Conservation laws of stress-energy tensor in Yang-Mills theory

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Abstract

In this paper we present a new identity to associate the conservation laws of stress-energy tensor with the field equations in Yang-Mills theory. The Lorentz force is included with a consistent formulation as in Maxwell theory.

Key words: Stress-energy tensor, conservation law, Lorentz force, Yang-Mills theory.

1 Introduction

The conservation laws for stress-energy tensor are important in physics and have been researched in many theories such as in differential geometry by using the well known Bianchi identities [1] [2]. In this paper we discuss a new identity to show bilinear relations by using the Hodge star operator, then apply the identity to study the conservation laws in Yang-Mills theory in association with the conservation laws in Maxwell theory.

The stress-energy tensor $T^{\mu\nu}$ will be discussed to satisfy an identity $\partial_\nu T^{\mu\nu}+g^{\mu\nu}f_\nu = Z^\mu$ for a tensor Z^μ, where $g^{\mu\nu}$ is the inverse of the Minkowski metric and f_ν is the Lorentz force term. We show in this paper that the equation

$$D(D{*}F) = 0, \tag{1.1}$$

implies $Z^\mu = 0$, where $F = DA = dA - A\wedge A$ is the Yang-Mills field, $A = A_\nu\, dx^\nu$ is the connection, D is the covariant derivative, and $*$ is the Hodge star. The conservation laws $\partial_\nu T^{\mu\nu}+g^{\mu\nu}f_\nu = 0$ are then implied if (1.1)

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is true. The equation (1.1) is satisfied for the self-dual case $*F = F$ because $DF = 0$. In the self-dual case, the Lorentz force term also vanishes. It is then interesting to find non self-dual solution for (1.1) so that the conservation laws have non-zero Lorentz force term. The twistor space theory [3] works to find solutions for the self-dual Yang-Mills equation. And the integrable systems so reduced can be further developed to find non self-dual solution. We will discuss this separately based on the results in [4] [5].

The condition $Z^\mu = 0$ also includes other cases. This condition extends the role of the density conservation law. It will be discussed that $Z^\mu = 0$ is a bilinear equation in terms of density components j_ν and connection components A_ν. The density conservation law is traditionally a linear equation for j_ν. Here j_ν's are multiplied by the A_ν's. Equation (1.1) is a typical case for $Z^\mu = 0$ and it is linear in j_ν.

This paper is organized as follows. We focus on the derivative formulas for the physical equations in this short report, and the physical parameters or constants are not included here. In next section we discuss the identity in differential form in Yang-Mills theory for the conservation laws in accordance with the identity in Maxwell theory. In section 3, we convert the differential form to vector form to compare with the equations in Maxwell theory.

2 Identity for stress-energy tensor

Let us first talk about an identity for the conservation laws in Maxwell theory. It is well known in Maxwell theory [6] that for the field $F = dA = \frac{1}{2}F_{\mu\nu}\, dx^\mu \wedge dx^\nu$ we can easily get the following identity

$$d\,(\,F_\nu \wedge *F\,) - \frac{1}{2}\partial_\nu(F \wedge *F) + F_\nu \wedge d*F = A_\nu\, d^2 *F, \tag{2.1}$$

where $A = A_\mu\, dx^\mu$ is $U(1)$ connection, $*$ is the Hodge star and $F_\nu = \partial_\nu A - dA_\nu = F_{\nu\alpha}\, dx^\alpha$. The right hand side $A_\nu\, d^2 *F$ in (2.1) vanishes because $d^2 = 0$. By multiplying $g^{\mu\nu}$ on both sides of the equation above and using contraction, we get the conservation laws for the stress-energy tensor $T^{\mu\nu}$ in Maxwell theory,

$$\partial_\nu T^{\mu\nu} + g^{\mu\nu}\, f_\nu = 0, \tag{2.2}$$

for $\mu = 0, 1, 2, 3$, where $g^{\mu\nu}$ is g^{-1}, the inverse of the Minkowski metric g which is $(-, +, +, +)$. The stress-energy tensor $T^{\mu\nu}$ in Yang-Mills theory discussed next is similar to the above after properly changing d to D.

Consider the Yang-Mills field $F = dA - A \wedge A$ where $A = A_\mu \, dx^\mu$ is the connection. Here we use $F = dA - A \wedge A$ instead of $F = dA + A \wedge A$ for convenience in further discussions for solution and gauge invariance by using Lax pair [4] [5]. The new identity we introduce in this paper is the following,

$$\mathrm{tr}\,(\, d\,(\, F_\nu \wedge *F\,) - \frac{1}{2}\partial_\nu (F \wedge *F) + F_\nu \wedge D*F\,) = \mathrm{tr}\,(\, A_\nu \, D(D*F)\,), \qquad (2.3)$$

where $F_\nu = \partial_\nu A - dA_\nu - [A_\nu, A] = F_{\nu\alpha}\, dx^\alpha$. The proof is straightforward by using $D\omega_1 = d\omega_1 - A \wedge \omega_1$, $D\omega_2 = d\omega_2 - A \wedge \omega_2 + \omega_2 \wedge A$, and $D\omega_3 = d\omega_3 - A \wedge \omega_3 - \omega_3 \wedge A$ for k-form ω_k.

