a moment.

Another remarkable attempt to explain inertia as of gravitational origin was made by Sciama [16]. He considered a gravitational field including an induction term (a gravitoelectric field), given by

$$\mathbf{E} = G(\nabla\varphi + \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}),$$

[1]The relative anisotropy of inertia expected by such potentials due to the contribution of e.g. the Milky Way to a particle's inertia is roughly $10^{-9}$, while the latest upper bound from experiment is $10^{-34}$[20]

with

$$\varphi(\mathbf{r}) = \int_{\mathbb{R}^3} \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\, d^3\mathbf{r}',$$
$$\mathbf{A}(\mathbf{r}) = \int_{\mathbb{R}^3} \frac{\rho(\mathbf{r}')\beta(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\, d^3\mathbf{r}'$$

the gravitoelectric potential and vector potential. He made the crucial observation that then an inertial term automatically arises from the gravitational field when transforming into any accelerated frame moving with a velocity v(t), since the vector potential A transforms as

$$\mathbf{A} \to \mathbf{A} + \varphi\frac{\mathbf{v}}{c}.$$

In a toy model using the gravitoelectromagnetic equations, he postulated that a particle always moves in a way that in its rest frame, the total gravitational field is zero. He could then show that the equations of motion read in a frame moving relative to it with a velocity v(t)

$$\frac{\varphi}{c^2}\frac{\partial \mathbf{v}}{\partial t} = \nabla\varphi + \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}, \tag{1.7}$$

where the term on the left side takes the role of the inertia term $m^*\mathbf{a}$, with the inertial mass then given by

$$m^* = m\frac{\varphi}{c^2}.$$

In the same way, in any frame moving again relative to this one, an additional inertial force is generated by the induction term on the right side of (1.7). Thus, Sciama had shown how inertia could be derived from a gravitoelectric field. Another crucial observation is that the gravitational constant cancels out on both sides of the equation (1.7). This implies that no gravitational constant has to be introduced in Sciama's toy model a priori. Instead, by dividing both sides of (1.7) by $\varphi/c^2$ and comparing the first term with Newton's law, Sciama obtained as an expression for the gravitational constant

$$G = \frac{c^2}{\varphi}. \tag{1.8}$$

As we will see later, his idea is precisely how inertia arises from the gravitational field in a theory built on a velocity-dependent gravitational potential. The reason why one does not need to postulate the gravitational constant, but can instead derive an expression for it is, as in Sciama's theory, the theory's ability to describe gravity and inertia in a unified law. Although it is well known that Gravitoelectromagnetism is present in linearized general relativity, Sciama's idea has never been built into a complete theory, neither non-relativistic nor relativistic.

A theory that *is* in principle capable of successfully implementing both Mach's principle and the idea of inertia being of gravitational origin is the inertia-free mechanics of H.J. Treder [5, 17, 18], on which we want to draw attention and build up our work. It uses the Riemann potential as a velocity-dependent gravitational potential, which was originally used by Riemann in his theory of electromagnetism [19, p. 325 f]

$$V_{Riemann} = -\frac{Gm_i m_j}{r_{ij}}(1 - \frac{\mathbf{v}_{ij}^2}{c^2}), \tag{1.9}$$

with $\mathbf{v}_{ij} = \mathbf{v}_i - \mathbf{v}_j$, which again takes the role of both kinetic and potential energy. It can implement both Mach's principle (after a suitable extension to rotational invariance) and the idea of inertia having a gravitational origin, without the short-coming of an anisotropic inertial mass. The theory will yield a unified description of gravity and inertia, the latter arising from the velocity-dependent part of the gravitational potential (1.9). Consequently, it will be possible to formulate classical mechanics without postulating the weak equivalence principle, a gravitational constant, or any concept of inertial mass or inertial forces a priori. We will show that all four can be derived from the theory. The inertial frames of reference are determined by all other particles in the universe, as demanded by Mach's principle. The exact Newtonian inertial forces will appear in any non-inertial frame, for translational and rotational acceleration. In the lowest order v/c we will re-obtain Newtonian mechanics, in the higher orders the theory gives rise to Gravitoelectromagnetism. Ultimately, we will derive an expression for the gravitational constant from the theory, allowing for an explanation of the apparent weakness of gravity.

## 2. The inertia-free mechanics

In this section, we want to present the inertia-free mechanics of H.J. Treder [5, 17, 18]. As we already saw, the Newtonian kinetic energy

$$T = \sum_i \frac{m_i}{2}\mathbf{v}_i^2$$

doesn't meet the requirements of Mach's principle. A theory incorporating it must necessarily depend on purely relative quantities, thus the kinetic energy must have the form (1.1). Further, we saw that it must not depend on the rates of changes of the distances $\dot{r}_{ij}$ , since this leads to mass anisotropy. Therefore, it must have the form

$$T = \sum_{i>j} m_i m_j f(r_{ij})\mathbf{v}_{ij}^2,$$

with f some arbitrary function. If we now implement the idea of inertia being of gravitational origin, we choose $f(r_{ij}) = b\frac{G}{c^2 r_{ij}}$ with b some dimensionless number, G the gravitational constant, and c the speed of light[2]). This makes the kinetic energy a velocity-dependent part of the gravitational potential

$$T = b\sum_{i>j} \frac{Gm_i m_j}{c^2 r_{ij}} \mathbf{v}_{ij}^2. \tag{2.1}$$

Together with the usual Newtonian gravitational potential

$$V = \sum_{i>j} \frac{Gm_i m_j}{r_{ij}}, \tag{2.2}$$

this yields the velocity-dependent Riemann potential, the Lagrange function of the inertia-free mechanics

$$L = \sum_{i>j} \frac{Gm_i m_j}{r_{ij}}(1 + \beta_{ij}^2), \tag{2.3}$$

with $\beta = \mathbf{v}/c$. This Lagrangian is invariant under any transformation $\mathbf{r} \to \mathbf{r} + \mathbf{g}(t)$. We will later (section 5) show how it can be extended to also be invariant under the full transformation (1.5) and as a consequence will completely satisfy Mach‘s principle.

Treder has shown that the energy corresponding to this Lagrangian is E=T+V with T and V given by (2.1) and (2.2), respectively. This quantity denotes the energy of the universe and is conserved, it holds

$$\frac{dE}{dt} = 0.$$

The generalized momentum of some particle k following from the Lagrange function (2.3) is

$$\mathbf{P}_k = \frac{\partial L}{\partial \mathbf{v}_k} = m_k^* \mathbf{v}_k - \frac{2bm_k G}{c}\mathbf{A}_k, \tag{2.4}$$

with the gravitoelectric potential and vector potential of the particles

$$\varphi_k = \sum_{j\neq k} \frac{m_j}{r_{kj}}, \tag{2.5}$$

$$\mathbf{A}_k = \sum_{j\neq k} \frac{m_j}{r_{kj}}\beta_j \tag{2.6}$$

and the inertial mass

$$m_k^* = \frac{2bG\varphi_k}{c^2} m_k. \tag{2.7}$$

[2]It must be noted that choosing c to be the speed of light here is not necessary and in fact, there is no strict physical reason for doing so. But it is convenient to do so, since it is undesirable to introduce another arbitrary velocity which doesn't appear anywhere else. Further, the structure of the relativistic theory, which has the theory presented here as classical limit also shows that it is indeed the speed of light.

This equation provides a relation between the inertial mass and the gravitational mass $m_k$. It shows, that the inertial mass (2.7) is induced by the gravity of all other masses in the universe. At the same time, it is isotropic, as demanded by the experiment. This can be seen by its scalar character[3]).

By demanding the strict equivalence of inertial and gravitational mass $m_k^* = m_k$, like is done in Newtonian theory, Treder obtained as a self-consistency condition of the theory

$$\frac{2bG\varphi_k}{c^2} = 1 \tag{2.8}$$

He interpreted this equation in the way that it determined the average gravitational potential of the universe for a *given* gravitational constant. By *demanding* $m_k^* = m_k$, Treder applied the weak equivalence principle. Apart from leading to problems with the equations of motion and ambiguities, this requirement is unnecessary. It will come out of the theory by itself, as a result of it correctly describing inertia as of gravitational origin. This will automatically yield a relation between inertial and gravitational mass and ultimately allow us to formulate classical mechanics without a priori introducing a gravitational constant. It, too, can be derived from the theory itself.

Further, equation (2.4) implies that the total momentum of the universe is zero

$$\mathbf{P} = \sum_k \mathbf{P}_k = 0, \tag{2.9}$$

since this is a symmetric sum over an antisymmetric quantity in the particle labels. Treder also derived an equation of motion from the Lagrangian for a simplified model of two particles moving in front of a distant background consisting of the other particles. In the next section, we will derive the exact equations of motion, therefore we don't present it here.

## 3. The equation of motion

In this section, we want to derive the equations of motion following from the Lagrangian (2.3). We will show that it is possible to formulate classical mechanics with postulating neither the weak equivalence principle nor a gravitational constant, nor any concept of inertial mass a priori. Instead, we will derive all three from the theory. We show that Newtonian mechanics is re-obtained in the lowest order $\beta$.

[3]If one used a Lagrangian based on the Weber potential (1.6) instead of the Riemann potential (1.9), then (2.7) would have tensorial character and thus be anisotropic.

As a correction, Gravitoelectromagnetism arises in the higher orders and a Lorentz-type force equation can be obtained. In the Lagrangian (2.3)[4])

$$L = \sum_{i>j} \frac{Gm_i m_j}{r_{ij}}(1+\beta_{ij}^2), \tag{3.1}$$

the gravitational constant appears in both terms, kinetic and potential energy. It is therefore nothing more than a constant factor, which doesn't change the equations of motion. We can drop it and write

$$L = \sum_{i>j} \frac{m_i m_j}{r_{ij}}(1+\beta_{ij}^2). \tag{3.2}$$

Consequently, it is not necessary to a priori introduce any gravitational constant, it will come out naturally later. It is important to notice that this step is possible *independent of the choice of units.* We have not specified any specific system of units to set G=1. It is a pure consequence of kinetic and potential energy both being proportional to G.

By using $\beta_{ij}^2 = \beta_i^2 + \beta_j^2 - 2\beta_i \cdot \beta_j$ and gathering together all terms involving the k th particle, one obtains for its Lagrangian

$$L_k = \frac{1}{2} m_k^* \mathbf{v}_k^2 + m_k \varphi_k - 2m_k \beta_k \cdot \mathbf{A}_k + \sum_{j\neq k} \frac{m_k m_j}{r_{kj}} \beta_j^2 \tag{3.3}$$

The first three terms in this expression are the Lagrangian for a particle in a gravitoelectromagnetic field, with a factor of 2 at the magnetic term. It is interesting to notice that the gravitomagnetic contribution to (3.3) arises due to the dependence of the Lagrangian (3.2) on the *relative* velocities. The inertial "mass" of the particle is given by[5])

$$m_k^* = \frac{2\varphi_k}{c^2} m_k. \tag{3.4}$$

We have thus derived an expression for the inertial mass. It is a scalar, showing again that inertial mass is isotropic. Notice, that we haven't introduced any concept of inertial mass a priori. All that appeared in the velocity-dependent gravitational potential were, by definition, gravitational masses. With the relation (3.4) between gravitational and inertial mass we have also derived the weak equivalence principle, that (inertial) mass and weight (gravitational mass) are locally in identical ratio for

[4]For simplicity and it being the natural choice, we set b=1. Treder used the value of b=3/2 to get the correct value for the perihelion shift of Mercury. Since we only have a non-relativistic theory which is to be generalised relativistically, we don't bother with getting the correct value here. Nevertheless, we already get 2/3 of the correct value 43" from a purely classical theory.

[5]The unit of this expression is not the one of a mass since we dropped G in the Lagrange function (3.2). If we kept it, the units would be correct, but G will cancel out in the equations of motion anyway. Consequently, nothing of what is said about the inertial mass in the following is altered by this "wrong" units.

all bodies. We want to remind at this point, that this is not possible in Newtonian mechanics. There is no reason to justify that inertial and gravitational mass in the Newtonian Lagrangian

$$T = \sum_i \frac{m_i^*}{2} \mathbf{v}_i^2 + \sum_{i>j} \frac{G m_i m_j}{r_{ij}} \tag{3.5}$$

are proportional or even equal to each other. It has to be *postulated* by applying the equivalence principle, for which no theoretical a priori justification exists. In the theory presented here, it is a consequence of the kinetic energy being a velocity-dependent part of the gravitational potential, as expressed by the Lagrangian (3.2).

For the generalized momentum, we obtain from (3.3)

$$\mathbf{P}_k = \frac{\partial L}{\partial \mathbf{v}_k} = m_k^* \mathbf{v}_k - \frac{2m_k}{c} \mathbf{A}_k. \tag{3.6}$$

Applying the Euler-Lagrange equations

$$\frac{d}{dt} \frac{\partial L}{\partial \mathbf{v}_k} = \frac{\partial L}{\partial \mathbf{r}_k}$$

to (3.3), one obtains the equation of motion

$$m_k^* \frac{\partial \mathbf{v}_k}{\partial t} = \mathbf{E}_k - 2m_k \sum_{j \neq k} \beta_{kj} \times \mathbf{B}_{kj}. \tag{3.7}$$

Here, the gravitoelectric and magnetic fields are given by

$$\mathbf{E}_k := - \sum_{j \neq k} \frac{m_j}{r_{kj}^3} \mathbf{r}_{kj} (1 - \beta_{kj}^2) + \frac{2}{c} \frac{\partial \mathbf{A}_k}{\partial t}, \tag{3.8}$$

$$\mathbf{B}_{kj} := \nabla_k \times \mathbf{A}_k + \beta_k \times \nabla_k \varphi_k \tag{3.9}$$