It is known in Yang-Mills theory [1] [2] that $*F = \frac{1}{4}\epsilon_{\mu\nu\alpha\beta} F^{\mu\nu} dx^\alpha \wedge dx^\beta$ where $F^{\alpha\beta} = g^{\alpha\mu}\, F_{\mu\nu}\, g^{\nu\beta}$. Also there is $F_\alpha^{\ \nu} = g^{\nu\beta} F_{\beta\alpha} = F^{\nu\gamma} g_{\gamma\alpha}$ where the left (first) index α is row index and right (second) index ν is column index for $F_\alpha^{\ \nu}$. We have

$$\begin{aligned} & g^{\mu\nu}\, \mathrm{tr}\, \left(d\,(\, F_\nu \wedge *F\,) - \frac{1}{2}\partial_\nu (F \wedge *F) + F_\nu \wedge D*F \right) \\ &= \partial_\nu \left(F^{\mu\alpha}\, F_\alpha^{\ \nu} - \frac{1}{4}\, g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \right) \mathrm{dvol} + g^{\mu\nu}\, f_\nu \, \mathrm{dvol} \\ &= (\partial_\nu T^{\mu\nu} + g^{\mu\nu}\, f_\nu)\, \mathrm{dvol}. \end{aligned}$$

By multiplying $g^{\mu\nu}$ on both sides of (2.3), the equation (2.3) then becomes

$$\partial_\nu T^{\mu\nu} + g^{\mu\nu}\, f_\nu = Z^\mu, \qquad (2.4)$$

where $Z^\mu \, \mathrm{dvol} = g^{\mu\nu}\, \mathrm{tr}\,(A_\nu \, D(D*F))$.

Therefore if the equations

$$\mathrm{tr}\,(A_\nu \, D(D*F)) = 0, \qquad (2.5)$$

for $\nu = 0, 1, 2, 3$ are satisfied, then the conservation laws $\partial_\nu T^{\mu\nu} + g^{\mu\nu}\, f_\nu = 0$ for $\mu = 0, 1, 2, 3$ hold. Specially when $D(D*F) = 0$, these equations are satisfied. In self-dual case $*F = F$, there is $D(D*F) = 0$ by using the first Bianchi identity $DF = 0$. In (2.3) it is required that the Minkowski metric is invariant. The twistor space theory discussed in [3] can be applied to find self-dual solution expressed by Painlevé functions and keeps metric invariant using confluent Killing vectors. The theories can be extended to find non self-dual solutions to be discussed separately based on the results in [4] [5].

3 Conservation laws in vector form

Now let us change the conservation laws discussed in last section in differential form into vector form in order to connect the tensor notation with the vector formulas. We have

$$F = DA = dA - A \wedge A = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu \tag{3.1}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - [A_\mu, A_\nu]. \tag{3.2}$$

Let us write F in matrix form as always in physics [1] [2] [6],

$$(F_{\mu\nu}) = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -B_3 & B_2 \\ -E_2 & B_3 & 0 & -B_1 \\ -E_3 & -B_2 & B_1 & 0 \end{pmatrix}, \tag{3.3}$$

where E_j and B_j are matrices, mathematically similar to the electric and magnetic fields in Maxwell theory. The density 1-form $j = \rho\, dx^0 - J_1\, dx^1 - J_2\, dx^2 - J_3\, dx^3$ satisfies [1] [2]

$$*j = -\rho\, dx^1 \wedge dx^2 \wedge dx^3 + J_1\, dx^0 \wedge dx^2 \wedge dx^3 + J_2\, dx^0 \wedge dx^3 \wedge dx^1 + J_3\, dx^0 \wedge dx^1 \wedge dx^2$$

as defined in the Yang-Mills equation $D{*}F = *j$. We can get the expression for the Lorentz force terms

$$\mathrm{tr}\,(F_0 \wedge D * F) = -\,\mathrm{tr}\,(\mathbf{J} \cdot \mathbf{E})\,\mathrm{dvol}, \tag{3.4}$$

$$\mathrm{tr}\,(\,(F_1, F_2, F_3)^T \wedge D * F\,) = \mathrm{tr}\,(\rho\,\mathbf{E} + \mathbf{J} \times \mathbf{B})\,\mathrm{dvol}, \tag{3.5}$$

where $\mathbf{J} = (J_1, J_2, J_3)^T$, $\mathbf{E} = (E_1, E_2, E_3)^T$, $\mathbf{B} = (B_1, B_2, B_3)^T$, and $\rho\,\mathbf{E} = (\rho\, E_1, \rho\, E_2, \rho\, E_3)^T$. Notice that $F_\nu = F_{\nu\alpha}\, dx^\alpha$ is corresponding to ν-th row in (3.3).