The partial time derivative in (3.8) means that only the velocity in A is to be differentiated in time. If one divides eq. (3.7) by the inertial mass $m_k^*$ , one obtains

$$\frac{\partial \mathbf{v}_k}{\partial t} = \frac{c^2}{2\varphi_k} (\mathbf{E}_k - 2 \sum_{j \neq k} \beta_{kj} \times \mathbf{B}_{kj}). \tag{3.10}$$

In the lowest order v/c this reduces to

$$\frac{\partial \mathbf{v}_k}{\partial t} = \frac{c^2}{2\varphi_k} \nabla_k \varphi_k, \tag{3.11}$$

which is Newton's law of gravity with the gravitational constant given by[6])

$$G_k = \frac{c^2}{2\varphi_k}. \tag{3.12}$$

It comes out naturally and does not have to be put in by hand. Equations (3.10 & 3.11) again show the weak equivalence principle: No inertial mass appears in it,

[6]An equation like (3.12) has been obtained in various works on Mach's principle [13, 16], cf. also eq (1.8).

implying the universality of free fall. Both are a direct consequence of the inertial mass being induced by gravity, according to (3.4). We will discuss this in more detail in section 7.

As a correction to Newton's law of gravity, we get Gravitoelectromagnetism, as can be seen by the full equation (3.10). It can also be written as

$$m_k \frac{\partial \mathbf{v}_k}{\partial t} = \mathbf{F}_k, \tag{3.13}$$

with the gravitoelectromagnetic force

$$\mathbf{F}_k = m_k G_k (\mathbf{E}_k - 2 \sum_{j \neq k} \beta_{kj} \times \mathbf{B}_{kj}). \tag{3.14}$$

Unlike in the conventional Lorentz force, the magnetic part of the force here depends on the relative velocities. The equation of motion (3.14) is invariant under any transformation $\mathbf{r} \to \mathbf{r} + \mathbf{g}(t)$, as was the Lagrangian (3.2) it was derived from[7]). Therefore, it is indeed valid in arbitrary linear accelerated frames of reference. This can be seen from the fact that the potentials (2.5 & 2.6) behave under such a transformation as

$$\mathbf{A}_k \to \mathbf{A}_k + \varphi_k \frac{\mathbf{V}}{c}, \tag{3.15}$$

$$\varphi_k \to \varphi_k, \tag{3.16}$$

and therefore the fields (3.8 & 3.9) as

$$\mathbf{E}_k \to \mathbf{E}_k + \frac{2\varphi_k}{c^2} \frac{\partial \mathbf{V}}{\partial t}, \tag{3.17}$$

$$\mathbf{B}_k \to \mathbf{B}_k, \tag{3.18}$$

with$\mathbf{V}(t) := d\mathbf{g}/dt$. This results in the claimed invariance of (3.14) under $\mathbf{r} \to \mathbf{r} + \mathbf{g}(t)$.

## 4. The origin of inertia and the relativity of linear acceleration

In this section, we want to show how the inertial frames of reference are defined by the motion of all particles in the universe, as demanded by Mach's principle. Also, we show that any linear acceleration relative to those frames yields the exact Newtonian inertial forces and, further, that these forces are of gravitational origin. Since our Lagrangian is by now only invariant under transformations $\mathbf{r} \to \mathbf{r} + \mathbf{g}(t)$, we will restrict ourselves to linear accelerations here. In the next section, we will show how it can be made invariant under the complete set of transformations (1.5) and show, that the same then also applies to rotational accelerations.

[7]This, of course, also applies to (3.7) and (3.10)

In the force (3.14) acting on the particle k, we can identify the inertial force as that dependent on the accelerations of the other particles

$$\mathbf{F}_k^{inert} = \frac{2m_k G_k}{c} \frac{\partial \mathbf{A}_k}{\partial t}. \quad (4.1)$$

Indeed, according to the transformation law (3.15), this force behaves under a transformation $\mathbf{v} \to \mathbf{v} + \mathbf{V}(t)$ as

$$\mathbf{F}_k^{inert} \to \mathbf{F}_k^{inert} + m_k \frac{\partial \mathbf{V}}{\partial t}. \quad (4.2)$$

It therefore automatically yields the additional Newtonian inertial force when transformed into a new frame, accelerating relative to the old one. From (4.1), it can also clearly be seen that the inertial force is, like the inertial mass, of gravitational origin. It is just the gravitoelectric induction force. The inertial frames are those in which this force vanishes, which is equivalent to the condition

$$\frac{\partial \mathbf{A}_k}{\partial t} = 0. \quad (4.3)$$

The left side is dependent on the acceleration of all other particles in the universe. Therefore, the inertial frames are determined by the motion of all other particles in the universe, as demanded by Mach's principle. Especially, any frame accelerated relative to the one defined by (4.3) will experience an inertial force

$$\mathbf{F}_k^{inert} = m_k \frac{\partial \mathbf{V}}{\partial t}, \quad (4.4)$$

which means that the well-known Newtonian inertial force arises in any frame accelerated relative to the rest frame defined by the universe via (4.3).

The inertial force (4.1) also gives rise to a "dragging" effect: If some particle j accelerates, this induces a drag force on particle k equal to

$$\mathbf{F}_{kj}^{inert} = \frac{2m_k m_j G_k}{c^2 r_{kj}} \frac{\partial \mathbf{v}_j}{\partial t}.$$

If now the whole universe would accelerate uniformly with $\partial \mathbf{V}/\partial t$, then the whole inertial force on the particle k would be again equal to (4.4). The universe drags the particle with it. Even though the particle is resting in the rest frame of absolute space, it experiences the same inertial force as if it were in a non-inertial frame accelerating with $-\partial \mathbf{V}/\partial t$. In Newtonian mechanics, the particle's acceleration, in this case, would be zero: It would not recognize the motion of the universe. The inertial frame of reference is again defined by (4.2), which means that we have to transform into a frame accelerating with $\partial \mathbf{V}/\partial t$ to satisfy the condition. The inertial frame is the one co-accelerating with the universe. It can thus be seen, that the role of absolute space as the universal inertial frame has been removed. As demanded by Mach's critique, it no longer plays any role in when inertial forces arise, but only the accelerations relative to the other particles in the universe matter.

The above-mentioned is a consequence of the relativity of linear acceleration, which is realized in the theory. It is dynamically equivalent if a particle is accelerating, or the rest of the universe is accelerating in the opposite direction. The reason for this is that only accelerations relative to all other particles in the universe enter the equation of motion (3.7). This was already shown by Treder for a simplified model. If we bring the inertial force (4.1) to the other side in (3.7), we can write the left side as

$$m_k^* \frac{\partial \mathbf{v}_k}{\partial t} - \frac{2m_k}{c}\frac{\partial \mathbf{A}_k}{\partial t} = \sum_{j\neq k} \frac{2m_k m_j}{c^2 r_{kj}} \frac{\partial \mathbf{v}_{kj}}{\partial t}. \tag{4.5}$$

Indeed, this expression contains only accelerations relative to the other particles.