Denote $E^2 = \mathrm{tr}\,(\mathbf{E} \cdot \mathbf{E})$ and $B^2 = \mathrm{tr}\,(\mathbf{B} \cdot \mathbf{B})$. By using the Minkowski metric $(g^{\mu\nu}) = (-, +, +, +)$, we have

$$T^{00} = \mathrm{tr}\, F^{0\alpha} F_\alpha{}^0 - \frac{1}{4}\, g^{00}\, \mathrm{tr}\, F_{\alpha\beta}\, F^{\alpha\beta} = E^2 + \frac{1}{2}(-E^2 + B^2) = \frac{1}{2}(E^2 + B^2). \tag{3.6}$$

The stress-energy tensor can be expressed as the following matrix,

$$(T^{\mu\nu}) = \begin{pmatrix} \frac{1}{2}(E^2 + B^2) & p_1 & p_2 & p_3 \\ p_1 & -\sigma_{11} & -\sigma_{12} & -\sigma_{13} \\ p_2 & -\sigma_{21} & -\sigma_{22} & -\sigma_{23} \\ p_3 & -\sigma_{31} & -\sigma_{32} & -\sigma_{33} \end{pmatrix}, \tag{3.7}$$

where $(p_1, p_2, p_3)^T = \text{tr}\,(\mathbf{E}\times\mathbf{B})$ and

$$\sigma_{ij} = \text{tr}\,(E_iE_j + B_iB_j) - \frac{1}{2}(E^2+B^2)\delta_{ij}. \tag{3.8}$$

Denote $u = \frac{1}{2}(E^2+B^2)$ and $t = x^0$. The identity (2.3) or (2.4) is then changed into vector form,

$$\frac{\partial u}{\partial t} + \nabla\cdot\mathbf{p} + \text{tr}\,(\mathbf{J}\cdot\mathbf{E}) = Z^0, \tag{3.9}$$

$$\frac{\partial \mathbf{p}}{\partial t} - \nabla\cdot\sigma + \text{tr}\,(\rho\,\mathbf{E} + \mathbf{J}\times\mathbf{B}) = \mathbf{Z}, \tag{3.10}$$

where $\nabla = (\partial_1, \partial_2, \partial_3)$, $\mathbf{p} = (p_1, p_2, p_3)^T = \text{tr}\,(\mathbf{E}\times\mathbf{B})$, $\nabla\cdot\sigma = \nabla\cdot(\sigma_1, \sigma_2, \sigma_3)$, $\sigma_i = (\sigma_{1i}, \sigma_{2i}, \sigma_{3i})^T$ and $\mathbf{f} = \text{tr}\,(\rho\,\mathbf{E} + \mathbf{J}\times\mathbf{B})$ is the Lorentz force for the Yang-Mills field analogous to the Maxwell field case [6]. Also,

$$Z^0 = -\,\text{tr}\,(\,(D_0\rho + \mathbf{D}\cdot\mathbf{J})\,A_0\,) \tag{3.11}$$

$$\mathbf{Z} = \text{tr}\,(\,(D_0\rho + \mathbf{D}\cdot\mathbf{J})\,\mathbf{A}\,), \tag{3.12}$$

where $\mathbf{A} = (A_1, A_2, A_3)^T$, $\mathbf{D} = (D_1, D_2, D_3)^T$, and $D_\nu M = \partial_\nu M - A_\nu M + MA_\nu$ for $M = \rho$ or $M = J_\nu$. When Z^0 and $\mathbf{Z} = (Z^1, Z^2, Z^3)^T$ vanish, equations (3.9) and (3.10) become the conservation laws in vector form according to Maxwell theory.

The energy $u = E^2 + B^2$ is related to the Hamiltonian function of the Painlevé equation if we use twistor space. By scaling the variables, the Hamiltonian function becomes the X variable discussed in [5]. The different domains of the X variable is discussed in [5] in association with the phase transition problems in physics.

We see that the D^2 term discussed above is good to simplify the problems. As a discussion, in gravitational field theory there is also D^2 term, that is in the second Bianchi identity $R\wedge\beta = D(D\beta)$ for Ricci tensor R in differental form and coordinate frame β, for example see [2]. There is also identity different from Bianchi identities and Jacobi identity by using Riemann geometry to associate the conservation laws with some conditions. The identities so discussed can help to research the relations between the conservation laws as well as the connections to the field equations.

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