We can also show how the whole inertia term of a particle $m_k^* \partial \mathbf{v}_k / \partial t$ can be derived from what appears as purely a vector potential in its rest frame. The mechanism is the same that had been proposed by Sciama [16]. Indeed, the whole momentum (3.6) is just the vector potential $\mathbf{A}_k^{'}$ as seen in the particle's rest frame

$$\mathbf{P}_k^{'} = -2m_k \mathbf{A}_k^{'}. \tag{4.6}$$

Further, the right side of (4.5) just is the time derivative of the vector potential seen in this frame

$$\sum_{j\neq k} \frac{2m_k m_j}{c^2 r_{kj}} \frac{\partial \mathbf{v}_{kj}}{\partial t} = -\frac{2m_k}{c}\frac{\partial \mathbf{A}_k^{'}}{\partial t}. \tag{4.7}$$

Thus, in this frame, the equation of motion reads

$$\mathbf{E}_k^{'} - 2m_k \sum_{j\neq k} \beta_{kj}^{'} \times \mathbf{B}_{kj}^{'} = 0. \tag{4.8}$$

This equation expresses that the total gravitational field in the particle's rest frame is zero; this is what had been postulated by Sciama. If we now transform into an arbitrary moving frame, the particle's velocity in this system is $\mathbf{v}_k$ . We get according to the transformation law (3.15)

$$\mathbf{P}_k = \frac{2m_k \varphi_k}{c^2}\mathbf{v}_k - \frac{2m_k}{c}\mathbf{A}_k = m_k^* \mathbf{v}_k - \frac{2m_k}{c}\mathbf{A}_k \tag{4.9}$$

and for the equation of motion (4.8) again with (3.17 - 3.18)

$$m_k^* \frac{\partial \mathbf{v}_k}{\partial t} = \mathbf{E}_k - 2m_k \sum_{j\neq k} \beta_{kj} \times \mathbf{B}_{kj}. \tag{4.10}$$

One can see how the inertia term arises from what appears as purely the vector potential in the particle's rest frame.

## 5. The relativity of rotation

As was stated in the beginning, the Lagrangian (3.2) is only invariant under translations $\mathbf{r} \to \mathbf{r} + \mathbf{g}(t)$, but not under rotations, and therefore not under the full transformation (1.5). In this section, we want to show how the theory can be extended to be invariant under arbitrary transformations (1.5) and thus fully incorporate Mach's principle. We will then be able to obtain the same results that we obtained for linear acceleration in the previous section, also for rotational acceleration.

To extend the Lagrangian, we adopt an idea used by Lynden-Bell & Katz in their approach to a theory incorporating Mach's principle [11]. We write

$$T' = \sum_{i>j} \frac{m_i m_j}{c^2 r_{ij}} (\mathbf{v}_{ij} - \mathbf{\Omega} \times \mathbf{r}_{ij})^2 \tag{5.1}$$

and minimize for $\mathbf{\Omega}$

$$\mathbf{M} := \frac{dL}{d\mathbf{\Omega}} = 0. \tag{5.2}$$

This yields

$$\mathbf{J} = I\mathbf{\Omega}, \tag{5.3}$$

$$\mathbf{J} := \sum_{i>j} \frac{m_i m_j}{c^2 r_{ij}} \mathbf{r}_{ij} \times \mathbf{v}_{ij}, \tag{5.4}$$

$$I := \sum_{i>j} \frac{m_i m_j}{c^2 r_{ij}} (\mathbf{r}_{ij}^2 \delta - \mathbf{r}_{ij} \otimes \mathbf{r}_{ij}). \tag{5.5}$$

As in the paper of Lynden-Bell & Katz, J is the angular momentum of the universe around its centre of mass, I its moment of inertia around the centre of mass[8]). Equation (5.2) expresses that the total angular momentum M of the universe is zero, just like the regular momentum P according to (2.9). The presence of the additional $r_{ij}$ terms doesn't change the behaviour of (5.1) under rotations compared to the expression obtained by Lynden-Bell & Katz

$$T' = \sum_{i>j} \frac{m_i m_j}{2M} (\mathbf{v}_{ij} - \mathbf{\Omega} \times \mathbf{r}_{ij})^2, \tag{5.6}$$

since distances remain invariant. Therefore, the kinetic energy (5.1) is also invariant under arbitrary rotations. Again adding the Newtonian potential V, we obtain the Lagrangian

$$L' = T' - V = \sum_{i>j} \frac{m_i m_j}{c^2 r_{ij}} [1 + (\mathbf{v}_{ij} - \mathbf{\Omega} \times \mathbf{r}_{ij})^2]. \tag{5.7}$$

[8] Since the gravitational constant had been dropped in the Lagrangian (3.2) and consequently also in the kinetic energy (5.1), the units are again different from the usual units of J and I.

Using the relation $\mathbf{\Omega} \cdot \mathbf{J} = \mathbf{\Omega} \cdot I\mathbf{\Omega}$, which follows from (5.3), we can write this Lagrangian in a compact form as

$$L' = L - \mathbf{\Omega} \cdot \mathbf{J}, \tag{5.8}$$

where L is the Lagrangian (3.2). This is our final Lagrangian, fully incorporating Mach's principle and the idea of inertia being of gravitational origin. With the definitions

$$\mathbf{v}' := \mathbf{v} - \mathbf{\Omega} \times \mathbf{r}, \tag{5.9}$$

$$\mathbf{A}'_k := \sum_{j \neq k} \frac{m_j}{r_{kj}} \beta'_j, \tag{5.10}$$

the generalized momentum derived from this is now (see appendix)

$$\mathbf{P}'_k = m^*_k \mathbf{v}'_k - \frac{2m_k}{c} \mathbf{A}'_k. \tag{5.11}$$

From this, one can see that the total linear momentum of the universe is again zero, since (5.11) is still an antisymmetric quantity in the particle labels so we have

$$\mathbf{P}' = \sum_k \mathbf{P}'_k = 0. \tag{5.12}$$

An important thing to notice here is that the equation for vanishing angular momentum (5.2) as well as the one for linear momentum (5.12) are invariant under the general set of transformations (1.5). Thus, they have an absolute meaning in the sense that momentum and angular momentum of the universe are exactly equal to zero in *any* frame of reference.

For the equation of motion for some particle k one obtains from this (see appendix)

$$m^*_k \frac{\partial \mathbf{v}_k}{\partial t} = m^*_k (\dot{\mathbf{\Omega}} \times \mathbf{r}_k + 2\mathbf{\Omega} \times \mathbf{v}_k + (\mathbf{\Omega} \times \mathbf{r}_k) \times \mathbf{r}_k) + m_k \mathbf{E}'_k - 2m_k \sum_{j \neq k} \beta'_{kj} \times \mathbf{B}'_{kj}. \tag{5.13}$$

The gravitoelectric and magnetic fields are the ones introduced in (3.8 & 3.9), now as seen in a frame rotating with an angular frequency $\mathbf{\Omega}$

$$\mathbf{E}'_k := - \sum_{j \neq k} \frac{m_j}{r^3_{kj}} \mathbf{r}_{kj} (1 - \beta'^2_{kj}) + \frac{2}{c} (\frac{\partial \mathbf{A}'_k}{\partial t} - \mathbf{\Omega} \times \mathbf{A}'_k), \tag{5.14}$$

$$\mathbf{B}'_{kj} := \nabla_k \times \mathbf{A}'_k + \beta'_k \times \nabla_k \varphi_k. \tag{5.15}$$

The inertial "mass" is again given by (3.4)

$$m^*_k = \frac{2\varphi_k}{c^2} m_k.$$

If one divides by $2\varphi_k/c^2$, one can write eq. (5.13) in the form

$$m_k \frac{\partial \mathbf{v}_k}{\partial t} = \mathbf{F}_k^{inert,rot} + \mathbf{F}_k^{GEM}, \tag{5.16}$$

$$\mathbf{F}_k^{inert,rot} = m_k(\dot{\mathbf{\Omega}} \times \mathbf{r}_k + 2\mathbf{\Omega} \times \mathbf{v}_k + (\mathbf{\Omega} \times \mathbf{r}_k) \times \mathbf{r}_k), \tag{5.17}$$

$$\mathbf{F}_k^{GEM} = m_k G_k(\mathbf{E}'_k - 2\sum_{j\neq k} \beta'_{kj} \times \mathbf{B}'_{kj}). \tag{5.18}$$

Here, $\mathbf{F}_k^{GEM}$ is the gravitoelectromagnetic force introduced in (3.14), as seen in a frame rotating with an angular frequency $\mathbf{\Omega}$. $\mathbf{F}_k^{inert,rot}$ is an additional inertial force, caused by the rotation of the universe. It exactly agrees with the Newtonian expression. $G_k$ is again the gravitational constant, given by (3.12). We can see, again, that the inertial frames of reference are determined by all other particles in the universe, as demanded by Mach's principle: The inertial force (5.17) vanishes in exactly that frame of reference where $\mathbf{\Omega} = 0$. This is according to (5.3) equivalent to the condition

$$I^{-1}\mathbf{J} = 0, \tag{5.19}$$

The left side of this equation is purely dependent on relative quantities between all particles in the universe, cf. equations (5.4 & 5.5).

Since the Lagrangian (5.7) is independent of the rotation of the frame chosen to write it in, so is the equation of motion (5.16). $\mathbf{\Omega}$ is the angular frequency of the universe perceived in this frame. Consequently, the exact Newtonian inertial forces arise automatically in any frame rotating relative to the one defined by (5.19). On the other hand, the force on the right side of (5.16) still transforms according to (4.2)

$$\mathbf{F}_k^{inert,rot} + \mathbf{F}_k^{GEM} \to \mathbf{F}_k^{inert,rot} + \mathbf{F}_k^{GEM} + m_k \frac{\partial \mathbf{V}}{\partial t} \tag{5.20}$$

under a linear transformation $\mathbf{v} \to \mathbf{v}+\mathbf{V}(t)$, as can easily be checked. Therefore, also the inertial force for linear acceleration still arises, as was discussed in the previous section. We have thus shown that all Newtonian inertial forces arise automatically from the theory in arbitrary accelerated frames.

The above also again implies, like was discussed for linear acceleration, that a particle at rest in the rest-frame of absolute space would experience the same centrifugal forces if the rest of the universe were rotating, then it would experience if itself would be rotating with the same angular frequency. In Newtonian mechanics, this is not the case: The particle would experience no force if just the universe were rotating. This is a manifestation of the relativity of rotation. It is dynamically equivalent if the particle is rotating or the universe is rotating in the other direction. This is mathematically expressed by the fact that only rotations relative to the universe enter the Lagrange function (5.7).

This also answers Mach's criticism of Newton's bucket: The Newtonian inertial forces arise in any frame rotating relative to the one defined by all other masses in the universe via (5.19). The water in the bucket is pushed upwards because it is rotating relative to the universe, not absolute space like in Newtonian theory. If now the walls of the bucket would hypothetically become thicker and thicker, ultimately consisting of the matter of the entire universe, then the water wouldn't be pushed up against the walls at all anymore, because the inertial frame would be that co-rotating with the universe, which would in this case be the walls of the bucket. But this is exactly the one, in which the water rests.

*Relation to the theory of Barbour & Bertotti, respectively Lynden-Bell & Katz:*

Finally, we want to point out the difference between our theory and the ones developed by Barbour & Bertotti [10] and Lynden-Bell & Katz [11]. Since both theories are mathematically equivalent, we only deal with Lynden-Bell & Katz' theory here, since their structure is very similar to the theory developed here. But everything applies to the theory of Barbour & Bertotti, too. Lynden-Bell & Katz used as kinetic energy for their Lagrangian

$$T' = \sum_{i>j} \frac{m_i m_j}{2M} (\mathbf{v}_{ij} - \mathbf{\Omega} \times \mathbf{r}_{ij})^2, \tag{5.21}$$

where M denotes the mass of the universe. The potential V then is just the Newtonian potential (2.2) (including! the gravitational constant). Thus, the Lagrangian of their theory is

$$L' = T' - V. \tag{5.22}$$

$\mathbf{\Omega}$ is now obtained in the same way as in our theory, by demanding (5.2). It results in equations similar to (5.3-5.5)

$$\mathbf{J} = I\mathbf{\Omega}, \tag{5.23}$$

$$\mathbf{J} = \sum_{i>j} \frac{m_i m_j}{2M} \mathbf{r}_{ij} \times \mathbf{v}_{ij}, \tag{5.24}$$

$$I = \sum_{i>j} \frac{m_i m_j}{2M} (\mathbf{r}_{ij}^2 \delta - \mathbf{r}_{ij} \otimes \mathbf{r}_{ij}). \tag{5.25}$$

From the Lagrangian (5.21), one can derive the equations of motion analogous to the derivation of (5.13) (see appendix) to obtain

$$m_k \frac{\partial \mathbf{v}_k}{\partial t} = m_k(\ddot{\mathbf{S}} + \dot{\mathbf{\Omega}} \times \mathbf{r}'_k + 2\mathbf{\Omega} \times \mathbf{v}'_k + (\mathbf{\Omega} \times \mathbf{r}'_k) \times \mathbf{r}'_k) - \frac{\partial V}{\partial \mathbf{r}_k}, \tag{5.26}$$

$$\mathbf{r}'_k := \mathbf{r}_k - \mathbf{S},$$

$$\mathbf{S} = \frac{1}{M} \sum_i m_i \mathbf{r}_i.$$

S is the position vector of the center of mass. Now, Lynden-Bell and Katz demanded, in addition to (5.2), that also the conditions

$$\mathbf{J} = 0, \tag{5.27}$$

$$\ddot{\mathbf{S}} = 0, \tag{5.28}$$

hold, which corresponds to a non-rotating universe $\mathbf{\Omega} = 0$, and that S is moving uniformly. In this case, their mechanics reduce to Newtonian mechanics, as can easily be seen by the equation of motion (5.26). However, neither of these demands is independent of the used frame. They are only invariant under Galilean transformations and thus only hold in unaccelerated frames. Therefore, they were not able to correctly obtain the inertial forces. However, their theory is capable of doing so, since it is, without the conditions (5.27 & 5.28), valid in arbitrary accelerated frames (it is invariant under the transformations (1.5) ). Those conditions are then just the *definitions* of the inertial frames of reference in the sense of Mach, like the equations (4.3) and (5.19) are for the theory presented here. In any frame accelerated relative to these, the inertial forces arise automatically as can be seen by the full equation of motion (5.26): The last three terms in the brackets on the right side are the centrifugal, Coriolis and Euler forces in any frame in which the perceived $\mathbf{\Omega}$ is non-zero (cf. eq. 5.16 & 5.17 for the theory presented here); the first term and therefore the whole right-hand side of (5.26) transforms as

$$m_k \ddot{\mathbf{S}} \to m_k \ddot{\mathbf{S}} + m_k \frac{\partial \mathbf{V}}{\partial t}$$

under a transformation $\mathbf{v} \to \mathbf{v} + \mathbf{V}(t)$ and thus yields the inertial force for linear acceleration (cf. eq. 5.20 for the theory presented here).

Further, it must be emphasized that demanding the conditions (5.27 - 5.28) is unnecessary for the universe to have zero angular and linear momentum. As is the case in our theory, the momentum and angular momentum of the universe vanishes, as expressed by (5.2) and (5.12). They are invariant under arbitrary transformations

(1.5) and thus hold in any accelerated frame of reference. The same is also true for Lynden-Bell's theory: The condition (5.2) for vanishing angular momentum holds for his theory, too (since it was, like in our theory, demanded for the derivation of (5.23-5.25)) and one can easily check that the momentum in his theory reads

$$\mathbf{P}'_k = \frac{\partial L'}{\partial \mathbf{v}_k} = \sum_{j\neq k} \frac{m_k m_j}{M}(\mathbf{v}_{kj} - \mathbf{\Omega} \times \mathbf{r}_{kj}). \tag{5.28}$$

Thus, it also fulfills (5.12), the total momentum of the universe is zero.

However, despite fulfilling Mach's principle and being able to explain the inertial forces, the theory does not explain inertia as of gravitational origin. The kinetic energy (5.21) is not proportional to the gravitational potential and thus the gravitational constant. It is therefore not possible to eliminate G from the Lagrangian. It still has to be put in by hand via the Newtonian potential appearing in the Lagrangian (5.22) and thus, the same G appears in the equation of motion (5.26). The same also applies to the inertial mass. Since there is no connection between the expression for the kinetic energy and the gravitational potential, there is no theoretical justification to assume that the masses in the kinetic energy are gravitational masses. There still is an a priori concept of inertial mass in Lynden-Bell's theory and the weak equivalence principle has to be postulated, so that $m_k^* \propto m_k$ holds.

## 6. Frame-dragging as a consequence of Mach's principle

Like in the linear case, the relativity of rotation gives rise to a dragging effect, which agrees qualitatively with the frame-dragging effect predicted by General relativity. Assume therefore some mass distribution around the origin, rotating with a velocity $\mathbf{v} = \omega \times \mathbf{r}$. According to (5.10), this gives rise to a vector potential

$$\mathbf{A}'_k = \sum_{j\in V} \frac{m_j}{r_{kj}} \omega' \times \mathbf{r}_j = \int_V \rho(\mathbf{r}_j) \frac{\omega' \times \mathbf{r}_j}{r_{kj}}\, d^3\mathbf{r}_j, \tag{6.1}$$

where $\omega' = \omega - \mathbf{\Omega}$ is the angular frequency of the mass distribution relative to the universe and V is its volume. We assume the distribution to be continuous and therefore replaced the sum by an integral in the second step. This expression agrees with the one obtained in linearized gravity for the same situation, except for the dependence on the *relative* angular frequency, instead of the absolute one. Therefore, from (6.1) all the well-known results can be derived for any mass distribution.

For the equation of motion, we neglect anything beyond the first order in the particle velocity $\mathbf{v}_k$ and the source velocities $\mathbf{v}_j$ , as well as the inertial forces caused by the rotation of the universe (which we assume to be small). Then we obtain

$$\frac{\partial \mathbf{v}_k}{\partial t} = G_k(\mathbf{E}'_k - 4\beta'_k \times \mathbf{B}'_k + \delta\mathbf{E}'_k), \tag{6.2}$$

$$\mathbf{E}'_k = \nabla_k \varphi_k + \frac{2}{c}(\frac{\partial \mathbf{A}'_k}{\partial t} - \mathbf{\Omega} \times \mathbf{A}'_k), \tag{6.3}$$

$$\mathbf{B}'_k = \nabla_k \times \mathbf{A}'_k, \tag{6.4}$$

$$\begin{aligned}\delta\mathbf{E}'_k = 2\nabla_k \int_V \rho(\mathbf{r}_j) \frac{\beta'_k \cdot (\omega' \times \mathbf{r}_j)}{r_{kj}}\, d^3\mathbf{r}_j \\ -2\nabla_k \times \int_V \rho(\mathbf{r}_j) \frac{\beta'_k \times (\omega' \times \mathbf{r}_j)}{r_{kj}}\, d^3\mathbf{r}_j.\end{aligned} \tag{6.5}$$

The equation of motion is again independent of the rotation of the frame chosen to write it in, as was already pointed out in the previous section. It agrees with the one obtained from linearized GR in the same order considered, apart from the last term in (6.2) (an additional acceleration that comes out of this theory). The potentials also agree with the ones obtained in linearized GR. There is, however, one crucial difference: The vector potential and the equation of motion, and therefore also the whole dragging effect, depend on the *relative* angular velocity $\omega' = \omega - \mathbf{\Omega}$ of the mass distribution to the universe, as well as the velocity $\mathbf{v}'_k = \mathbf{v}_k - \mathbf{\Omega} \times \mathbf{r}_k$. In GR, it depends on the absolute angular velocity $\omega$. This is again a manifestation of Mach's principle and the relativity of rotation.

Thus, we have shown that already a classical theory implementing Mach's principle and the idea of inertia being a gravitational effect gives rise to the frame-dragging effect. No curved spacetime or relativistic physics is necessary for it. As we have seen, it is just a consequence of the inertial frames of reference being defined by all other masses in the universe, as demanded by Mach's principle. If a mass distribution is rotating, the inertial frame for a particle close to it is one slightly co-rotating, depending on the distance and the mass of the distribution. Thus, the particle will also slightly corotate.

## 7. The gravitational constant

We have shown in section 3 that the kinetic and potential energy is proportional to the gravitational potential, and therefore the gravitational constant, which allows for its elimination from the Lagrangian (3.1). This is not possible with the Newtonian kinetic energy (3.5) or, as we saw, even the Machian one obtained by Lynden-Bell & Katz (5.21). Those kinetic energies are both not proportional to the gravitational potential and therefore don't depend on G. We can conclude that what finally allowed us to obtain an expression for the gravitational constant was the ability of the theory to correctly explain inertia as of gravitational origin.

Equation (3.12)

$$G_k = \frac{c^2}{2\varphi_k} \tag{7.1}$$

also allows us to shed light on the physical reason for the existence of the constant G. It is not the gravitational field itself, which has G built into it, but the inertial masses (3.4)

$$m_k^* = \frac{2\varphi_k}{c^2} m_k \tag{7.2}$$

have built in the factor 1/G, which is a direct result of their gravitational origin. Indeed, 1/G is the factor relating the inertial mass and the gravitational mass. Its value represents how large the inertial mass caused by the entire universe is. The gravitational constant is the inverse of this value (cf. the derivation of eq. (3.10 & 3.11) from (3.7)).

This also explains why the gravity appears to be such a weak force. One can approximately calculate the value of (7.1). For an approximately homogeneous universe, we have

$$\varphi_k \approx \int_V \frac{\rho(\mathbf{r}_j)}{r_{kj}} d^3\mathbf{r}_j \approx 4\pi\rho_0 \int_0^{R_u} r\, dr = \frac{3M_u}{2R_u}, \tag{7.3}$$

where $M_u, R_u$ are the mass and radius of the observable universe. For the gravitational constant, this yields

$$G = \frac{R_u c^2}{3M_u}. \tag{7.4}$$

Since there is such a huge amount of matter in the universe, particles have a very large inertial mass, and consequently, only experience very small gravitational accelerations. If there was considerably less matter in the universe, gravitational accelerations would be predicted to be much stronger. E.g. if the universe consisted only of the Milky Way, then they would be roughly $10^7$ times stronger than they are in our universe[9]), at least if c would keep its known value in such a situation. This conclusion was also reached by Sciama & Treder for similar expressions for G as (7.4) [16, 18].

It is well known that expression (7.4) is confirmed by observation. Plugging in the observed values $M_u \approx 10^{53} kg$ , $R_u \approx 4 \cdot 10^{26} m$ and $c \approx 3 \cdot 10^8 ms^{-1}$, one gets

$$G \approx 8 \cdot 10^{-11} m^3 kg^{-1} s^{-2}. \tag{7.5}$$

[9]If the universe indeed consisted only of the Milky Way we had $M_u \approx 10^{41} kg$ and $R_u \approx 10^{21} m$ , and therefore roughly $G \approx 6 \cdot 10^{-4}$ , $10^7$ times the known value.

Indeed, $GM_u/R_uc^2 \sim 1$ is one of the unexplained “cosmological coincidences”. In common theories, it is a coincidence; in the theory presented here, it is a confirmed prediction. If the matter content of the universe or its density were considerably different from what is observed, common theories would remain valid, where against the theory presented here would be disproved.

Equation (7.5) is an agreement to a very good accuracy since the mass and the radius of the universe are only known by orders of magnitude, and a homogeneous universe is only a rough approximation. Also, this is the result of a non-relativistic theory. The main contribution to the integral (7.3) comes from the most distant masses in the universe, for which retardation effects are expected to be non-negligible. Those can only be treated properly in a relativistic theory of what was presented here. It is therefore very unlikely that this relation is just a coincidence, rather than being anchored in an underlying theory like the one presented here.

## 8. Conclusion

We have shown how the phenomenon of inertia in non-relativistic mechanics can be explained in a unified theory of gravity and inertia, fully implementing Mach's principle, with inertia arising from a velocity dependence of the gravitational potential. Other than Newton's theory, the one presented here is not only valid in a preferred set of inertial frames of reference but also in arbitrary accelerated frames. The gravitational force automatically produces the correct Newtonian inertial forces for translational and rotational motion in any non-inertial frame, without having to postulate them. As a consequence of the theory's ability to explain inertia as of gravitational origin, it does not require to introducing the concept of inertial mass, the weak equivalence principle, or the gravitational constant a priori. Instead, we were able to derive all three from the theory itself, ultimately allowing us to explain the weakness of gravity. This also shows that the appearance of the unexplained constant G in common theories is tied to their inability to explain inertia as of gravitational origin correctly. The above leads us to the conviction that what was presented here is the correct formulation of classical non-relativistic mechanics.

Therefore, any full theory including relativistic and quantum phenomena should have the theory presented here as its classical limit and not rely on any notion of inertial mass a priori. Interestingly, this fact alone already, that inertial mass is no longer an intrinsic property of the particles, but a result of their gravitational mass and the gravitational field, has far-reaching consequences for the nature of mass and the elementary particles. This directly shows a way towards a quantum-relativistic version of the theory presented here. It implies, however, a very different origin of relativistic and quantum effects and thus suggests a new and different approach to a unified theory of gravity, relativistic and quantum phenomena, as well as the structure of the elementary particles. This, we will discuss in a subsequent paper.

## 9. Acknowledgements

We want to thank Dr. Julian Barbour and Dr. Tim Koslowski for reading and commenting on this work. Further, we want to thank Dr. Julian Barbour for extensive and helpful discussions.

## Appendix A (Derivation of the equations of motion)

We want to derive the expression for the equations of motion for the Lagrangian (5.8)

$$L' = L - \mathbf{\Omega} \cdot \mathbf{J} \tag{A.1}$$

with

$$\mathbf{J} = I\mathbf{\Omega} \tag{A.2}$$

and I, $\mathbf{J}$ given by (5.4 - 5.5). We first calculate

$$\frac{\partial}{\partial \mathbf{v}_k}\mathbf{\Omega} \cdot \mathbf{J} = 2\mathbf{\Omega} \cdot \frac{\partial \mathbf{J}}{\partial \mathbf{v}_k}, \tag{A.3}$$

$$\frac{\partial}{\partial \mathbf{r}_k}\mathbf{\Omega} \cdot \mathbf{J} = 2\mathbf{\Omega} \cdot \frac{\partial \mathbf{J}}{\partial \mathbf{r}_k} - \mathbf{\Omega} \cdot \frac{\partial I}{\partial \mathbf{r}_k}\mathbf{\Omega}. \tag{A.4}$$

To obtain them, it was used that $I^T = I$ and thus also $(I^{-1})^T = I^{-1}$ hold. Further, it was made use of the identity

$$\frac{\partial}{\partial \mathbf{r}_k}\mathbf{\Omega} = I^{-1}(\frac{\partial \mathbf{J}}{\partial \mathbf{r}_k} - \frac{\partial I}{\partial \mathbf{r}_k}\mathbf{\Omega}), \tag{A.5}$$

which is obtained by differentiating both sides of (A.2) with respect to $\mathbf{r}_k$ . Plugging in the definitions (5.4 - 5.5) of J and I and executing the derivatives, one obtains

$$\mathbf{\Omega} \cdot \frac{\partial \mathbf{J}}{\partial \mathbf{v}_k} = \sum_{j \neq k} \frac{m_k m_j}{c^2 r_{kj}} \mathbf{\Omega} \times \mathbf{r}_{kj}, \tag{A.6}$$

$$\mathbf{\Omega} \cdot \frac{\partial \mathbf{J}}{\partial \mathbf{r}_k} = \sum_{j \neq k} \frac{m_k m_j}{c^2 r_{kj}} \mathbf{v}_{kj} \times \mathbf{\Omega} - \sum_{j \neq k} \frac{m_k m_j}{c^2 r_{kj}^3} \mathbf{r}_{kj} \mathbf{\Omega} \cdot (\mathbf{r}_{kj} \times \mathbf{v}_{kj}), \tag{A.7}$$

$$\mathbf{\Omega} \cdot \frac{\partial I}{\partial \mathbf{r}_k}\mathbf{\Omega} = -\sum_{j \neq k} \frac{2 m_k m_j}{c^2 r_{kj}} \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}_{kj}) - \sum_{j \neq k} \frac{m_i m_j}{c^2 r_{ij}^3} \mathbf{r}_{kj} (\mathbf{\Omega} \times \mathbf{r}_{kj})^2. \tag{A.8}$$

Equation (A.6) together with (A.3) yield for the generalized momentum

$$\mathbf{P}'_k = \frac{\partial L}{\partial \mathbf{v}_k} - \frac{\partial}{\partial \mathbf{v}_k}\mathbf{\Omega}\cdot\mathbf{J} = \sum_{j\neq k}\frac{2m_k m_j}{c^2 r_{kj}}(\mathbf{v}_{kj} - \mathbf{\Omega}\times\mathbf{r}_{kj}). \qquad (A.9)$$

Using the definitions (5.9 & 5.10) as well as the expression (3.4) for the inertial mass, one obtains the claimed expression (5.11).

From this, one easily obtains

$$\frac{d}{dt}\mathbf{P}'_k = -\sum_{j\neq k}\frac{2m_k m_j}{c^2 r_{kj}^3}\mathbf{v}'_{kj}\cdot(\mathbf{v}'_{kj}\cdot\mathbf{r}_{kj})$$

$$+\sum_{j\neq k}\frac{2m_k m_j}{c^2 r_{kj}}(\dot{\mathbf{v}}_{kj} - \dot{\mathbf{\Omega}}\times\mathbf{r}_{kj} - \mathbf{\Omega}\times\mathbf{v}_{kj}), \qquad (A.10)$$

where we made use of the fact that $\mathbf{v}'_{kj}\cdot\mathbf{r}_{kj} = \mathbf{v}_{kj}\cdot\mathbf{r}_{kj}$ since $\mathbf{r}_{kj}\cdot(\mathbf{\Omega}\times\mathbf{r}_{kj}) = 0$ . Using (A.4), as well as (A.7 & A.8), one obtains for the generalized force

$$\frac{\partial L'}{\partial \mathbf{r}_k} = -\sum_{j\neq k}\frac{2m_k m_j}{c^2 r_{kj}}(\mathbf{\Omega}\times(\mathbf{\Omega}\times\mathbf{r}_{kj}) + \mathbf{v}_{kj}\times\mathbf{\Omega}) - \sum_{j\neq k}\frac{m_i m_j}{c^2 r_{ij}^3}\mathbf{r}_{kj}(1+\beta'^2_{kj}). \quad (A.11)$$

Equating (A.10) and (A.11) and collecting terms yields

$$\sum_{j\neq k}\frac{2m_k m_j}{c^2 r_{kj}}\frac{\partial \mathbf{v}_{kj}}{\partial t} = \sum_{j\neq k}\frac{2m_k m_j}{c^2 r_{kj}}(\dot{\mathbf{\Omega}}\times\mathbf{r}_{kj} + 2\mathbf{\Omega}\times\mathbf{v}_{kj} + (\mathbf{\Omega}\times\mathbf{r}_{kj})\times\mathbf{\Omega})$$

$$+\sum_{j\neq k}\frac{2m_k m_j}{c^2 r_{kj}^3}\mathbf{v}'_{kj}\times(\mathbf{v}'_{kj}\times\mathbf{r}_{kj}) + \sum_{j\neq k}\frac{m_i m_j}{c^2 r_{ij}^3}\mathbf{r}_{kj}(1-\beta'^2_{kj}). \qquad (A.12)$$

Now, the second term on the right side can be written as

$$\sum_{j\neq k}\frac{2m_k m_j}{c^2 r_{kj}^3}\mathbf{v}'_{kj}\times(\mathbf{v}'_{kj}\times\mathbf{r}_{kj}) = -2m_k\sum_{j\neq k}\beta'_{kj}\times\mathbf{B}'_{kj}, \qquad (A.13)$$

with the definition of B introduced in (5.15). Further, all remaining terms proportional to $\mathbf{r}_j$ and $\mathbf{v}_j$ and the 3$^{\text{rd}}$ term on the right can be collected to yield

$$\sum_{j\neq k}\frac{2m_k m_j}{c^2 r_{kj}}(\dot{\mathbf{v}}_j + \dot{\mathbf{\Omega}}\times\mathbf{r}_j + 2\mathbf{\Omega}\times\mathbf{v}_j + (\mathbf{\Omega}\times\mathbf{r}_j)\times\mathbf{\Omega}) - \sum_{j\neq k}\frac{m_i m_j}{c^2 r_{ij}^3}\mathbf{r}_{kj}(1-\beta'^2_{kj})$$

$$= -\sum_{j\neq k}\frac{m_j}{r_{kj}^3}\mathbf{r}_{kj}(1-\beta'^2_{kj}) + \frac{2}{c}(\frac{\partial \mathbf{A}'_k}{\partial t} - \mathbf{\Omega}\times\mathbf{A}'_k) = m_k\mathbf{E}'_k, \qquad (A.14)$$

with the definition of E introduced in (5.14). In the remaining terms, proportional to $\mathbf{r}_k$ and $\mathbf{v}_k$ , the summations can be carried out using

$$\sum_{j\neq k}\frac{2m_k m_j}{c^2 r_{kj}} = \frac{2m_k\varphi_k}{c^2} = m_k^*. \qquad (A.15)$$

Putting all three together into (A.12) yields the claimed equation of motion (5.13).

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